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THE FIRST, THE SECOND, AND THE THIRD LIOUVILLE FORMULA AND PERIODICAL SOLUTIONS OF LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER

Miloje Rajović[1] and Rade Stojiljković[2]

[1] *Faculty of Mechanical Engineering, Kraljevo, Serbia*
[2] *School for Pre-School Teachers Gnjilane, Bujanovac, Serbia*
(e-mail: rajovic.m@maskv.edu.yu)

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Abstract. In this paper, the need for reviving the classical theory of Liouville is emphasized, in order to determine only periodical solutions of linear differential equations this time.

This problem, being one of the most important problems in the practice of differential equations (Mechanics, Mechanical engineering, Electronics) due to fast and in a way impulsive development of these sciences, has not been sufficiently systematically solved.

1. HISTORICAL AND TERMINOLOGICAL INTRODUCTION

There are formulae in the theory of linear differential equation connecting any (unknown) particular integral with a known particular integral, and the coefficients of the equation. Those formulae are of the following type

$$y_i = y_k \int f_i \left(y_j e^{\int a_1(x)}; \dots; y_k e^{\int a_n(x)} \right) dx$$

called Liouville formulae.

For the linear homogeneous differential equation of the first order

$$y' + a(x)y = 0,$$

there is the known solution

$$y = Ce^{-\int a(x)dx}, \tag{1}$$

being a proto-Liouville formula.

For the linear nonhomogeneous differential equation of the first order

$$y' + a(x)y + b(x) = 0,$$

there is the formula

$$y = e^{-\int a(x)dx}\left(C - \int b(x)e^{\int a(x)dx}\right)$$

containing two integrals of Liouville type

$$y_1 = e^{-\int a(x)dx},\ y_2 = \int b(x)e^{\int a(x)dx}, \tag{2}$$

and giving solution through coefficients only.

We emphasize that we have not seen anywhere the conditions for $a(x)$ and $b(x)$, in order for solutions to be periodical.

For the linear homogeneous differential equation of the second order

$$y'' + a(x)y' + b(x)y = 0 \tag{3}$$

there is the basic theorem:

If $y_1(x)$ is a known particular integral of the equation, then the second particular integral could be obtained in accordance with formula

$$y_2 = y_1 \int \frac{e^{-\int a(x)dx}}{y_1^2} dx. \tag{4}$$

This formula is called Liouville formula. It is derived from Fundamental theorem on general solution:

$$y = C_1 y_1 + C_2 y_2 \tag{5}$$

and determinant form of the equation (1)

$$\begin{vmatrix} y'' & y' & y \\ y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \end{vmatrix} = 0 \tag{6}$$

after introduction of Wronskian

$$W(x) = W\left(y_1, y_2\right) = \begin{vmatrix} y_1' & y_1 \\ y_2' & y_2 \end{vmatrix} = y_1'y_2 - y_1y_2' \neq 0. \tag{7}$$

After comparison of (1), (1) and (1) the following formula is acquired

$$a(x) = \frac{-\frac{dW}{dx}}{W}; \tag{8}$$

the formula is also called Liouville formula for Wronskian. After the substition

$$y = y_1 z, \tag{9}$$

the formula transforms to formula (1).

2. THE SECOND LIOUVILLE FORMULA

The above first Liouville formula connects two solutions y_1 and y_2 explicitely, and through single, the first, coefficients $a(x)$ of the equation (1) only, whereas the second coefficient is not explicitely visible, being implicitly hidden in the solutions y_1 and y_2 instead.

It is less known (we have not found it in the available literature at least) that a formula of the type (1) with both coefficients $a(x)$ and $b(x)$ present is possible as well. It is easily derived by development of the determinant (1) and comparison of (1), implying

$$b(x) = \frac{1}{W}\begin{vmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{vmatrix}.$$

From Wronskian (1) and (1), there is

$$W = e^{-\int a(x)dx},$$

wherefrom

$$y_1''y_2' - y_2''y_1' = \begin{vmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{vmatrix} = b(x)e^{-\int a(x)dx}$$

After division by $y_2'^2$ and integration,there is

$$\frac{y_1''y_2' - y_2''y_1'}{y_2'^2} = \frac{b(x)e^{-\int a(x)dx}}{y_2'^2} = \frac{d}{dx}\left(\frac{y_1'}{y_2'}\right)$$

$$\frac{y_1'}{y_2'} = \int \frac{b(x)e^{-\int a(x)dx}}{y_2'^2}dx$$

$$y_1 = \int y_2'\left(\int \frac{b(x)e^{-\int a(x)dx}}{y_2'^2}dx\right)dx. \tag{10}$$

In the obtained formula, a particular integral is expressed by means of the other particular integral and both coefficients $a(x)$ i $b(x)$, through three integrations. The formula provides for the influence of the coefficients on the solution.

