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AFFINE INVARIANT CONTRACTIONS OF SIMPLICES

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Abstract. It is known that every hyperbolic Iterated Function System containing affine mappings can be represented in barycentric form by which it becomes affine invariant. Some properties were surveyed and some new ones were established. Examples of fractal sets supplement the theory.

1. INTRODUCTION

Parametric fractal sets are known to change their shape upon continuous change of one or more parameters. One way of introducing shape parameters is to let the surrounding space being dependent on them, in a predictable way. Then, these parameters may be used to model animated fractal sets and to show their generation and continuous transformation from or into smooth sets. This animation may be used as an effective educational tool for better understanding of fractal architecture and its application.

The usual way of introducing "laboratory" fractals is constructive approach, usually defined in terms of *Iterated Function Systems* (IFS) [1]. An IFS, $F = \{\mathbb{R}^m; w_1, \ldots, w_n\}$ $(n \geq 2)$ is a set of mappings $\{w_1, \ldots, w_n\}$, of the metric space $(\mathbb{R}^m, d)$ $(m \geq 1)$ into itself. Let $\tilde{\mathbb{R}}^m$ be the set of all nonempty compact subsets of $\mathbb{R}^m$. Attached to F is the *Hutchinson's operator* $W_F : \tilde{\mathbb{R}}^m \to \tilde{\mathbb{R}}^m$, defined as $W_F(\cdot) = \bigcup\limits_{i=1}^{n} w_i(\cdot)$. If all mappings w_i are contractions, the IFS F is called *hyperbolic*, and than, as a consequence, W_F is also a contraction of the complete metric space $(\tilde{\mathbb{R}}^m, h)$, where h is the Hausdorff's metric induced by d. Using the Banach's theorem, iterations $\{W_F^{(k)}(B)\}$ converge to a fixed point A for any $B \in \tilde{\mathbb{R}}^m$ [7]. The fixed point, called *attractor* of the IFS F, may have a non-integer Hausdorff's dimension, an object that we usually call *fractal set* [1]. Also, note that the contractions w_i from F need not to be nonlinear to get fairly complex fractal attractors. Typical choice of w_i is the simplest mapping - an affine one

$$w(\mathbf{x}) = A\mathbf{x} + \mathbf{b}, \ \mathbf{x} \in \mathbb{R}^{m-1}, \tag{1}$$

where A is an $(m-1) \times (m-1)$ real matrix and $\mathbf{b} \in \mathbb{R}^{m-1}$.

A number of scientific or technology fields is evidenced using fractals in different aspects ranging from earthquake prediction to graphic design and computer art. People experiment with different IFS settings to get nice looking figures of strange forms resembling tree foliage, water surface, mist, clouds, moss, rock, formations, tissue textures etc. But, from the point of modelling, the IFS method faces difficulties. The biggest problem appears to be unpredictability of the form of fractal attractor generated by an IFS. The next one is change of attractor's form, which is almost impossible task.

The aim of this note is to offer some practical tools that may help in making fractal sets more flexible in the sense that one may model them interactively in the similar way designers do with free form curves, surfaces and volumes. Some similar ideas were developed by Zair and Tosan (see [8] and references cited there).

2. THE AIFS

Let the set of points $\{T_1, \ldots, T_m\}$ from $\mathbb{R}^{m-1}$ define a non-degenerate $(m-1)$-dimensional simplex $\mathbf{T} \subset \mathbb{R}^{m-1}$. Let $\{S_1, \ldots, S_n\}$ be the set of $m \times m$ real, nonsingular, row-stochastic matrices (all rows sum up to 1).

Definition 1. *The system $F_T = \{\mathbf{T} \subset \mathbb{R}^{m-1}; S_1, \ldots, S_n\}$, $n \geq 2$, is called affine invariant $(m-1)$-dimensional IFS (AIFS), and it is hyperbolic if the linear map defined by $S_i : \mathbf{T} \to \mathbf{T}$ is a contraction.*

Note that $S_i : \mathbf{T} \to \mathbf{T}$ is a contraction if and only if $S_i(\mathbf{T})$ belongs to the interior of $\mathbf{T}$ up to the orthogonal transformation. Here, the linear map of the space $\mathbb{R}^{m-1}$ into itself, performed by S_i should be understood as the map of the vector $\tilde{\mathbf{x}}$ of barycentric coordinates of the point $x \in \mathbb{R}^{m-1}$, w.r.t. simplex $\mathbf{T}$, into $\tilde{\mathbf{x}}^T S_i$. Thus, the revisited Hutchinson's operator $W : \mathbb{R}^{m-1} \to \mathbb{R}^{m-1}$, associated with the AIFS F_T, will be

$$W(\cdot) = \bigcup_{i=1}^{n} (\cdot)^T S_i \, .$$

It is proven in [3] that, if the system F has an attractor, so does F_T and vice versa. Formally, the attractor of F_T is given by $\mathrm{att}(F_T) = W^\infty(\mathbf{T})$.

