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MULTIQUASIGROUPS AND WEIGHTED PROJECTIVE PLANES

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Abstract. We introduce and study the notion of weighted projective planes which is a generalization of the usual projective plane. We compare them with totally symmetric (n, m)-quasigroups. We prove that a weighted projective plane $S(2, n+1, n^2+n+1)$ iz equivalent to a totally symmetric $(2, n-1)-$quasigroup.

1. INTRODUCTION

An incidence structure is a triple $D = (V, B, I)$, where V and B are disjoint sets and $I \subseteq V \times B$. The element of V are called *points*, and the elements of B are called *blocks*. If A is a point of V, the set of all blocks incident with A is denoted by (A). Thus

$$(A) = \{b : b \in B, AIb\}.$$

Moreover, for $A_1, A_2, ..., A_n$, the set of all the blocks incident with all the points A is denoted by $(A_1, A_2, ..., A_n)$. Thus

$$(A_1, A_2, ..., A_n) = \{b : b \in B, A_i Ib \ \text{ for all } \ i \in N_n\},$$

where N is the set of all positive integers and $N_n = \{1, 2, ..., n\}$. Dually, for $b, b_1, b_2, ..., b_n \in B$,

$$(b) = \{A : A \in V, AIb\},$$

$$(b_1, b_2, ..., b_n) = \{A : A \in V, AIb \text{ for all } i \in N_n\}.$$

We consider only the incidence structures where distinct blocks have distinct sets of points. We identify each block b with the set (b) and identify the incidence relation with the membership relation $\in$.

2. SOME DEFINITION AND RESULTS

Definition 1. *An incidence structure $D = (V, B, I)$ is called projective plane if and only if it satisfies the following axioms:*

(P.1) Any two distinct points are joined by exactly one line.

(P.2) Any distinct lines intersect in a unique point.

(P.3) There exists a quadrangle, i.e. 4 points no three of which are on a common line.

The following theorem is proved in [1].

Theorem 1. *Let $D = (V, B, I)$ be a finite projective planes. Then there exists a natural number n, called the order of D, satisfying:*

a) $|(P)| = |(g)| = n + 1$ for all $P \in V$ and $g \in B$;

b) $|V| = |B| = n^2 + n + 1$.

The finite projective plane of order n will be denoted by $S(2, n + 1, n^2 + n + 1)$.

Theorem 2. *For each prime power q, there exists projective plane of order q.*

Proof. Let F be the Galois field on q elements and W the vector space of dimension 3 over F. Choose as points all 1-dimensional subspaces and as lines all 2-dimensional subspaces. Using the dimension formula of linear algebra, one checks that the axioms (P.1) and (P.2) are satisfied. For (P.3), one may choose the points e_1F, e_2F, e_3F and $(e_1 + e_2 + e_3)F$ where e_1, e_2, e_3 is any basis of W. The fact that F is the field on q elements is only needed to show that the resulting projective plane has order q: the number of 1-dimensional subspaces of W is then $\frac{(q^3-1)}{(q-1)} = q^2 + q + 1$. $\square$

It is not known whether there exist projective planes of any other order. The following famous non-existence result is proved in [1].

Theorem 3. **[Bruck/Ryser 1949]** *Let $n \in N, n \equiv 1$ or 2 mod 4. If there exists an odd prime $p \equiv 3 \mod 4$ dividing the squarefree part of n, then there is no $S(2, n+1, n^2+n+1)$.*

The existence question for an $S(n, n+1, n^2+n+1)$ is open for $n = 12, 15, 18, 20, 24, 26, 28, 34, ...$

For $n = 6, 14, 21, 22, 30, 33, 38, ...$ there is no $S(n, n+1, n^2+n+1)$.

Definition 2. *An finite incidence structure $D = (V, B, I)$ is called block design with parameters $v, k, \lambda \in N$, if and only if it satisfies the following axioms:*

(D. 1) $|V| = v$;

(D. 2) $|(P, Q)| = \lambda$, *for any two distinct points* $P, Q \in V$;

(D. 3) $|(b)| = k$, *for any block* $b \in B$.

A block design with parameters $v, k, 1$ is called a *Steiner system* and is denoted by $S(2, k, v)$.

It is easy to see that any $S(2, n+1, n^2+n+1)$ with $n \geq 2$ is a projective plane of order n.

3. WEIGHTED PROJECTIVE PLANES

The following definition generalizes the notion of finite projective planes $S(2, n+1, n^2+n+1)$.

