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A CONTRIBUTION TO THE PROBLEM OF COEXISTENCE OF TWO PERIODICAL SOLUTIONS OF THE HILL'S EQUATION

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Abstract. There are 17 theorems on characteristics of periodical solutions of the Hill's equation presented in this paper. These theorems were not presented either in the known monograph of Magnus and Winkler [1], or in Kamke's monograph [2], and [3].
An elementary approach to the most important equation in the oscillations theory – the Hill's equation, is given in this paper as opposed to the famous monograph [4] where the problem of periodicity of solutions is treated by means of the Floquet theorem. The approach is based on simple yet in the literature inadequately emphasized features of periodicity.
The particularly important question will be: when the integral of the only coefficient of the equation $\int b(x)dx$ is periodical, and when it is not. Depending on this, there are given various conditions for existence of entire and discontinuous solutions of the equation (1).
We will not deal with the particular case of (1), the equation $y'' + (\lambda + Q(x))\, y = 0$, $\lambda \neq 0$, where $b(x) = \lambda + Q(x)$, and $\int b(x)dx = \lambda x + \int Q(x)dx$ is not periodical, a case especially important for boundary problems solved by Floquet theorem in the way that there are two series of constants for λ $\{\lambda_n\}$, and $\{\lambda'_n\}$, whose alternating combination is crucial for the stability of the solutions.
Therefore, the paper is more based on and related to quadratural aspects.

1. INTRODUCTION

The issue of the equation $y''+b(x)y=0$, especially if $b(x)$ is a periodical function in which case the equation is called the Hill's equation, could not be considered totally resolved, although its solving under general assumptions has lasted for 150 years already; the issue of the harmonic oscillations $y''+n^2y=0$ is ever actual problem dating from the very beginnings of differential and integral calculi, particularly in practice.

In this paper we give some basically elementary theorems we have not met neither in theory nor in practice, related to the form of general periodical solutions of the Hill's equation (formulae (16)).

2. FUNDAMENTAL THEOREMS

In the famous monograph [1], for the Hill's equation

$$y''+b(x)y=0. \tag{1}$$

where $b(x)$ is a periodical function, a great attention is paid to the issue of concurrent existence of two linearly independent periodical and entire solutions with a common period. The theory hitherto (the classical Floquet theorem [2]) as well as practice had shown that (1) had only one periodical solution in a general case. That is the so called problem of coexistence.

It is of crucial importance that all the solutions are periodical for the sake of stability, since monotone solutions are likely to be unstable.

Unfortunately, the entire approach to this issue is not popular, due to its apparatus, for simple and technical practice.

We will try to give a simple and elementary approach to this issue, yet not any less strict and general than the latter.

In order to accomplish this, we have to specify some of our fundamental theorems, whose detailed exposure will be given at some other place.

Theorem 1. (The Basic Theorem) *In order to the differential equation (1) have periodical solutions, the neccessary condition is that the coefficient is periodical with the period* ω:

$$b(x+\omega) = b(x)$$

The following classical (known) theorem is important for the above mentioned:

Theorem 2. *For an integral of an analytic and periodical function to be a periodical function*

$$\int_0^x p(x)dx = P(x),\ P(x+\omega) = P(x),$$

necessary and sufficient is that the following is valid for the Fourier sum of $p(x)$

$$p(x) = a_0 + \sum_1^\infty a_k \cos kx + \sum_1^\infty b_k \sin kx: \tag{2}$$

(i) a_0=0
(ii) the sum $\sum\limits_1^\infty \frac{b_k}{k}$ *is convergent.*

For example, $\int \cos nxdx = \frac{1}{n}\sin nx + C$ is a periodical function, while $\int \cos^2 nxdx = \int \frac{1+\cos 2nx}{2}dx = \frac{x}{2} + \frac{\sin 2nx}{4n} + C$ is not, as $a_0 = \frac{n}{2\pi}\int\limits_0^{\frac{2\pi}{n}} \cos^2 nxdx = \frac{1}{2} \neq 0$.

However, the theorem on periodical integrals of square of periodical functions is also important, which has not been emphasized anywhere:

Theorem 3. *Integral of square of an entire and analytical periodical function*

$$\int p^2(x)dx$$

cannot be periodical.