Unlikely the IFS notation, where the attractor is uniquely determined by the IFS, different AIFS may produce the same attractor. Explanation is simple. The mapping $\mathbf{T} \mapsto S(\mathbf{T})$ splits into m sub-mappings

$$\mathbf{T} \mapsto S(\mathbf{T}) = \{\mathbf{s}_1\mathbf{T}, \ldots, \mathbf{s}_m\mathbf{T}\},$$

where $\mathbf{s}_i$ denotes the i-th row of the matrix S. In some cases, the order in the set $S(\mathbf{T})$ has no influence on the attractor's shape, so, the rows of S may be permuted.

The following properties are important in fractal modelling [4]-[3], [8]:

Affine invariance: For any affine transform $\mathcal{A}:\mathbb{R}^{m-1}\to\mathbb{R}^{m-1}$

$$\mathcal{A}(\mathrm{att}(F_T))=\mathrm{att}(F_{\mathcal{A}(\mathrm{T})})\,;$$

Convex hull (CH) property: If matrices S_i have non-negative entries, the attractor of the AIFS belongs to the interior of $\mathbf{T}$, $\mathrm{att}(F_T)\subset \mathrm{int}\mathbf{T}$. It can be shown that this inclusion has the following refinement:

$$\mathrm{int}\mathbf{T}\supset \mathrm{int}W(\mathbf{T})\supset \mathrm{int}W^2(\mathbf{T})\supset\cdots\supset \mathrm{int}W^{\infty}(\mathbf{T})=\mathrm{att}(F_T);$$

Continuity property: The attractor $\mathrm{att}(F_T)$ is a continuous curve in $\mathbb{R}^{m-1}$, if matrices S_i, may be after permutation of rows, satisfy

$$\mathbf{e}_1^T S_1=\mathbf{e}_1^T,\ \mathbf{e}_m^T S_n=\mathbf{e}_m^T,\ \text{and}\ \mathbf{e}_m^T S_i=\mathbf{e}_1^T S_{i+1},\ (i=1,\,\ldots,\,n-1);$$

Interpolation property: If the attractor $\mathrm{att}(F_T)$ is a continuous curve and there exists i such that $\mathbf{e}_m^T S_i=\mathbf{e}_1^T S_{i+1}=\mathbf{e}_j^T$, then it interpolates the j-th vertex of $\mathbf{T}$;

Symmetry property: Let the simplex $\mathbf{T}\subset\mathbb{R}^{m-1}$ be symmetric w.r.t. some orthonormal transformation $\mathcal{A}_0:\mathbb{R}^{m-1}\to\mathbb{R}^{m-1}$, i.e. $\mathcal{A}_0(\mathbf{T})=\mathbf{T}$. If all the matrices S_i satisfy $\Pi S_i\Pi=S_{n-i+1}$, where Π is permutation matrix $\Pi=[\delta_{i,m-j+1}]$ ($\delta_{i,j}$-Kronecker's delta), $\mathrm{att}(F_T)$ is also symmetric, i.e., $\mathcal{A}_0(\mathrm{att}(F_T))=\mathrm{att}(F_T)$;

Smooth shapes: The smooth curved forms like (parametric) polynomials and splines can be generated as attractor of specific AIFS that is usually called (mainly in CAGD) *subdivision schemes* or *algorithms* [2].

Examle 1. Consider a two-term, four dimensional hyperbolic AIFS $\{\mathbf{T}\subset\mathbb{R}^{m-1};\,S_1,\,S_2\}$, where S_1 and S_2 are well known subdivision matrices [2], $S_1=\left[2^{-i+1}\begin{pmatrix} i-1\\ j-1\end{pmatrix}\right]_{i,j=1}^{m}$, $S_2=\left[2^{-m+i}\begin{pmatrix} m-i\\ m-j\end{pmatrix}\right]_{i,j=1}^{m}$. The associated Hutchinson's operator, $W=S_1\cup S_2$ defines De Casteljau subdivision, and its iterates W, W^2, W^3 and W^{∞} map simplex T into decreasing inclusive-isotonic sequence of sets that converge to the attractor $W^\infty(\mathbf{T})$, known as Bézier curve. This is a smooth set, or, in other words its fractal dimension is integer number $m-1$. Note that Bézier curve has all the properties listed above.