Definition 3. *An finite incidence structure $D = (V, B, I)$ is called weighted projective plane with parameters $n^2 + n + 1, k, 1 \in N$, if for any $b \in B$ there is a mapping $f_b : (b) \longrightarrow N$, and if and only if it satisfies the following axioms:*

(WD. 1) $|V| = n^2 + n + 1$*;*

(WD. 2) $|(P, Q)| = 1$*, for any two distinct points* $P, Q \in V$*;*

(WD. 3) $k_b = n + 1$*, for any block* $b \in B$*, where:*

a. *the image $f_b(A)$ is denoted by t_{Ab}, and is called the weight of the point A in the block b,*

b. *for $A \in V$, its weight is $t_A = \sum_{A \in b_i} t_{Ab_i}$ and*

c. *for $b \in B$, the number $k_b = \sum_{A_i \in b} t_{A_i b}$ is called the size of the block b.*

Example 1. Every usual projective plane $S(2, n + 1, n^2 + n + 1)$ is a weighted projective plane where the mapping $f_{(b)} : (b) \longrightarrow N$ for all blocks $b \in B$ is defined by

$$f_b(A) = t_{Ab} = 1, \quad \text{for all } A \in b.$$

For any point $A \in V$ the weight of A is

$$t_A = \sum_{A \in b_i} t_{Ab_i} = r = n + 1$$

where r is the number of blocks which contain A. For any block $b \in B$

$$k_b = \sum_{A_i \in b} t_{A_i b} = k = n + 1.$$

From Example 1 it follows that the weighted projective plane is a generalization of the usual finite projective plane.

Theorem 4. *Let $D = (V, B, I)$ be a weighted projective plane with parameters $n^2 + n + 1, n + 1, 1 \in N$. Then*

$$n + 1 = k_b - \sum_{A_i \in b} (t_{Aib} - 1) = t_A - \sum_{A \in b_i} (t_{Ab_i} - 1).$$

Proof. For any point $A \in V$, we have

$$(A) = \{b_1, b_2, ..., b_{n+1}\}, \quad t_A = t_{Ab_1} + t_{Ab_2} + ... + t_{Ab_{(n+1)}}.$$

$$\begin{aligned} t_A - \sum_{A \in b_i} (t_{Ab_i} - 1) &= (t_{Ab_1} + t_{Ab_2} + ... + t_{Ab_{(n+1)}}) - (t_{Ab_1} + t_{Ab_2} + ... + t_{Ab_{(n+1)}}) \\ &+ (1 + 1 + ... + 1) = n + 1. \end{aligned}$$

Similary, for any block $b \in B$ we have

$$(b) = \{A_1 + A_2 + ... + A_{n+1}\}, \quad k_b = t_{A_1 b} + t_{A_2 b} + ... + t_{A_{(n+1)} b}.$$

$$\begin{aligned} k_b - \sum_{A_i \in b} (t_{A_i b} - 1) &= (t_{A_1 b} + t_{A_2 b} + ... + t_{A_{(n+1)} b}) - (t_{A_1 b} + t_{A_2 b} + ... + t_{A_{(n+1)} b}) \\ &+ (1 + 1 + ... + 1) = n + 1. \end{aligned}$$

For usual projective plane we have $f_b(A) = t_{Ab} = 1$, for all $A \in V, b \in B$ such that $A \in b$. From Theorem 1 we have

$$n + 1 = t_A - \sum_{A \in b_i} (t_{Ab_i} - 1) = t_A - 0 = t_A,$$

$$n + 1 = k_b - \sum_{A_i \in b} (t_{A_i b} - 1) = k_b - 0 = k_b.$$

Hence, we have that the weight of the point A is equal to the number of blocks $b \in B$ which contain A and the size of the block b is equal to the number of point $A \in V$ such that $A \in b$. □

Definition 4. *A weighted projective plane $S' = (V', B, \in)$ is an extension of a weighted projective plane $S = (V, B, \in)$, if $V \subseteq V'$ and for each $b \in B$ there is $b' \in B'$ such that $(b) \subseteq (b')$, and for each $A \in (b), t_{Ab'} = t_{Ab}$.*

Definition 5. *An extension $(V', B', \in)$ of a weighted projective plane with parameters $n^2 + n + 1, n + 1, 1$ defined by*

a) $V' = V$;

b) $B' = B \cup B''$ *where* $B'' = \{\{A^{n+1}\} : A \in V\}$*, and*

c) For each $A \in V, t_A = r + n + 1$*, where* r *is the number of block in* B *containing* A*,*

is called a complete weighted projective plane with parameters $n^2+n+1, n+1, 1$*, and is denoted by* $S'(2, n+1, n^2+n+1)$*.*

4. MULTIQUASIGROUPS AND WEIGHTED PROJECTIVE PLANES

Next we compare complete weighted projective plane $S'(2, n+1, n^2+n+1)$ with the notion of totally symmetric $(2, n-1)$-quasigroup given below.

