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MINIMUM PRINCIPLE FOR QUADRATIC SPLINE COLLOCATION DISCRETIZATION OF A CONVECTION-DIFFUSION PROBLEM

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Abstract. In this paper the quadratic spline difference scheme for a convection-diffusion problem is derived. With the suitable choice of collocation points we provide the discrete minimum principle. The numerical results implies the uniform convergence of order $\mathcal{O}(n^{-2}\ln^2 n)$.

1. INTRODUCTION

We consider the two parameter singularly perturbed boundary value problem

$$\begin{aligned} Ly := \varepsilon y''(x) + a(x)y'(x) - b(x)y(x) &= f(x), \quad x \in (0,1), \\ y(0) = p_0, \qquad y(1) &= p_1, \end{aligned} \tag{1}$$

where a, b and f are sufficiently smooth functions, $0 < \varepsilon \ll 1$ and

$$A \geq a(x) \geq a > 0, \quad B \geq b(x) \geq b > 0, \quad x \in [0,1].$$

Problem (1) is a convection-diffusion problem with boundary layer of width $\mathcal{O}(\varepsilon)$ near $x = 0$.

In [5] a reaction-diffusion problem is considered and numerically treated with spline collocation method on a slightly modified Shishkin mesh with standard collocation points. Paper [8] deals with convection-diffusion problem (1) where $b = 0$ and considers discretization of the problem using quadratic splines as approximation functions on a piecewise uniform Shishkin mesh. Since those method is non-monotone, apart from the consistency error, the construction and analysis of the Green's grid function had to be carried out in order to prove the uniform convergence. Therefore, the proof from [8] is complicate. In [6] and [7] two-parameter singularly perturbed problem is considered and the position of the collocation points are chosen so that minimum principle is achieved on a standard Shishkin mesh. In this paper we use technique from [5],[6] and [7] for the problem (1) and prove uniform convergence based on the inverse monotonicity.

Problem (1) satisfies the minimum principle.

Lemma 1. *If $g \in C^2[0,1]$ such that $Lg \leq 0,\ x \in (0,1)$ and $g(0) \geq 0,\ g(1) \geq 0$, then $g(x) \geq 0,\ x \in [0,1]$.*

2. SOLUTION DECOMPOSITION, MESH AND SCHEME

Lemma 2. *Let $a, b, f \in C^2[0,1]$. Then (1) has unique solution $y \in C^4[0,1]$*

$$y = v + w, \tag{2}$$

where

$$\begin{cases} Lv &= f, \\ Lw &= 0 \end{cases}$$

and

$$\begin{cases} |v^{(k)}(x)| &\leq M(1 + \varepsilon^{2-k} e^{-\delta x/\varepsilon}), \\ |w^{(k)}(x)| &\leq M\theta^k e^{-\theta x}, \end{cases} \qquad k = 0, 1, 2, 3, 4,$$

θ is positive root of the equation $\varepsilon\theta^2 - a\theta - b = 0$. δ and M are constant independent of ε and $\delta > 0$.

The above decomposition is given in [2] and [3], but the estimates for $|v^k(x)|$ for $k = 3, 4$ are obtained by the comparison of the results from [3] and [1].

We seek the solution of (1) in the form of quadratic spline $u(x)$ on the mesh $\triangle_n$ from [3]:

$$\triangle_n : x_0 = 0,\ x_1, x_2, \dots, x_n = 1,$$

where

$$x_i = \begin{cases} \frac{2\sigma}{n}\, i, & i \le \frac{n}{2} \\ \sigma + \frac{2}{n}(i - \frac{n}{2})(1-\sigma), & \frac{n}{2} \le i \le n \end{cases}$$

$$\sigma = \min\left\{\frac{1}{2}, \frac{2}{\theta}\ln n\right\}.$$

We suppose that $\sigma = \frac{2}{\theta}\ln n$ since otherwise we use standard procedure on equidistant mesh.

