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THE SECOND ISOMORPHISM THEOREM ON ORDERED SET UNDER ANTIORDERS

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Abstract. In this article we give two new characteristics of quasi-antiorder relation on ordered set under antiorder.
The new results in this article is co-called the second isomorphism theorem on ordered sets under antiorders: Let $(X, =, \neq, \alpha)$ be an ordered set under antiorder α, ρ and σ quasi-antiorders on X such that $\sigma \subseteq \rho$. Then the relation σ/ρ, defined by

$$\sigma/\rho = \{(x(\rho \cup \rho^{-1}), y(\rho \cup \rho^{-1}) \in X/(\rho \cup \rho^{-1}) \times X/(\rho \cup \rho^{-1}) : (x, y) \in \sigma\},$$

is a quasi-antiorder on $X/(\rho \cup \rho^{-1})$ and $(X/(\rho \cup \rho^{-1}))/((\sigma/\rho) \cup (\sigma/\rho)^{-1}) \cong X/(\sigma \cup \sigma^{-1})$ holds.
Let $\mathbf{A} = \{\tau : \tau$ is quasi-antiorder on X such that $\tau \subset \sigma\}$. Let $\mathbf{B}$ be the family of all quasi-antiorder on X/q, where $q = \sigma \cup \sigma^{-1}$. We shall give connection between families $\mathbf{A}$ and $\mathbf{B}$. For $\tau \in \mathbf{A}$, we define a relation $\psi(\tau) = \{(aq, bq) \in X/q \times X/q : (a, b) \in \tau\}$. The mapping $\psi : \mathbf{A} \to \mathbf{B}$ is strongly extensional, injective and surjective mapping from $\mathbf{A}$ onto $\mathbf{B}$ and for $\tau, \mu \in \mathbf{A}$ we have $\tau \subseteq \mu$ if and only if $\psi(\tau) \subseteq \psi(\mu)$.

Let $(X, =, \neq)$ be a set in the sense of book [1], [2] and [6], where $\neq$ is a binary relation on X which satisfies the following properties:

$$\neg(x \neq x), x \neq y \Rightarrow y \neq x, x \neq z \Rightarrow x \neq y \vee y \neq z, x \neq y \wedge y = z \Rightarrow x \neq z$$

called *apartness* (A. Heyting). The apartness is *tight* (W. Ruitenburg) if $\neg(x \neq y) \Rightarrow x = y$ holds. Let Y be a subset of X and $x \in X$. The subset Y of X is *strongly extensional* in X if and only if $y \in Y \Rightarrow y \neq x \vee x \in Y$ ([3], [6]). If $x \in X$, we defined ([3]) $x \# Y \Leftrightarrow (\forall y \in Y) y \neq x$.

Let $f : (X, =, \neq) \to (Y, =, \neq)$ be a function. We say that it is:

(a) f is *strongly extensional* if and only if $(\forall a, b \in X)(f(a) \neq f(b) \Rightarrow a \neq b)$;

(b) f is an *embedding* if and only if $(\forall a, b \in X)(a \neq b \Rightarrow f(a) \neq f(b))$.

Let $\alpha \subseteq X \times Y$ and $\beta \subseteq Y \times Z$ be relations. The *filled product* ([3]) of relations α and β is the relation

$$\beta * \alpha = \{(a, c) \in X \times Z : (\forall b \in Y)((a, b) \in \alpha \vee (b, c) \in \beta)\}.$$

A relation $q \subseteq X \times X$ is a *coequality* relation on X if and only if holds:

$$q \subseteq \neq, q \subseteq q^{-1}, q \subseteq q * q.$$

If q is a coequality relation on set $(X, =, \neq)$, we can construct factor-set $(X/q, =_q, \neq_q)$ with

$$aq =_q bq \Leftrightarrow (a, b) \# q, aq \neq_q bq \Leftrightarrow (a, b) \in q.$$

M. Bozic and this author were first defining and studied notion of coequality relation in 1985. For more information on this relation readers can see in the paper [3].