Definition 6. *Let* Q *be nonempty set,* n *and* m *positive integers, and*

$$f : (x_1, x_2, ..., x_n) \longrightarrow f(x_1, x_2, ..., x_n)$$

a mapping from Q^n *into* Q^m*. Then we say that* $Q(f)$ *is an* (n, m)*-groupoid.*

An (n, m)-groupoid $Q(f)$ is said to be an (n, m)-quasigroup iff the following statement is satisfied:

(A). For each "vector" $(a_1, a_2, ..., a_n) \in Q^n$ and each injection φ from $N_n = \{1, 2, ..., n\}$ into N_{n+m} there exists unique "vector" $(b_1, b_2, ..., b_{n+m}) \in Q^{n+m}$ such that $b_{\varphi(1)} = a_1, ..., b_{\varphi(n)} = a_n$ and

$$f(b_1, b_2, ..., b_n) = (b_{n+1}, b_{n+2}, ..., b_{n+m}).$$

In the paper [3] an (n, m)-quasigroup is interpreted as an (n, m)-quasigroup relation.

Definition 7. *An* $(n+m)$*-ary relation* $\rho \subseteq Q^{n+m}$ *is called* (n, m)*-quasigroup relation iff the following statement is satisfied:*

(A). For each "vector" $(a_1, a_2, ..., a_n) \in Q^n$ *and each injection* φ *from* $N_n = \{1, 2, ..., n\}$ *into* N_{n+m} *there exists unique "vector"* $(b_1, b_2, ..., b_{n+m}) \in Q^{n+m}$ *such that* $b_{\varphi(1)} = a_1, ..., b_{\varphi(n)} = a_n$ *and*

$$(b_1, b_2, ..., b_{n+m}) \in \rho.$$

The following theorem is proved in [3].

Theorem 5. *An (n,m)- groupoid (Q,f) is an (n,m)-quasigroup iff the $(n+m)$-ary relation $\rho \subseteq Q^{n+m}$ defined by*

$$(x_1, x_2, ..., x_{n+1}) \in \rho \Longleftrightarrow f(x_1, x_2, ..., x_n) = (x_{n+1}, x_{n+2}, ..., x_{n+m})$$

is an (n,m)-quasigroup relation.

Definition 8. *An (n,m)-quasigroup is called totally symmetric, iff*

$$f(x_1, x_2, ..., x_n) = (x_{n+1}, x_{n+2}, ..., x_{n+m}) \Longleftrightarrow f(y_1, y_2, ..., y_n) = (y_{n+1}, y_{n+2}, ..., y_{n+m}),$$

for any $(x_1, x_2, ..., x_{n+m}) \in Q^{n+m}$ and any permutation $(y_1, y_2, ..., y_{n+m})$ of $(x_1, x_2, ..., x_m)$. The $(n+m)$-ary relation $\rho \subseteq Q^{n+m}$ in this case is called totally symmetric.

Theorem 6. *Every complete weighted projective plane $S'(2, n+1, n^2+n+1)$ defines a totally symmetric $(2, n-1)$-quasigroup relation $\rho \subseteq V^{n+1}$, where*

$$(A_1, A_2, ..., A_{n+1}) \in \rho \Longleftrightarrow \{A_1, A_2, ..., A_{n+1}\} \in B.$$

Conversely, any totally symmetric $(2, n-1)$-quasigroup relation $\rho \subseteq V^{n+1}$ satisfying $(A, A, ..., A) = (A^{n+1}) \in \rho$ for any $A \in V$, defines a complete weighted projective plane $S'(2, k, n^2+n+1)$, where

$$\{A_1, A_2, ..., A_{n+1}\} \in B \Longleftrightarrow (A_1, A_2, ..., A_{n+1}) \in \rho.$$