We have not seen the proof anywhere; it is an elementary one (it exists most likely), and is based on the Fourier sum (2). From the sum there is

$$\int_0^x p^2(x)dx = \int_0^x \left(a_0^2 + \left(\sum_1^\infty a_k \cos kx \right)^2 + \left(\sum_1^\infty b_k \sin kx \right)^2 \right) dx +$$
$$+2a_0 \sum_1^\infty \int_0^x a_k \cos kx + 2a_0 \sum_1^\infty \int_0^x b_k \sin kx + 2 \sum_{k=1}^\infty \sum_{i=1}^\infty a_k b_i \int_0^x \cos kx \sin ixdx$$

Calculations of the above integrals give, among the others, linear functions $Ax+B$, which are not periodical.

The same is valid if $p(x)$ is discontinuous with finite discontinuities of the first order, when $p^2(x)$ is positive, and it has "removed" or "blank" points that do not affect the integral.

However, this does not apply if $p(x)$ is not analytical, having discontinuity in the form of poles. For example, the following integrals of squared elementary discontinuous periodical functions are periodical:

$$\int \frac{dx}{\cos^2 x} = \int \left(\frac{1}{\cos x} \right)^2 dx = \tan x + C \qquad \int \frac{dx}{\sin^2 x} = \int \left(\frac{1}{\sin x} \right)^2 dx = -\cot x + C$$
$$\int \frac{dx}{\sin x} = \int \left(\frac{1}{\sqrt{\sin x}} \right)^2 dx = \int \frac{\sin x}{1 - \cos^2 x} dx = \ln \tan \frac{x}{2} + C$$
$$\int \frac{dx}{\cos x} = \int \left(\frac{1}{\sqrt{\cos x}} \right)^2 dx = \int \frac{\cos x}{1 - \sin^2 x} dx = \ln \tan \left(\frac{x}{2} + \frac{\pi}{2} \right) + C$$

This is an important case for the Hill's equation.

Of course, if we look at periodicity through several periodicity intervals

$$(0, \omega), \ (\omega, 2\omega), \ (2\omega, 3\omega), ..., \ (n\omega, (n+1)\omega),$$

which only makes sense, then in the discontinuities of the periodical function $p^2(x)$ the improper integral, in the sense of the Cauchy principal value, should be regarded.

3. REDUCTION TO THE FIRST ORDER EQUATION

Let us go back to the equation (1). Substitution

$$\frac{y'}{y} = z \tag{3}$$

$(y' = zy\ ,\ y'' = z'y + zy' = z'y + z\,(yz) = y\,(z' + z^2))$
transforms the equation (1) to the Riccati equation of the first order

$$y\left(z' + z^2 + b(x)\right) = 0$$

and if the trivial solution $y = 0$ is neglected, we get

$$z' + z^2 = -b(x) \tag{4}$$

If we are looking for a periodical solution $y(x)$ of the equation (1), then according to the Basic Theorem 1, $b(x)$ must be a periodical function as well (or a constant). Yet, since the first derivative $y'(x)$ of a periodical function $y(x)$ is a periodical function as well, and since a quotient of two periodical functions is a periodical function too, (3) implies that $z(x)$ must also be a periodical function. Therefore, the problem transforms to looking for periodical solutions of an equation of the first order – the Riccati equation. Since the integration in (3) gives

$$\frac{dy}{y} = z(x)dx,\ \ln y = \int z(x)dx$$

$$y = e^{\int z(x)dx} \tag{5}$$

hence, in order $y(x)$ to be periodical: $z(x)$ must be periodical, and both $\int z(x)dx$ and $e^{\int z(x)dx}$ must remain periodical. For this, there is a need for some other theorems, which we also have not met in the literature, and which we proved for this purpose.

Theorem 4. *In order to the integral of a periodical function $p(x)$*

$$\int_0^x p(x)dx = P(x)$$

make a periodical function too, it is necessary and sufficient that $p(x)$ *has zeros of an odd degree within the interval of a single period* $(0, \omega)$.