Example 2. (*Lévy curve*). To the contrast, consider a two-dimensional AIFS $\{\mathbf{T};\, S_1,\, S_2\}$ defined by

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & -0.5 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

Due to the negative items, the AIFS has no convex hull property. Further, since $\mathbf{e}_1^T S_1 = \mathbf{e}_1^T$, $\mathbf{e}_3^T S_2 = \mathbf{e}_3^T$, and $\mathbf{e}_3^T S_1 = \mathbf{e}_1^T S_2$, the AIFS has continuity property. But, despite the absence of CH property, changing the polygon $\{T_1\, T_2\, T_3\}$ changes the shape of the fractal attractor in the similar way it changes the shape of Bézier's curve. The attractor of this AIFS is called Levy curve [7]. Some other examples of AIFS without CH property are given in [2].

3. HOW TO MODEL?

There are several possibilities for free-form modelling actions. First, one can easily model transformation $\mathbf{T} \to S_i(\mathbf{T})$ by selecting items of matrices S_i that are barycentric coordinates of $S_i(\mathbf{T})$ with respect to $\mathbf{T}$. For example, the simplex $\mathbf{T}$ from $\mathbb{R}^3$, can be subdivided into four sub-simplices having vertices in the middle of each edge emanated from each vertex $\mathbf{T}$ plus this vertex. In this case, the first subdivision matrix has typical form

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix},$$

and the other three will be just permutations of S_1. The attractor of such AIFS is known as the Sierpinski pyramid. Second, the shape of the simplex can be changed by changing the lengths of its edges. It will produce the Bézier's controllability effect that can be stable if the AIFS has the CH property, or unstable if it hasn't. In the case of *Barnsly's Fern* [1], the absence of CH property occurs, so that a slight perturbation of the simplex height causes a dramatic change of the shape of the *Fern*.

And finally, the third possibility is to combine two or more AIFS with the same dimension (same number of mappings) in the sense that matrices A, B, C, ... from different AIFS's may be combined by some "blending" function f that preserves row-stochasticity, so that $S = f(A, B, C, \ldots)$. A typical example is the convex combination, like the one of the Fern and Sierpinski's pyramid.

In all three cases there is need to know AIFS form of an IFS and vice versa. The following theorems solve this problem.

Theorem 1. *Let w be an affine contraction* (1). *The corresponding linear mapping is defined by the $m \times m$ real row-stochastic matrix S, given in block form by*

$$S = \left[\begin{array}{c|c} (A + \mathbf{b}J^{\mathrm{T}})^{\mathrm{T}} & J - J^{\mathrm{T}}(A^{\mathrm{T}} - \mathbf{b}J^{\mathrm{T}}) \\ \hline \mathbf{b}^{\mathrm{T}} & 1 - J^{\mathrm{T}}\mathbf{b} \end{array}\right], \tag{3}$$

where $J = [1\ 1\ \ldots\ 1]^{\mathrm{T}} \in \mathbb{R}^{m-1}$.

Proof. In [5, 6] it was proven that the matrices S, A and vector $\mathbf{b}$ are related by

$$S = S_p S_w^T S_l = S_p S_w^T (S_p)^{-1}, \tag{4}$$

where the projection matrix S_p and lifting matrix S_l are given by

$$S_p = \left[\begin{array}{c|c} I_{m-1} & J \\ \hline \mathbf{0}^{\mathrm{T}} & 1 \end{array}\right], \qquad S_l = S_p^{-1} = \left[\begin{array}{c|c} I_{m-1} & -J \\ \hline \mathbf{0}^{\mathrm{T}} & 1 \end{array}\right], \tag{5}$$

(I_m is identity matrix) and

$$S_w = \left[\begin{array}{c|c} A & \mathbf{b} \\ \hline \mathbf{0}^{\mathrm{T}} & 1 \end{array}\right]. \tag{6}$$

By relations (4), (5) and (6), provided that $A = [a_{ij}]_{m-1}$, $\mathbf{b} = [b_i]_{m-1}$, $S = [s_{ij}]_m$ the following transformation follows:

$$(A,\ \mathbf{b}) \to S : \begin{cases} s_{ij} = a_{ji} + b_j, & i,\, j = 1, \ldots, m-1, \\ s_{mj} = b_j, & j = 1, \ldots, m-1, \\ s_{im} = 1 - \sum\limits_{j=1}^{m-1} s_{ij}, & i = 1, \ldots, m; \end{cases} \tag{7}$$