The function $u(x) \in C^1[0,1]$ has the form

$$u(x) = u_i + (x - x_i)u_i' + \frac{(x - x_i)^2}{2}u_i'', \quad x \in [x_i, x_{i+1}], \tag{3}$$

where $u_i = u(x_i)$, and

$$u_{i+1} = u_i + h_{i+1}u_i' + \frac{h_{i+1}^2}{2}u_i'', \qquad h_i = x_i - x_{i-1} \tag{4}$$

$$u_{i+1}' = u_i' + h_{i+1}u_i''. \tag{5}$$

We chose collocation points in a nonstandard way:

$$\xi_i = \alpha_{1i}x_{i-1} + (1 - \alpha_{1i})x_i, \text{ on } [x_{i-1}, x_i],$$

$$\eta_i = \alpha_{2i}x_i + (1 - \alpha_{2i})x_{i+1}, \text{ on } [x_i, x_{i+1}]$$

where $\quad 0 \le \alpha_{1i}, \alpha_{2i} \le 1.$

The collocation equations have the form

$$\varepsilon u''(\xi_i) + a(\xi_i)u'(\xi_i) - b(\xi_i)u(\xi_i) = f(\xi_i), \quad \text{for} \quad \xi_i \in [x_{i-1}, x_i]$$

$$\varepsilon u''(\eta_i) + a(\eta_i)u'(\eta_i) - b(\eta_i)u(\eta_i) = f(\eta_i), \quad \text{for} \quad \eta_i \in [x_i, x_{i+1}]$$

$i = 1, \dots, n$, where $u''(\xi_i) = u''_{i-1}, \quad u''(\eta_i) = u''_i$. In the construction of the scheme, apart from collocation equations (4) and (5) we use the following equations

$$u'(\xi_i) = u'_{i-1} + (1-\alpha_{1i})h_i u''_{i-1}, \quad u'(\eta_i) = u'_i + (1-\alpha_{2i})h_{i+1}u''_i,$$

$$u(\xi_i) = u_{i-1} + (1-\alpha_{1i})h_i u'_{i-1} + (1-\alpha_{1i})^2 \frac{h_i^2}{2} u''_{i-1},$$

$$u(\eta_i) = u_i + (1-\alpha_{2i})h_{i+1}u'_i + (1-\alpha_{2i})^2 \frac{h_{i+1}^2}{2} u''_i.$$

In order to shorten the notation we omit the indices and use the following notation $f^+ = f(\eta_i), f^- = f(\xi_i), \dots$

After some calculations we have

$$u'_{i+1} = \frac{2(u_{i+1}-u_i)}{h_{i+1}} + \frac{1}{h_{i+1}P_i}((u_{i+1}-u_i)Q_i + h_{i+1}^2 b^+ u_i + h_{i+1}^2 f^+) \tag{6}$$

$$u'_i = \frac{1}{h_{i+1}P_i}((u_{i+1}-u_i)Q_i + h_{i+1}^2 b^+ u_i + h_{i+1}^2 f^+), \tag{7}$$

where

$$P_i = -2\varepsilon + a(\eta_i)(2\alpha_{2i}-1)h_{i+1} - b(\eta_i)\alpha_{2i}(1-\alpha_{2i})h_{i+1}^2,$$

$$Q_i = -2\varepsilon - 2a(\eta_i)(1-\alpha_{2i})h_{i+1} + b(\eta_i)(1-\alpha_{2i})^2 h_{i+1}^2.$$

We can determine u'_i from (6) by puting collocation points ξ_i in (6) instead of η_i and obtain

$$u'_i = \frac{2(u_i-u_{i-1})}{h_i} - \frac{1}{h_i D_i}((u_i-u_{i-1})\Omega_i + h_i^2 b^- u_{i-1} + h_i^2 f^-), \tag{8}$$

where

$$D_i = -2\varepsilon + 2a(\xi_i)h_i(2\alpha_{1i}-1) - b(\xi_i)h_i^2(1-\alpha_{1i})^2,$$

$$\Omega_i = -2\varepsilon - 2a(\xi_i)h_i(1-\alpha_{1i}) + b(\xi_i)h_i^2(1-\alpha_{1i})^2.$$