Theorem 5. *The integral of any periodical function is in general case equal to a sum of a linear function* $L(x)$ *(monotone function) and a periodical function* $P(x)$*:*

$$\int_0^x p(x)dx = L(x) + P(x) \tag{6}$$

If the conditions of the theorem 4 are to be met, it implies from (6)

$$L(x) \equiv Const.$$

This means that a solution of the Hill's equation, by means of Riccati equation of lower order, can have the following form

$$y(x) = e^{L(x)+P(x)} \tag{7}$$

where linear function $L(x)$ reduces to a constant. However, the exponential function given in (7) is a positive function then, periodical though, without zeroes. It is the case of a non-characteristic periodical function whose integral

$$I = \int e^{\int z(x)dx} = \int e^{P(x)}dx \tag{8}$$

has no zeroes and is not a periodical function since it is monotone. Since it is not periodical function, it is undesirable in the practice. This case is also resolved by one of our theorems, which we have not seen in the literature:

Theorem 6. *In order to an integral of type (8) determines a periodical and oscillating function, it is necessary and sufficient that function* $P(x)$ *within it represents a logarithm of a periodical function* $\Pi(x)$*:*

$$P(x) = \ln \Pi(x)$$

i.e. from (8)

$$\int z(x)dx = \ln \Pi(x);$$

alternatively: the solution of the Riccati equation has to be a logarithmic derivative of a periodical function

$$z = \frac{\Pi'(x)}{\Pi(x)}.$$

This means that the solutions of the Riccati equation which have zeroes and poles and are periodical should be looked for.

Further, from (4) we get the normal form

$$z' = -z^2 - b(x)$$

implying

$$z(x) = -\int z^2 dx - \int b(x)dx \tag{9}$$

Let us look for an entire (analytical) and periodical solution $z(x)$, the one having a Fourier sum of the type (2). The theorem 3 then implies that the integral of squared function $\int z^2 dx$ cannot be periodical. Let the coefficient $b(x)$ be such that the integral

$$\int b(x)dx$$

is periodical as well.

From (9) then implies

$$z(x) + \int b(x)dx = -\int z^2 dx.$$

On the left side there is a sum of two periodical functions, i.e. a periodical function again; on the right side there is a non-periodical function, therefore this is not possible. From this discrepancy it could be seen that the Riccati equation does not have an entire and periodical solution $z(x)$, and since (5) implies that the solution of (1) $y(x)$ is entire as well if $z(x)$ is entire, therefore the Hill's equation does not have an entire and periodical solution in this case. From above it is implied:

Theorem 7. *If the periodical coefficient in the Hill's equation (1) has the feature that its integral $\int b(x)dx$ is also a periodical function, then the Hill's equation cannot have entire periodical solutions, i.e. solutions having a convergent expansion into Fourier sum.*

4. CONSEQUENCE - TWO APPROACHES TO RESEARCH OF HILL'S EQUATION

There are only two possible cases left:

(i) $\int b(x)dx$ in (9) is aperiodical

(ii) $z(x)$ is not an entire and periodical function at all, i.e. the theorem 3 is not valid for $\int z^2 dx$ – it is possible that integral of squared periodical function is a periodical function too (see the examples at the beginning). $z(x)$ is then a meromorphic function with poles.

4.1. $\int b(x)dx$ IS APERIODICAL

So that the function $z(x)$ in (9) is to remain periodical, the periodical coefficient $b(x)$ must have a feature that the cause of non-periodicity in $\int z^2 dx$ cancels with the cause of non-periodicity in $\int b(x)dx$. According to the theorem 5, the causes of non-periodicity are some linear functions, defined with (6). There is

$$z(x) = -\left(L_1(x) + P_1\right) - \left(L_2(x) + P_2\right),$$

and so that $z(x)$ can be periodical, there must be

$$-L_1(x) = L_2(x).$$

Since in

$$-\int z^2(x)dx = -\int \left[a_0 + \sum a_k \cos kx + \sum b_k \sin kx\right]^2 dx$$

the linear member must begin as $-a_0^2 x$, and if L_2 is a linear function

$$L_2(x) = kx + C_2,$$

it implies that $-a_0^2 = -k$, that is k must be positive, whereas C_1 and C_2 are not important for periodicity of P_1 and P_2. Accordingly, the following must be valid