A relation α on X is antiorder ([3], [4]) on X if and only if

$$\alpha \subseteq \neq, \alpha \subseteq \alpha * \alpha, \neq \subseteq \alpha \cup \alpha^{-1}, \alpha \cap \alpha^{-1} = \emptyset.$$

Antiorder relation on set with apartness the first defined in article [4].

Let $f : (X, =, \neq, \alpha) \to (Y, =, \neq, \beta)$ be a strongly extensional function of ordered sets under antiorders. f is called *isotone* if

$$(\forall x, y \in S)((x, y) \in \alpha \Rightarrow (f(x), f(y)) \in \beta);$$

f is called *reverse isotone* if and only if

$$(\forall x, y \in S)((f(x), f(y)) \in \beta \Rightarrow (x, y) \in \alpha).$$

The strongly extensional mapping f is called an *isomorphism* if it is injective and embedding, onto, isotone and reverse isotone. X and Y called *isomorphic*, in symbol $X \cong Y$, if exists an isomorphism between them.

As in [3] a relation $\tau \subseteq X/ \times X$ is a *quasi-antiorder* on X if and only if

$$\tau \subseteq \alpha(\subseteq \neq), \tau \subseteq \tau * \tau, \tau \cap \tau^{-1} = \emptyset.$$

The first proposition gives some information on quasi-antiorder:

Lemma 1. *If τ is a quasi-antiorder on X, then the relation $q = \tau \cup \tau^{-1}$ is a coequality relation on X.*

Proof. (1) $q = \tau \cup \tau^{-1} \subseteq \neq \cup \neq^{-1} = \neq \cup \neq = \neq$ (because $\neq^{-1} = \neq$).

(2) $q = \tau \cup \tau^{-1} = \tau^{-1} \cup \tau = (\tau \cup \tau^{-1})^{-1} = q^{-1}$.

(3) $q = \tau \cup \tau^{-1} \subseteq (\tau * \tau) \cup (\tau^{-1} * \tau^{-1}) \subseteq (\tau * \tau) \cup (\tau * \tau^{-1}) \cup (\tau^{-1} * \tau) \cup (\tau^{-1} * \tau^{-1}) = \tau * (\tau \cup \tau^{-1}) \cup \tau^{-1} * (\tau \cup \tau^{-1}) = (\tau \cup \tau^{-1}) * (\tau \cup \tau^{-1}) = q * q$. $\square$

Now we give the second results. Let $(X, =, \neq, \alpha, \tau)$ be a ordered set under antiorder α and let τ be a quasi-antiorder on X.

Lemma 2. *Let τ be a quasi-antiorder relation on X, $q = \tau \cup \tau^{-1}$. Then the relation θ on X/q, defined by*

$$(aq, bq) \in \theta \Leftrightarrow (a, b) \in q,$$

is an antiorder on X/q. The mapping $\pi : X \to X/q$ strongly extensional surjective reverse isotone mapping and holds $\theta \circ \pi = \tau$, in which case θ is equal to $\tau \circ \pi^{-1}$.

Proof. We shall check only linearity of the relation θ: Let aq and bq be arbitrary elements of set X/q such that $aq \neq bq$. Then $(a, b) \in q = \tau \cup \tau^{-1}$. Thus $(a, b) \in \tau \vee (b, a) \in \tau$. Therefore, we have $(aq, bq) \in \theta \vee (bq, aq) \in \theta$. $\square$

For the next proposition we need a lemma in which we will give explanation about two coequalities α and β on a set $(X, =, \neq)$ such that $\beta \subseteq \alpha$.