¿From (7) and (8) we obtain scheme

$$\begin{aligned} \mathcal{L}_n u_i : &= r_i^- u_{i-1} + r_i^c u_i + r_i^+ u_{i+1} = q^- f(\xi_i) + q^+ f(\eta_i), \ i = 1(1)n-1 \\ u_0 &= p_0, \qquad u_n = p_1, \end{aligned} \tag{9}$$

where

$$r_i^- = \frac{S_i}{2D_i}, \quad r_i^+ = \frac{h_i Q_i}{2h_{i+1}P_i}, \quad r_i^c = -1 + \frac{h_i h_{i+1}}{2P_i} b(\eta_i) - \frac{Q_i h_i}{2h_{i+1}P_i} + \frac{\Omega_i}{2D_i},$$

$$q_i^- = -\frac{h_i^2}{2D_i}, \qquad q_i^+ = -\frac{h_i h_{i+1}}{2P_i},$$

$$S_i = -2\varepsilon + 2a(\xi_i)h_i\alpha_{1i} + b(\xi_i)h_i^2\alpha_{1i}^2.$$

We want to determine α_{1i} and α_{2i} so that the corresponding matrix has L-form, i.e.

$$r_i^- \geq 0, \quad r_i^+ \geq 0, \quad r_i^c \leq 0.$$

We put $\alpha_{2i} = \frac{1}{2}$ except in the case when $i = n/2$ when $\alpha_{2,n/2} = \frac{1}{2} + \frac{h_{n/2+1}}{2^k}$ where $k = \log_2 x + 1$ and x is the solution of the equation

$$b(\eta_{n/2})x^2 - 4a(\eta_{n/2})x - b(\eta_{n/2})h_{n/2+1}^2 = 0.$$

Then for $n \geq n_0$ (n_0 is independent of ε) we have

$$Q_i = -2\varepsilon - a(\eta_i)h_{i+1} + b(\eta_i)\frac{1}{4}h_{i+1}^2 \leq 0$$

and $P_i = -2\varepsilon - b(\eta_i)\frac{1}{4}h_{i+1}^2 \leq 0$, which gives $r_i^+ \geq 0$.

We determine α_{1i} from the condition $r_i^- \geq 0$. This condition is fulfilled if $S_i \leq 0$ and $D_i \leq 0$. Since

$$D_i \leq 0 \qquad \text{if} \qquad \alpha_{1i} \leq \frac{1}{2},$$

we can find α_{1i} so that $S_i \leq 0$. Then $r_i^c < 0$ and the matrix of the system (9) is inverse monotone.

Lemma 3. (Discrete minimum principle) *If W is mesh function and $L_n W_i \leq 0, i = 1(1)n-1,$ and $W_0 \geq 0, W_n \geq 0$ then $W_i \geq 0$ for $i = 0, 1, \ldots n$.*

We use the following discrete decomposition

$$u = V + W,$$

where V is the approximation for v and W approximate w and

$$\mathcal{L}_n W = 0, \;\; W(0) = w(0), \;\; W(1) = w(1).$$

Theorem 1. *We have the following bound for* W :

$$|W(x_j)| \leq M \prod_{i=1}^{j} (1+2\theta_L h_i)^{-1} := \psi_j, \quad \psi_0 = M,$$

where θ_L *is positive root of the equation* $2\varepsilon\theta_L^2 - a\theta_L - b = 0$.