$$-\int b(x)dx = -kx - C_2$$

$$-\int z^2 dx = -a_0^2 x + C_1$$

implying

$$\begin{aligned} b(x) &= k + P_1'(x) \\ z^2(x) &= a_0^2 + P_2'(x) \end{aligned} \tag{10}$$

where k depends on the Fourier coefficients in the like (2) expansion of $b(x)$, with the like (i) and (ii) conditions of the same theorem 2 applied

$$k = k\left(a_n, b_n\right)$$

a_0^2 is also dependant on the like Fourier coefficients for $\int z^2(x)dx$, in accordance with the theorem 3:

$$a_0^2 = a_0^2\left(A_n, B_n\right),$$

where, in order to cancel aperiodicity due to monotone linear functions in (9), a relation of the following type must apply

$$a_0\left(A_n, B_n\right) = \sqrt{k\left(a_n, b_n\right)}$$

where k and a_0^2 are the starting coefficients in related like (2) Fourier expansions for $b(x)$ and $z^2(x)$, and their integrals. The functions $P_1'(x)$ and $P_2'(x)$ are derivatives of periodical functions, consequently having no additive constants. Therefore, the following theorem applies

Theorem 8. *The Hill's equation (1) with a periodical coefficient $b(x)$ could have entire and periodical solutions, if the relations given above apply for the Fourier coefficients (a_n, b_n) of the given periodical function $b(x)$ and the unknown coefficients (A_n, B_n) of entire and periodical solution of the auxiliary Riccati equation (4), the function $z(x)$.*

If $z(x)$ is entire function and represents a periodical solution, (5) implies that the solution of the Hill's equation is entire and periodical as well, for $z(0) = 0$ (i.e. $y(0) \neq 0$ and $y'(0) = 0$).

Example 1. The equation $y''+(\sin x)\, y = 0$ has related Riccati equation $-\sin x = z' + z^2$ wherefrom through its first integral

$$\cos x - z = \int z^2 dx$$

the following is concluded: if z is a presumed entire and periodical solution, then $\int z^2 dx$ cannot be entire and periodical (according to the theorem 3), and the last equality is not possible. It implies that the equation has no entire and periodical solution.

Theorem 9. *The Hill's equation*

$$y'' + (K + Q'(x))\, y = 0 \tag{11}$$

where $Q'(x)$ is a derivative of a periodical function (implying that it has no additive constant), could have entire and periodical solutions.

Theorem 10. *The Mathieu equation*

$$y''+ (a + b\cos 2x)\, y = 0$$

beside the trivial case $a = n^2$, $b(x) = 0$ (which makes up for ordinary harmonic oscillations), could have other periodical solutions as well.

This is the basis for expansion of Mathieu functions in the famous monograph [5]. There remains only one logically possible case, discussed below.

4.2. THE SOLUTION OF THE RICCATI EQUATION (4) IS NOT BOTH ENTIRE AND PERIODICAL

This means that there is no convergent trigonometric series for z. Therefore, might be a meromorphic solution. In order to obtain it, let us start with the boundary case in (11), from which aperiodicity of $\int b(x)dx$ starts, in case if $Q'(x) \equiv 0$. There remains $b(x) = K$. The boundary Riccati equation then writes

$$z' + z^2 = -K.$$

Variables may be split in it:

$$z' = -z^2 - K, \; \frac{dz}{-K - z^2} = dx$$

and its solution could be acquired through integration. The integration gives periodical function if K is a positive constant only: $K = n^2$ (if $K < 0$ then the solution is monotone). Now we get

$$\frac{1}{n}\arctan\frac{z}{n} = -x + C,$$

wherefrom

$$z = n\tan\left(nC - nx\right),$$

which is a non-entire solution of the Riccati equation, since it has poles. In this case the corresponding solution of the Hill's equation writes (according to (5))

$$y_1 = e^{\int z dx} = e^{\int n\tan(nC - nx) dx} = e^{\int n\frac{\sin(nC-nx)}{\cos(nC-nx)}dx} = \cos\left(nC - nx\right)$$