Lemma 3. *([5]) Let α and β be coequality relations on a set X with apartness such that $\beta \subset \alpha$. Then the relation β/α on X/α, defined by $\beta/\alpha = \{(x\alpha, y\alpha) \in X/q \times X/q : (x,y) \in \beta\}$, is an coequality relation on X/α and there exists the strongly extensional and embedding bijection $f : (X/\alpha)/(\beta/\alpha) \to (X/\beta)$.*

So, at the end of this article, we are in position to give a description of connection between quasi-antiorder ρ and quasi-antiorder σ of set X with apartness such that $\sigma \subseteq \rho$.

Theorem 1. *Let $(X, =, \neq, \alpha)$ be a set, ρ and σ quasi-antiorders on X such that $\sigma \subseteq \rho$. Then the relation σ/ρ defined by*

$$\sigma/\rho = \{(x(\rho \cup \rho^{-1}), y(\rho \cup \rho^{-1})) \in X/(\rho \cup \rho^{-1}) \times X/(\rho \cup \rho^{-1}) : (x,y) \in \sigma\},$$

is a quasi-antiorder on $X/(\rho \cup \rho^{-1})$ and

$$(X/(\rho \cup \rho^{-1})/((\sigma/\rho) \cup (\sigma/\rho)^{-1})) \cong X/(\sigma \cup \sigma^{-1})$$

holds.

Proof.

(1) Put $q = (\rho \cup \rho^{-1})$, and construct the factor-set $(X/q, =_q, \neq_q)$. If a and b are elements of X, then

$(aq, bq) \in \sigma/\rho \Leftrightarrow (a,b) \in \sigma$
$\Rightarrow (a,b) \in \rho$ (because $\sigma \subseteq \rho$)
$\Rightarrow (a,b) \in q$
$\Leftrightarrow aq \neq_q bq$.

$(aq, cq) \in \sigma/\rho \Leftrightarrow (a,c) \in \sigma$
$\Rightarrow (\forall b \in X)((a,b) \in \sigma \vee (b,c) \in \sigma)$
$\Leftrightarrow (\forall bq \in X/q)((aq,bq) \in X/q \vee (bq,cq) \in q)$.

Suppose that $(\sigma/\rho) \cap (\sigma/\rho)^{-1} \neq \emptyset$, and let $(aq,bq) \in (\sigma/\rho) \cap (\sigma/\rho)^{-1}$. Then $(aq,bq) \in \sigma/\rho \wedge (bq,aq) \in \sigma/\rho$, and thus $(a,b) \in \sigma \wedge (b,a) \in \sigma$, i. e. $(a,b) \in$

$\sigma \cap \sigma^{-1}$. It is impossible. So, must be $(\sigma/\rho) \cap (\sigma/\rho)^{-1} = \emptyset$. Therefore, the relation σ/ρ is a quasi-antiorder on set X/q.

(2) From $\sigma \subseteq \rho$ it follows $\rho \cup \rho^{-1} \supseteq \sigma \cup \sigma^{-1}$. By lemma 1, there exist following coequality relations:

$q = \rho \cup \rho^{-1}$ on the set $(X, =, \neq)$,

$r = \sigma \cup \sigma^{-1}$ on the set $(X, =, \neq)$,

$r/q = (\sigma/\rho) \cup (\sigma/\rho)^{-1}$ on factor-set X/r,

and, by Lemma 3, there exists the strongly extensional and embedding bijective function

$$\varphi : (X/(\rho \cup \rho^{-1}))/((\sigma/\rho) \cup (\sigma/\rho)^{-1}) \to X/(\sigma \cup \sigma^{-1}).$$

(3) By Lemma 2, on factor-set X/r with $r = \sigma \cup \sigma^{-1}$ there exists the antiorder $\theta(\sigma)$ such that

$$(a(\sigma \cup \sigma^{-1}), b(\sigma \cup \sigma^{-1})) \in \theta(\sigma) \Leftrightarrow (a, b) \in \sigma,$$

and on factor-set X/q with $q = \rho \cup \rho^{-1}$ there exists the antiorder $\theta(\rho)$ such that

$$(a(\rho \cup \rho^{-1}), b(\rho \cup \rho^{-1})) \in \theta(\rho) \Leftrightarrow (a, b) \in \rho.$$