Proof. Let $S'(2, n+1, n^2+n+1)$ be a complete weighted projective plane, and $\rho \subseteq V^{n+1}$ be defined as above. From the definition it follows that if $(A_1, A_2, ..., A_{n+1}) \in \rho$ then either $|\{A_1, A_2, ..., A_{n+1}\}| = n+1$ or $A_1 = A_2 = ... = A_{n+1}$, and moreover, $(A_1, A_2, ..., A_{n+1}) \in \rho$ iff $(B_1, B_2, ..., B_{n+1}) \in \rho$ for any arbitrary permutation $(B_1, B_2, ..., B_{n+1})$ of $(A_1, A_2, ..., A_{n+1})$. Hence $(n+1)$-ary relation $\rho \subseteq Q^{n+1}$ is totally symmetric. For any two distinct point $A \neq B$, there is a unique block containing A, B, i.e. there is a unique $(A_1, A_2, ..., A_{n+1}) \in \rho$, such that $A, B \in \{A_1, A_2, ..., A_{n+1}\}$. And for any $A \in V$, the pair (A, A) is in the unique $(A, A, ..., A) = (A^{n+1}) \in \rho$. Hence, ρ is a totally symmetric $(2, n-1)$-quasigroup relation.

Conversely, let $\rho \subseteq V^{n+1}$ be a totally symmetric $(2, n-1)$-quasigroup relation satisfying the condition $(A, A, ..., A) = (A^{n+1}) \in \rho$ for any $A \in V$, and let $S'(2, n+1, n^2+n+1)$ be complete weighted projective plane defined as above. If $(A_1, A_2, ..., A_{n+1}) \in \rho$ and $A_i = A_j = A$ for some $i \neq j$, then, since $(A, A, ..., A) = (A^{n+1}) \in \rho$, it follows that $A_1 = A_2 = ... = A_{n+1} = A$. Hence, if $(A_1, A_2, ..., A_{n+1}) \in \rho$, then $|\{A_1, A_2, ..., A_{n+1}\}| = n+1$ or $A_1 = A_2 = ... = A_{n+1}$. Let $B = B' \setminus \{\{A_{n+1}\} : A \in V\}$. Then it is easy to see that $(V, B, \in)$ weighted projective plane $S(2, n+1, n^2+n+1)$, and $(V, B', \in)$ is its extension. Hence, $(V, B', \in)$ is a complete weighted projective plane $S'(2, n+1, n^2+n+1)$. $\square$

5. EXAMPLES

Example 2. A projective plane $(V, B, \in)$ of order 2 is a Steiner system $S(2, 2+1, 2^2+2+1)$. The weighted projective plane $(V', B', \in)$, where $V = V', B' = B \cup B''$ where $B'' = \{\{A^3\} : A \in V\}$, is a complete weighted projective plane $S'(2, 2+1, 2^2+2+1)$. The relation $\rho \subseteq V^{2+1}$ defined by $(A_1, A_2, A_{2+1}) \in \rho \Longleftrightarrow \{A_1, A_2, A_{2+1}\} \in B$ or $A_1 = A_2 = A_{2+1}$, is a totally symmetric $(2, 2-1)$-quasigroup relation satisfying the condition $(A, A, A) = (A^{2+1}) \in \rho$. The number of points is $|V| = 2^2+2+1 = 7$, the number of blocks is $|B'| = 2^2+2+1+7 = 14$, and $t_A = 3+3 = 6$, for all $A \in V$.

Example 3. A projective plane $(V, B, \in)$ of order 3 is a Steiner system $S(2, 3+1, 3^2+3+1)$. The weighted projective plane $(V', B', \in)$, where $V = V', B' = B \cup B''$ where $B'' = \{\{A^4\} : A \in V\}$, is a complete weighted projective plane $S'(2, 3+1, 3^2+3+1)$. The relation $\rho \subseteq V^{3+1}$ defined by $(A_1, A_2, A_3, A_{3+1}) \in \rho \Longleftrightarrow \{A_1, A_2, A_3, A_{3+1}\} \in B$ or $A_1 = A_2 = A_3 = A_{3+1}$, is a totally symmetric $(2, 3-1)$-quasigroup relation satisfying the condition $(A, A, A, A) = (A^{3+1}) \in \rho$. The number of points is $|V| = 3^2+3+1 = 13$, the number of blocks is $|B'| = 3^2+3+1+13 = 26$, and $t_A = 4+4 = 8$, for all $A \in V$.

6. CONCLUSION

This paper presents the results obtained by generalization of the finite projective planes. Similar generalization can be studied for the $t-$designs. Thus, a connection between the (n, m)-quasigroups and weighted $t-$designs is achieved.

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