The second solution y_2 of the Hill's equation, which now has the form $y'' + n^2 y = 0$, is easily determined by the substitution $y = y_1 W$, and it writes $y_2 = \sin\left(nC - nx\right)$. The general solution, after simple transformation, writes

$$y = C_1\cos x + C_2\sin x.$$

Thus, in a boundary case of the loss of periodicity for the coefficient $b(x)$, when $b(x) = n^2$, the Riccati equation has a periodical and discontinuous solution, and since the integral (5) converges in that case, the Hill's equation has an entire and periodical solution. In any other case of a periodical coefficient $b(x)$ in (10) the equation (1) could have a periodical solution, but it must not be entire one. There are the following theorems implied:

Theorem 11. *The Hill's equation (1) has an entire and periodical solution if $b(x) = n^2$ is a positive constant, and in that case (1) is a constant coefficients equation, determining harmonic functions.:*

Theorem 12. *If $b(x)$ is a discontinuous and periodical coefficient, then the Riccati equation (4) might have periodical solutions, if the improper integral*

$$v.p. \int_0^\omega b(x)dx$$

exists in the sense of the principal value.

Theorem 13. *The Hill's equation (1) could have only discontinuous periodical solutions if $b(x)$ is a discontinuous and periodical coefficient.*

5. AUXILLIARY THEOREMS AND THE FORM OF GENERAL SOLUTIONS

We will use the common classical theory of Liouville. It is known for the equation

$$y'' + a(x)y' + b(x)y = 0 \tag{12}$$

that two linearly independent particular integrals must be found in order to solve it:

$$y_1(x) \text{ and } y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} e^{-\int a(x)dx} dx. \tag{13}$$

Our special demand is that the both of them are periodical. Since for the Hill's equation (1) $a(x) = 0$, and the Liouville's formula

$$a(x) = -\frac{dW/dx}{W}$$

applies as well, where $W = W(x) = W(y_1, y_2)$ is the Wronskian of the equation (12), therefore it is valid for (1):

$$\frac{dW}{dx} = 0,\ W = C = 1 \text{ (without loss of generality)}.$$

Now the formula (13) for y_1 writes more simply

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} dx$$

and an integral of the theorem 3 type appears. Therefore the integral must be periodical, i.e.

$$\int \frac{1}{y_1^2(x)}dx = \pi(x) = \text{a periodical function.}$$

From the latter we get $\frac{1}{y_1^2(x)}dx = \pi'(x)$, or

$$y_1(x) = \frac{1}{\sqrt{\pi'(x)}} \tag{14}$$

and finally

$$y_2(x) = \frac{1}{\sqrt{\pi'(x)}}\pi(x) \tag{15}$$

a special form of periodical solutions of the Hill's equation.:

Theorem 14. *The Hill's equation has a specific and simple form of solution in the sense of periodicity of both linearly independent particular integrals:*

- one of the integrals is a reciprocal of a square root of derivative of a periodical function

- the second integral is the product of the periodical function and the above first integral.

Now, from (14) and (15), we have a simple form of general solution of the Hill's equation (1) when the both integrals are periodical:

$$y(x) = \frac{1}{\sqrt{\pi'(x)}}\left(C_1 + C_2\pi(x)\right). \tag{16}$$

Remark 1. Since it must be that $\pi'(x) \neq 0$, i.e. $\pi(x) \neq Const$, hence the trivial solution of the equation (1) $y = 0$ is obtained only for $C_1 = C_2 = 0$. Besides, it cannot be $\pi'(x) = Const$, because of $\pi(x) = Cx$, and it is not a periodical function. Therefore, the derivative $\pi'(x)$ must be an effective periodical function without an additive factor: $\pi'(x) \neq C + P(x)$; otherwise it would be $\pi(x) = Cx + \int P(x)dx$, a non-periodical function. It implies that $\pi'(x) = Const$ or $\pi'(x) = 0$ is a boundary case between periodicity and non-periodicity of solutions.

Remark 2. It is obvious from the form (14) that if $\pi(x)$ is an entire function, then $\pi'(x)$ is also entire function and it must have zeroes, and the solution most commonly have discontinuities.