3. THE THIRD LIOUVILLE FORMULA

Although the equation (1) is very important for research of conservative physical and technical processes, it does not apply to the most frequent phenomena of forced oscillations and other caused behaviours. That's why it is very important to investigate nonhomogeneous linear differential equation of the second order

$$y'' + a(x)y' + b(x)y = f(x). \tag{11}$$

For the equation is known: if (y_1, y_2) is a pair of linearly independent particular integrals of the corresponding linear homogeneous equation (1), then the solution of the equation (1) is obtained from the solution (1), if pressumption is made that C_1 i C_2 are not constants, but functions of x:

$$C_1 = C_1(x),\ C_2 = C_2(x).$$

This way, from

$$y = C_1(x)y_1 + C_2(x)y_2, \tag{12}$$

by means of Lagrange method of variation of constants, there is

$$\begin{aligned} C_1' y_1 + C_2' y_2 &= 0 \\ C_1' y_1' + C_2' y_2' &= f(x). \end{aligned} \tag{13}$$

The functions are found from the above

$$\begin{aligned} C_1(x) &= \int \frac{y_2}{W} f(x)dx + A \\ C_2(x) &= -\int \frac{y_1}{W} f(x)dx + B \end{aligned} \tag{14}$$

(where A and B are new integration constants).

Finally, the solution (1) of the equation (1) is

$$y = Ay_1 + By_2 + y_1 \int \frac{y_2}{W} f(x)dx - y_2 \int \frac{y_1}{W} f(x)dx. \tag{15}$$

From the general theory of differential equations, there is the known result that

$$Y_p = y_1 \int \frac{y_2}{W} f(x)dx - y_2 \int \frac{y_1}{W} f(x)dx \tag{16}$$

is a particular solution of the nonhomogeneous equation (1), if (y_1, y_2) are solution of the equation (1), for which the first (1) and the second (1) formula apply.

As the farest reach of the exact theory of linear differential equation is that the equation is solvable only if a particular integral is known somehow (it has been proved that this is a maximum), let's find y_1 i Y_p through a known $y_2(x)$. Replacing in (1) $Y_p = y_3$, and taking into consideration (1), (1) and (1), there is

$$\begin{aligned} y_3 \quad = \quad & y_1 \int e^{\int a(x)dx} \left(\int y_1' \left(\int \frac{b(x)e^{-\int a(x)dx}}{y_1'^2} dx \right) dx \right) f(x)dx \\ & - \left(\int y_1' \left(\int \frac{b(x)e^{-\int a(x)dx}}{y_1'^2} dx \right) dx \right) \cdot \int e^{\int a(x)dx} y_1 f(x)dx, \end{aligned} \tag{17}$$

wherefrom the third integral, i.e. the integral of the entire nonhomogeneous equation, is expressed through the first integral of homogeneous equation, its derivative and all the three coefficients $\{a(x), b(x), f(x)\}$ only.

The formula (1) contains four quadratures, which could independently influence periodicity of solution. This is the main issue for our further research. It is interesting that the important detail has not been researched in an appropriate extent.

4. THE PROBLEMS WITH RESEARCH OF PERIODICAL SOLUTIONS

It is astounding that, despite of tremendous importance of periodical solutions of the equations (1) and (1) in applications and technics, ther is no book on general aspect of periodicity of all solutions for equation (1) at least. Instead, research is spread in a vast number of various books and journals in mathematics, physics, technics, mechanics and all the natural sciences in a period of more than three centuries.

We have dared to do a little and humble try in this sense. In order to explain our formulae (1) and (1), we will gave several theorems of ours, whose simplicity disconcerts, whereas their absence in all the elementary courses of analysis and engineering mathematics as well as differential equations ([1], [2], [3]) brings up the conclusion that scientific aspect of this big problem has been going chaotically, instead of being consecutive, methodological and systematic. The best illustration for this big truth is that there are for example entire studies on solutions of differential equations of Lienard, Rayleigh, Van der Paul, Schrödinger etc, whereas there are not given neither nesseccary nor sufficient conditions for periodicity of solution of the equation (1).