Proof. Consider $\phi_j = \psi_j \pm W(x_j)$. Now

$$\mathcal{L}_n \phi_j = r^- \psi_{j-1} + r^c \psi_j + r^+ \psi_{j+1} \pm 0 \tag{10}$$

$$= M\psi_{j+1}[G(1+2\theta_L h_{i+1}) + \frac{\theta_L h_i}{D_i P_i}(S_i P_i - Q_i D_i) + 2\theta_L^2 \frac{h_i h_{i+1}}{D_i}(S_i + 2\varepsilon)$$
$$+ \frac{\theta_L h_i h_{i+1}^2 b^+}{P_i} - 2\frac{h_i h_{i+1}\beta}{D_i} + \frac{h_i h_{i+1} b^+}{2P_i} - 2\frac{h_i h_{i+1}\theta_L a}{D_i}],$$

where

$$G = -1 + \frac{S_i + \Omega_i}{2D_i} = \frac{h_i^2 b^-}{2D_i} \leq 0.$$

Crucial expressions for the analysis of (10) are $\tilde{A}$ and $\tilde{B}$. $\tilde{A}$ contains the values of the function $a(x)$ and $\tilde{B}$ contains the values of the function $b(x)$. Since

$$\tilde{A} = 2\theta_L[-2a^- h\varepsilon - 2a^+ h_{i+1}\varepsilon + a^- a^+ h_i h_{i+1}(-1+2\alpha_1)$$
$$-2h_{i+1}a(-2\varepsilon - \frac{b^+}{4}h_{i+1}^2)] \leq 0$$

if $\alpha_1 \leq \frac{1}{2} - \frac{h_i b^+}{4a^+} \leq \frac{1}{2} - \frac{h_i B}{4a}$, this situation is accepted in our treatment of the problem. But, in the opposite case we consider the expression β :

$$\beta = \tilde{A} + 4\theta_L^2 h_{i+1} P_i (S_i + 2\varepsilon).$$

By using the fact $\theta_L \geq \frac{a}{2\varepsilon}$ ([3]) we have that $\beta \leq 0$ if $4ah_i \geq \varepsilon$, which is fulfilled for $i = \frac{n}{2}+1, \frac{n}{2}+2, \ldots, n-1$. Further,

$$\tilde{B} = 2\alpha_1\theta_l(-2b^-\varepsilon h_i^2(2-\alpha_1) + b^- b^+ h_{i+1}^2 h_i^2\left(-\frac{3}{4} + \frac{1}{2}\alpha_1\right)$$
$$-h_{i+1}b^+(2\varepsilon + h_i(1-2\alpha_1) + b^-\alpha_1(1-\alpha_1)h_i^2)$$
$$+(2\varepsilon + \frac{b^+}{4}h_{i+1}^2)(4\beta h_{i+1} - b^- h_i(1+2\theta_L h_{i+1}))$$

and we have

$$\tilde{B} \leq 0 \qquad \text{if} \qquad 2\varepsilon \leq h_i,$$

which is satisfied for the larger mesh step. This implies that $\mathcal{L}_n \phi_j \leq 0,\ j = n/2 + 1, \ldots, n-1$.

If $i = 1(1)n/2$ we put $\alpha_1 = 1/2$ and then we have fine mesh which implies $\mathcal{L}_n \phi_j \leq 0$. Thus, using discrete minimum principle we obtain the result.$\diamondsuit$

Numerical results given in the next section implies that the error is of order $\mathcal{O}(n^{-2} \ln^2 n)$.

3. NUMERICAL RESULTS

We study the test problem

$$\varepsilon y''(x) + y'(x) - y(x) = -x, \quad x \in (0,1), \qquad y(0) = 1, \qquad y(1) = 0.$$

The exact solution is

$$y(x) = x + 1 + \frac{2e^{k_2/(2\varepsilon)}}{1 - e^{-k_1/\varepsilon}} \cdot e^{-xk_1/(2\varepsilon)} - \frac{2}{1 - e^{-k_1/\varepsilon}} \cdot e^{(1-x)k_2/(2\varepsilon)}$$

where $k_{1,2} = 1 \pm \sqrt{1 + 4\varepsilon}$. Tables contain errors, convergence orders and values of the parameter α_1.