Remark 3. Where is the place of the most important periodical functions – the simple harmonic oscillations? As seen, they are determined for $b(x) = Const$, and their Hill's equation writes $y'' + n^2 y = 0$.
Let it be in $\pi'(x) = \frac{1}{\cos^2 x}$ our solution (16). Then $\frac{1}{\sqrt{\pi'(x)}} = \cos x$. $\pi(x)$ itself is $\tan x$. The general solution is obtained from (14):

$$y = \cos x\,(C_1 + C_2 \tan x) = C_1 \cos x + C_2 \sin x.$$

Remark 4. In a similar fashion, we could obtain many other entire and periodical solutions.

The following theorems could be easily proved.

Theorem 15. *Every first order zero of the solution y_1 gives a first order pole of the coefficient $b(x)$.*

Theorem 16. *Every second order zero or first order pole of the solution y_1 gives a second order pole of the coefficient $b(x)$.*

Those theorems of recognition are important for practice. By them, a form of solution is constructed, and looked after exactly afterwards.

Theorem 17. (**The second Liouville formula for Hill's equation**) *There is the following relation between the solution y_1 and the coefficient $b(x)$:*

$$y_1(x) = \int \sqrt{\frac{b(x)}{\pi'(x)}}dx$$

where $\pi(x)$ is a periodical function.

Example 2. Consider a Hill's equation with discontinuous coefficient

$$y'' + \frac{1}{4}\left(1 + \frac{1}{\sin^2 x}\right) y = 0$$

The theoretical form of general solution, the addend $\frac{1}{\sin^2 x}$ and the known trigonometric relation between $\sin x$ and $\cot x$: $1 + \cot^2 x = \frac{1}{\sin^2 x}$ suggest that a particular integral of the type

$$y_1 = \sqrt{\sin x}$$

should be looked for. The direct test proves that it is a solution of the equation. The common procedures then give the second solution:

$$y_2 = \sqrt{\sin x}\ln\cot\frac{x}{2},$$

and the general solution is:

$$y = \sqrt{\sin x}\left(C_1 + C_2 \ln\cot\frac{x}{2}\right).$$

The solution is periodical. Therefore, y_1 and y_2 are in a periodical coexistence.

The corresponding Riccati equation (4) writes

$$P' + P^2 = -b(x) = -\frac{1}{4}\left(1 + \frac{1}{\sin^2 x}\right)$$

Its solution is not evident. However, from (5)

$$y_1 = e^{\int P dx} = \sqrt{\sin x},$$

wherefrom

$$\int P dx = \ln\sqrt{\sin x} \text{ and } P(x) = \frac{1}{2}\cot x.$$

Example 3. Consider a Hill's equation with discontinuous coefficient

$$y'' - \frac{3}{\sin^2 x \cos^2 x}\left(\frac{1}{4} + \sin^2 x\right) y = 0.$$

The corresponding Riccati equation (4) writes

$$P' + P^2 = -b(x) = \left(\frac{3}{\sin^2 x} + \frac{3}{\cos^2 x}\right)$$

and suggests that a periodical solution should be looked for in the form $\frac{\alpha}{\sin 2x} + \beta$. We find $\alpha = 3$, $\beta = 0$, and $P(x) = \frac{3}{\sin 2x}$. From (5), one solution is

$$y = e^{\int \frac{3}{\sin 2x} dx} = e^{\frac{3}{2}\int \frac{dx}{\sin x \cos x}} = e^{\frac{3}{2}\int \frac{\frac{dx}{\cos^2 x}}{\tan x}} = e^{\frac{3}{2}\int \frac{d(\tan x)}{\tan x}} = e^{\frac{3}{2}\ln \tan x} = e^{\ln \tan^{\frac{3}{2}} x} = \sqrt{\tan^3 x}.$$

The second solution is easily obtained in a common way, as well as the general solution

$$y_1 = \sqrt{\tan^3 x}\left(C_1 + C_2\left(\ln \sin x + \frac{1}{2\sin^2 x}\right)\right).$$

The solution is periodical, and the periodical solutions y_1 and y_2 are in coexistence.