Below are given some of our results, without proves.

Theorem 1. (The Fundamental Theorem) *In order for any differential equation*

$$F\left(x, y(x), y'(x), y'', ..., y^{(n)}; a_i(x)\right) = 0$$

to have periodical solution, the neccessary condition is that

1° either all the coefficients $a_i(x) = Const.$

2° or all the coefficients $a_i(x)$ are periodical with a mutual least period.

Theorem 2. *In order for the fundamental integral in the theory of linear differential equation*

$$y = e^{\int a(x)dx}$$

to be periodical, neccessary and sufficient is that the integral $\int a(x)dx$ is a logarithm of a periodical function

$$\int a(x)dx = \ln \Pi(x)$$

i.e. it must be

$$a(x) = -\frac{\Pi'(x)}{\Pi(x)} \tag{18}$$

Consequence 1. *As a derivative of an analytic and periodical function must have zeroes in a closed periodicity interval* $[0, \omega]$*, it means that the periodical coefficient* $a(x)$ *must change its sign.*

Theorem 3. *In order for the quadrature*

$$\int b(x) e^{\int a(x)dx} dx$$

to be periodical, it is neccessary that

1° $a(x)$ *and* $b(x)$ *are periodical functions with a same period*

and sufficient is that

2° $\int a(x)dx$ *is a logarithm of a periodical function,* $\int a(x)dx = \ln \Pi(x)$

3° *the integral* $\int b(x)\Pi(x)dx$ *is periodical as well (there is a special criterion on zeroes and signs for that).*

This way, the issues of periodicity of the results of formulae (1), (1), and (1) can be exhaustively analysed, in dependence of periodical threesome of coefficients $\{a(x), b(x), f(x)\}$ and appropriate noted integrals.

Naturally, it turns out that this is a big venture.

As an illustration of importance, we mention another important aspect of the issue.

5. PERIODICAL FORMS OF GENERAL SOLUTION OF LINEAR EQUATION AND THE HILL'S EQUATION

It is known that the equation (1) by the substitution

$$y = e^{-\int a(x)dx} \cdot z \tag{19}$$

transforms to its cannonical form

$$z'' + A(x)z = 0, \tag{20}$$

where

$$A(x) = b(x) - \frac{a'(x)}{2} - \frac{a^2(x)}{4}.$$

If both coefficients $a(x)$ and $b(x)$ are periodical, then $A(x)$ is periodical as well. The equation (1) is in that case called the Hill's equation ([4]). It has not been fully investigated until present. For example, in the monograph [5], the following results are missing.

Theorem 4. *General solution of the Hill's equation has the form*

$$z = \frac{1}{\sqrt{\pi'(x)}} \left(C_1 + C_2 \pi(x)\right). \tag{21}$$

It depends on a periodical function $\pi(x)$ and square root of its derivative only. Many new results for the Hill's equation could be derived from that.

Consequence 2. *The second Liouville formula applies to the Hill's equation (20). It gives one periodical solution*

$$z_1 = \int \sqrt{\frac{A(x)}{\pi'(x)}}\, dx. \tag{22}$$

Many important conclusions could be drawn from this, not existing in the monograph [5]. The monograph mostly employs the classical Floquet theorem, valid only for continuous coefficients $A(x)$, although the richness of periodicity is in improper and singular integrals of discontinuous functions, either convergent or not

$$\int \frac{dx}{\cos^2 x} = \tan x, \quad \int \cot x dx = \ln \sin x.$$

Besides, combinations of $y(x)$ with $a(x)$, $b(x)$, $f(x)$ in the formulae (1), (1), and (1) if discontinuous and infinite integrals appear, could still give both entire and analytical results. It is well-known from theory and practice. Moreover, general solution of the equation (1) for periodical case is obtained from (1) and (1):

Theorem 6. *General solution of linear homogeneous differential equation of the second order has the form*

$$y = \frac{\Pi(x)}{\sqrt{\pi'(x)}} \left(C_1 + C_2 \pi(x)\right) \tag{23}$$

where $\pi(x)$ *and* $\Pi(x)$ *are periodical functions.*

Those forms are much more illustrative about the periodical solution than classical Liouville form of solution

$$y = C_1 y_1 + C_2 y_1 \int \frac{e^{-\int a(x)dx}}{y_1^2} dx.$$

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