In Table 1 we present errors $E_n = \max_{0 \leq i \leq n} |y(x_i) - u_i|$ computed on the Shishkin mesh for various values of ε and n. The convergence rates of these errors are calculated in a standard way

$$P_n = \frac{\ln E_n - \ln E_{2n}}{\ln 2} \quad \text{for} \ \ n = 2^k, \ \ k = 5, 6, \ldots 10$$

and presented in Table 2.

Table 1. Errors E_n

$\varepsilon \setminus n$	32	64	128	256	512	1024
2^{-4}	4.725e-3	1.551e-3	3.857e-4	9.645e-5	2.409e-5	6.012e-6
2^{-5}	4.527e-3	1.609e-3	5.443e-4	1.776e-4	5.615e-5	1.732e-5
2^{-6}	4.721e-3	1.580e-3	5.331e-4	1.739e-4	5.498e-5	1.695e-5
2^{-7}	4.147e-3	1.664e-3	5.283e-4	1.721e-4	5.441e-5	1.678e-5
2^{-8}	4026e-3	1.485e-3	5.542e-4	1.713e-4	5.413e-5	1.669e-5
2^{-9}	3.894e-3	1.459e-3	5.056e-4	1.784e-4	5.400e-5	1.665e-5
2^{-10}	3.693e-3	1.434e-3	5.003e-4	1.659e-4	5.587e-5	1.663e-5
2^{-11}	3.324e-3	1.401e-3	4.961e-4	1.647e-4	5.270e-5	1.711e-5
2^{-12}	2.796e-3	1.344e-3	4.915e-4	1.639e-4	5.242e-5	1.631e-5
2^{-13}	5.223e-3	1.242e-3	4.843e-4	1.632e-4	5.224e-5	1.624e-5
2^{-14}	1.002e-2	1.059e-3	4.709e-4	1.621e-4	5.211e-5	1.620e-5

Table 2. Convergence rates P_n

$\varepsilon \setminus n$	32	64	128	256	512
2^{-4}	1.607	2.008	2.000	2.001	2.003
2^{-5}	1.492	1.564	1.616	1.661	1.697
2^{-6}	1.579	1.567	1.616	1.661	1.697
2^{-7}	1.317	1.655	1.618	1.662	1.697
2^{-8}	1.438	1.422	1.694	1.662	1.697
2^{-9}	1.416	1.529	1.503	1.724	1.697
2^{-10}	1.365	1.519	1.592	1.571	1.748
2^{-11}	1.246	1.498	1.590	1.644	1.623
2^{-12}	1.057	1.451	1.584	1.645	1.684
2^{-13}	2.072	1.358	1.569	1.643	1.685
2^{-14}	3.242	1.170	1.538	1.638	1.685

Table 3. Values of α_1

$\varepsilon \setminus n$	32	64	128	256	512	1024
2^{-4}	0.5	0.5	0.5	0.5	0.5	0.5
2^{-5}	0.5	0.5	0.5	0.5	0.5	0.5
2^{-6}	0.25	0.5	0.5	0.5	0.5	0.5
2^{-7}	0.125	0.25	0.5	0.5	0.5	0.5
2^{-8}	6.250e-2	0.125	0.25	0.5	0.5	0.5
2^{-9}	3.125e-2	6.250e-2	0.125	0.25	0.5	0.5
2^{-10}	1.562e-2	3.125e-2	6.250e-2	0.125	0.25	0.5
2^{-11}	7.812e-3	1.562e-2	3.125e-2	6.250e-2	0.125	0.25
2^{-12}	3.906e-3	7.812e-3	1.562e-2	3.125e-2	6.250e-2	0.125
2^{-13}	1.953e-3	3.906e-3	7.812e-3	1.562e-2	3.125e-2	6.250e-2
2^{-14}	9.766e-3	1.953e-3	3.906e-3	7.812e-3	1.562e-2	3.125e-2

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