Example 4. The Hill's equation

$$y'' - \left(\cos x + \sin^2 x\right) y = 0$$

has the corresponding Riccati equation

$$P' + P^2 = -b(x) = \cos x + \sin^2 x$$

where $P(x)$ should be a periodical function. It is obvious that it is valid to write $P = \sin x$, $P^2 = \sin^2 x$, $P' = \cos x$. Therefore, one solution is, according to (5), $y_1 = e^{\int P dx} = e^{-\cos x}$. The second solution is $y_2 = y_1 \int \frac{1}{y_1^2} dx = e^{-\cos x} \int e^{2\cos x} dx$, and it is not periodical since $e^{2\cos x}$ has no zeroes, and its integral is monotone, i.e. the integral cannot be a periodical function. The solution is entire and non-oscillating

$$y = e^{-\cos x}\left(C_1 + C_2 \int e^{2\cos x} dx\right).$$

This complies with our theory in the case 4.1., that is if $b(x)$ is entire function and $\int b(x)dx$ is non-periodical (as in this case), the general solution of (1) cannot be periodical.

Thus, there is no coexistence of periodicities. One solution is periodical, entire and elementary, and positive. The other solution is non-periodical, non-elementary, and positive.

Example 5. The Hill's equation

$$y'' - \frac{2}{\cos^2 x} y = 0$$

has the coefficient

$$b(x) = \frac{2}{\cos^2 x} = -\left(\frac{1}{\cos^2 x} + \frac{1}{\cos^2 x}\right) = -\left(\left(1 + \tan^2 x\right) + \tan' x\right),$$

that differs from the condition (4) for a single one. However, this still suggests checking if $y_1 = \tan x$ is a solution. It is easily confirmed. By means of Liouville's formula y_2 is obtained

$$\begin{aligned} y_2 = y_1 \int \frac{dx}{y_1^2} &= \tan x \int \frac{dx}{\tan^2 x} = \tan x \int \frac{\cos^2 x}{\sin^2 x} dx = \tan x \int \frac{1 - \sin^2 x}{\sin^2 x} dx \\ &= \tan x \left(\int \frac{dx}{\sin^2 x} - x\right) = \tan x\,(-\cot x - x) = -x \tan x - 1 \end{aligned}$$

The general solution is

$$y = C_1 \tan x + C_2\,(x \tan x + 1)\,,$$

and is not periodical, since the quadrature in y_2 is non-periodical: $\int \frac{dx}{\tan^2 x}$. One class of the solutions is periodical, the other one only oscillating.

The solutions are not in coexistence.

7. APPLICATIONS

The Hill's equation has been treated in several hundreds papers in mathematics and especially its applications in almost all of the natural sciences and engineering fields. According to [1], about 300 mathematical papers, and more than 700 applications had been published until 1965.

Thus, it would not be possible nowadays to collect all the literature without a widely organized venture, which could not be carried by individuals.

Let us mention only a few applications:

- Astronomy, Mechanics
- Physics, Engineering
- Theory of Electrical Circuits
- Metal Conductivity
- Cyclotron

Example 6. Pulse function. Related to a quantum theory problem (an electron in one-dimensional conductor), Kronig and Penney [6] were solving a special form of Hill's equation

$$y'' + (\lambda + Q(x))\, y = 0$$

where $Q(x)$ is a periodical function with the features:

(i) $Q(x) = -U_0,\ -b < x < 0$

(ii) $Q(x) = 0,\ 0 < x < a$

(iii) $Q(x + c) = Q(x),\ c = a + b$

Example 7. Equation of Frequency Modulation [7]

$$(1 + a \cos 2x)\, y'' + \lambda y = 0$$

$|a| < 1$ and $\lambda \in \mathbb{R}$ ($b(x)$ is quotient in this case).

References

[1] W. Magnus, W. Winkler, *Hill's Equation*, Interscience-Wiley, New York (1966).

[2] E. Kamke, *Handbook of Ordinary Differential Equations*, Nauka, Moscow (1971) (in Russian).

[3] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Interscience-Publishers, New York (1953), Vol. **1**, Chap. 5.

[4] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations,* McGraw-Hill, New York (1955).

[5] R. Campbell, *Théorie générale de l'Équation de Mathieu*, Masson et Cie editeurs, Paris (1955).

[6] R. Kronig, W. Penney, *Quantum Mechanics in Crystal Lattices*, Proc. Roy. Soc. London, **130** (1931), 499–513.

[7] E. Cambi, *Trigonometric Components of a Frequency-Modulated Waveuantum*, Proc. L.R.E., **36** (1948), 42–49.