These explicit formulae indicate 2×2-block structure of the matrix S. The $(1,1)$ block is $(m-1)$-square matrix. Setting $J = [1\ 1\ \dots\ 1]^{\mathrm{T}} \in \mathbb{R}^{m-1}$, this block may be put in the form $A^{\mathrm{T}} + J\mathbf{b}^{\mathrm{T}} = (A + \mathbf{b}J^{\mathrm{T}})^{\mathrm{T}}$. Underneath the block $(2,1)$ is an $(m-1)$ row-matrix that coincide with $\mathbf{b}^{\mathrm{T}}$. The "right" blocks, $(1,2)$ and $(2,2)$ are fully determined by the row-stochastic property of S. This gives S as in (3). $\square$

Theorem 2. *Let* $K = [I_{m-1}|\,\mathbf{0}]$ *and* $\mathbf{e}_m = [\,0\,0\dots\ 0\,1]^{\mathrm{T}} \in \mathbb{R}^m$. *Then,* $A = [a_{ij}]_{m-1}$ *and* $\mathbf{b} = [b_i]_{m-1}$ *are uniquely determined by the row stochastic matrix* $S = [s_{ij}]_m$. *More precisely,*

$$A = K\left(S - JS^{\mathrm{T}}\mathbf{e}_m\right)K^{\mathrm{T}}, \quad \mathbf{b} = KS^{\mathrm{T}}\mathbf{e}_m. \tag{8}$$

Proof. The transformation (7) is invertible, since it is linear system regarding a_{ij} and b_i with a nonsingular matrix. By inverting, one gets

$$S \to (A,\ \mathbf{b}):\quad \begin{cases} a_{ij} = s_{ji} - s_{mi}, & i,\ j = 1,\dots,m-1, \\ b_i = s_{mi}, & i = 1,\dots,m-1; \end{cases} \tag{9}$$

The product $S^{\mathrm{T}}\mathbf{e}_m$ gives us the last row of S as a single column. Then, multiplication from the left with J makes m copies of this column, filling up an $m \times m$ matrix. So, the matrix $S - JS^{\mathrm{T}}\mathbf{e}_m$ has $s_{ij} - s_{mj}$ as its (i,j)-th item. Then, the "sandwich operator" $K(\cdot)K^{\mathrm{T}}$ prunes the last row and column of the $m \times m$ matrix it is applied on. In this way the matrix A is obtained. In the similar way, vector $\mathbf{b}$ is obtained, as it is given in (8). $\square$

Note that (7) gives algorithm for transforming an IFS into AIFS, i.e. $(A,\ \mathbf{b}) \to S$, while (9) transforms AIFS into IFS, i.e. $S \to (A,\ \mathbf{b})$.

Example 3. To test the algorithms (7) and (9), let transform AIFS for Lévy curve (2) into IFS form. Applying (9) one has $a_{ij} = s_{ji} - s_{3i},\ b_i = s_{3i}\ \ i,\ j = 1,2$ for each matrix S. So that the IFS will be

$$A_1 = \begin{bmatrix} 1 & 0.5 \\ -1 & 0 \end{bmatrix},\ \mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\quad A_2 = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix},\ \mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By applying (7), on the IFS above, one yields $s_{ij} = a_{ji} + b_j,\ i,\ j = 1,2,\ s_{3j} = b_j,\ j =$

$1, 2$, $s_{i3} = 1 - s_{i1} - s_{i2}$, $i = 1, 2, 3$, for each of two contractions. This gives S_1 and S_2 as in (2).

4. CONCLUSION

The aim of this paper is to review some properties of the affine invariant iterated function systems (AIFS) that has been developed in [3], [4], [5] and [8] (and references cited there). In addition, we contribute explicit formulae for transforming IFS into AIFS and v.v., and quote three main methods for free-form-friendly manipulation by fractal attractors. In the first case, the simplex variant of the Collage's theorem allows rough modelling of the fractal set. This segment is very useful for learning the true nature of the fractal sets and some basic properties. The second method is suitable for minor interventions on finer details, or in the case when one wants to improve some form by changing the ratio of its parts, with a remark that modelling is as weaker as the model properties are more away from the "nice" properties of the Bézier's or spline model. The third "blending" technique may yield mixed shapes, or interpolate fixed fractal forms by simply changing one or several parameters. The effective is blending non-smooth and smooth forms, such as splines with the Von Koch's curve or similar. This aspect is especially educational and interesting for studying dimensions of fractal sets. All three methods may be used for animation purposes.

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