Also, on the factor-set $X/((\sigma/\rho) \cup (\sigma/\rho)^{-1})$ there exists the antiorder $\theta(\sigma/\rho)$ such that

$$(a((\sigma/\rho) \cup (\sigma/\rho)^{-1}), b((\sigma/\rho) \cup (\sigma/\rho)^{-1})) \in \theta(\sigma/\rho) \Leftrightarrow (ar, br) \in \theta(\sigma).$$

It is easily to verify that the mapping

$$\varphi : (X/(\rho \cup \rho^{-1}))/((\sigma/\rho) \cup (\sigma/\rho)^{-1}) \to X/(\sigma \cup \sigma^{-1})$$

is isotone and reverse isotone strongly extensional embedding bijection. $\square$

At the end of this consideration, we shall give the following proposition:

Theorem 2. *Let $(X, =, \neq, \alpha, \sigma)$ be an ordered set under an antiorder α, σ a quasi-antiorder on X such that $\sigma \cap \sigma^{-1} = \emptyset$. Let $\mathbf{A} = \{\tau : \tau$ is quasi-antiorder on X such that $\tau \subseteq \sigma\}$. Let $\mathbf{B}$ be the family of all quasi-antiorder on X/q, where $q = \sigma \cup \sigma^{-1}$. For $\tau \in \mathbf{A}$, we define a relation*

$$\psi(\tau) = \{(aq, bq) \in X/q \times X/q : (a, b) \in \tau\}.$$

The mapping $\psi : \mathbf{A} \to \mathbf{B}$ is strongly extenstional, injective and surjective mapping from $\mathbf{A}$ onto $\mathbf{B}$ and for $\tau, \mu \in \mathbf{A}$ we have $\tau \subseteq \mu$ if and only if $\psi(\tau) \subseteq \psi(\mu)$.

Proof.

(1) f is well-defined function:

Let $\tau \in \mathbf{A}$. Then $\psi(\tau)$ is a quasi-antiorder on X/q by Lemma 3.

Let $\tau, \mu \in \mathbf{A}$ with $\tau = \mu$. If $(aq, bq) \in \psi(\tau)$ then $(a, b) \in \tau = \mu$, so $(aq, bq) \in \psi(\mu)$. Similarly, $\psi(\mu) \subseteq \psi(\tau)$. Therefore, $\psi(\mu) =_q \psi(\tau)$, i. e. the mapping ψ is a function.

(2) ψ is an injection. Let $\tau, \mu \in \mathbf{A}$, $\psi(\tau) =_q \psi(\mu)$. Let $(a, b) \in \tau$. Since $(aq, bq) \in \psi(\tau) =_q \psi(\mu)$, we have $(a, b) \in \mu$. Similarly, we conclude $\mu \subseteq \tau$. So, $\mu = \tau$.

(3) ψ is strongly extensional. Let $\tau, \mu \in \mathbf{A}$, $\psi(\tau) \neq_q \psi(\mu)$ i. e. let there exists an element $(aq, bq) \in \psi(\tau)$ and $(aq, bq)\#\psi(\mu)$. Then, $(a, b) \in \tau$. Let (x, y) be an arbitrary element of μ. Then, $(xq, yq) \in \psi(\mu)$ and $(xq, yq) \neq (aq, bq)$. It means $xq \neq_q aq \vee yq \neq_q bq$, i. e. $(x, a) \in q \vee (y, b) \in q$. Therefore, from $x \neq a \vee y \neq b$ we have $(a, b) \in \tau$ and $(a, b) \neq (x, y) \in \mu$. Thus we have $\tau \neq \mu$. Similarly, from $(aq, bq)\#\psi(\tau)$ and $(aq, bq) \in \psi(\mu)$ we conclude $\tau \neq \mu$.

(4) ψ is onto. Let $M \in \mathbf{B}$. We define a relation μ on X as follows:

$$\mu = \{(x, y) \in X \times X : (xq, yq) \in M\}.$$

μ is a quasi-antiorder. In fact:

(I) Let $(x, y) \in \mu$. Since $(xq, yq) \in M \subseteq \neq_q$ on X/q, we conclude that $xq \neq_q yq$, i. e. $(x, y) \in q = \sigma \cup \sigma^{-1}$. Hence $(x, y) \in \sigma \subseteq \neq$ or $(y, x) \in \sigma \subseteq \neq$. Therefore, we have $x \neq y$. Let $(x, z) \in \mu$ i. e. let $(xq, zq) \in M$. Then $(xq, yq) \in M$ or $(yq, zq) \in M$ for arbitrary $yq \in X/q$ by cotransitivity of M. Thus $(x, y) \in \mu$ or $(y, z) \in \mu$. Therefore, the relation μ is a quasi-antiorder relation on X. Let $M \cap M^{-1} = \emptyset$. Suppose that $(a, b) \in \mu \cap \mu^{-1}$. Then $(aq, bq) \in M \cap M^{-1} = \emptyset$, which is impossible. So, $\mu \cap \mu^{-1} = \emptyset$.

(II) $\psi(\mu) = M$. Indeed:

$$(xq, yq) \in \psi(\mu) \Leftrightarrow (x, y) \in \mu \Leftrightarrow (xq, yq) \in M.$$

(III) $\mu \subseteq \sigma$. In the matter of fact, we have sequence
$(a, b) \in \mu \Leftrightarrow (aq, bq) \in \psi(\mu) = M$
$\Leftrightarrow (\pi(a), \pi(b)) \in \psi(\mu) = M$ ($\pi : X \to X/q$ is strongly extensional function)
$\Leftrightarrow (a, b) \in \pi^{-1}(\psi(\mu)) = \pi^{-1}(M)$ (by $\pi^{-1}(M) \subseteq q = Coker(\pi) = \sigma \cup \sigma^{-1}$) By Corollary 3. 1. the mapping $\pi : X \to X/q$ is a strongly extensional reverse isotone surjective mapping. Therefore, we have $\pi^{-1}(M) \subseteq \sigma$.

(5) Let $\tau, \mu \in \mathbf{A}$. We have $\tau \subseteq \mu$ if and only if $\psi(\tau) \subseteq \psi(\mu)$. Indeed: Let $\tau \subseteq \mu$ and $(xq, yq) \in \psi(\tau)$. Since $(x, y) \in \tau \subseteq \mu$, we have $(xq, yq) \in \psi(\mu)$. Opposite, let $\psi(\tau) \subseteq \psi(\mu)$ and $(x, y) \in \tau$. Since $(xq, yq) \in \psi(\tau) \subseteq \psi(\mu)$, we conclude that $(x, y) \in \mu$. $\square$

References

[1] E. Bishop, *Foundation of Constructive Analysis*; McGraw-Hill, NY (1967).

[2] R. Mines, F. Richman, W. Ruitenburg, *A Course of Constructive Algebra*, Springer-Verlag, New York (1988).

[3] D. A. Romano, *Coequality Relation, a Survey*, Bull. Soc. Math. Banja Luka, **3** (1996), 1–36

[4] D. A. Romano,*Semivaluation on Heyting Field*, Kragujevac Journal of Mathematics, **20** (1998), 24–40

[5] D. A. Romano, *A Theorem on Subdirect Product of Semigroups with Apartness*, Filomat, **14** (2000), 1–8

[6] A. S. Troelstra, D. van Dalen, *Constructivism in Mathematics, An Introduction, Volume II*, North-Holland, Amsterdam (1988).