Kragujevac J. Math. 30 (2007) 253–262.

ITERATIVE OPERATORS FOR FAREY TREE

Ljubiša Kocić[1], Liljana Stefanovska[2] and Sonja Gegovska-Zajkova[3]

[1] *Faculty of Electronic Engineering, University of Niš, Serbia*
(e-mail: kocic@elfak.ni.ac.yu)

[2] *Faculty of Technology and Metallurgy, SS Cyril and Methodius University, Skopje, R.Macedonia*
(e-mail: liljana@ian.tmf.ukim.edu.mk)

[3] *Faculty of Electrical Engineering, SS Cyril and Methodius University, Skopje, R.Macedonia*
(e-mail: szajkova@etf.ukim.edu.mk)

(Received October 18, 2006)

Abstract. The existence of an operator that maps rational number 1/2 into the array of Farey tree is proven. It is shown that this operator can be represented by combinatorial compositions of two simple real functions: $f : [0, 1] \to [1/2, 1]$, which is (0, 1)-rational and $\sigma : [0, 1] \to [0, 1]$, which is linear. Then, another operator, mapping rational $r \in (0, 1)$ into the branch of the Farey tree emanating from the node characterized by r is described.

1. INTRODUCTION

The Farey tree is a collection of sets (called *levels*) $FT = \{T_0, T_1, T_2, \dots\}$, where $T_0 = \{r_1 = 1/2\}$ is called *root of the tree.* The n-th level $T_n = \{r_{2^n}, \dots r_{2^{n+1}-1}\}$, $n = 0, 1, 2, \dots$, is the decreasing sequence of rationals $r_j \in (0, 1)$, like $T_1 = \{r_2 =$

$2/3$, $r_3 = 1/3\}$, $T_2 = \{r_4 = 3/4$, $r_5 = 3/5$, $r_6 = 2/5$, $r_7 = 1/4\}$, $\dots$. One can identify FT with the infinite binary graph which set of vertices is isomorphic with $\mathbb{Q}[0, 1]$, the set of rationals from the segment $[0, 1]$. In set-theoretic notation, Farey tree is collection of sets

$$\begin{aligned} FT \;=\; & \{\{1/2\}, \{1/3, 2/3\}, \{1/4, 2/5, 3/5, 3/4\}, \\ & \{1/5, 2/7, 3/8, 3/7, 4/7, 5/8, 5/7, 4/5\}, \\ & \{1/6, 2/9, 3/11, 3/10, 4/11, 5/13, 5/12, 4/9, 5/9, \\ & \ 7/12, 8/13, 7/11, 7/10, 8/11, 7/9, 5/6\}, \dots\}. \end{aligned} \tag{1}$$

Farey tree (1) plays an important role in Chaos Theory. For ex., it contains all *quasi-periodic routes to chaos*. In fact, if a dynamic system contains two periodic oscillators with different frequencies, f_1 and f_2, ($f_1 < f_2$), the regime in the system tries to preserve the state where the ratio f_1/f_2 is the simplest rational number, say 1. Then, $f_1 : f_2 = 1 : 1$ which is called *optimal resonance* or $1 : 1$ *mode-locking* regime. If this is not possible, the system "jumps" to the "reserve" mode-locking state, $f_1/f_2 = 1/2$. If, by some reason, this state is not possible, the system passes to the next simple mode-locking possibility, $f_1/f_2 = 2/3$ (or $f_1/f_2 = 1/3$), and so on, along the Farey tree. Among others, Farey tree contains the quickest route, so called *golden route to chaos*, the sequence of ratios of consecutive Fibonacci numbers converging to famous golden mean $\phi = (\sqrt{5} - 1)/2$.

Let $a_0, \dots, a_k$ denote coefficients in continuous fraction expansion (partial quotients) of a rational number r

$$r = [a_0, a_1, \dots, a_k] = \frac{1}{a_0 +}\,\frac{1}{a_1 +}\ \cdots\ \frac{1}{+\ a_k}, \quad a_i \in \mathbb{N}, \quad a_k \geq 2. \tag{2}$$

Using expansion (2), Cvitanović [1] gave the following formal definition of the Farey tree level:

Definition 1. *The n-th Farey tree level T_n is the monotonically increasing sequence of 2^n continued fractions $[a_0, a_1, \dots, a_k]$ whose entries $a_i \geq 1, i = 0, 1, \dots, k-1$, $a_k \geq 2$, add up to $n + 2$.*

For example,

$$\begin{aligned} T_2 &= \{[4],\ [2,\ 2], [1,\ 1,\ 2],\ [1,\ 3]\} = \{1/4,\ 2/5,\ 3/5,\ 3/4\}, \\ T_3 &= \{[5],\ [3,\ 2],\ [2,\ 1,\ 2],\ [2,\ 3],\ [1,\ 1,\ 3],\ [1,\ 1,\ 1,\ 2],\ [1,\ 2,\ 2],\ [1,\ 4]\} \\ &= \{1/5,\ 2/7,\ 3/8,\ 3/7,\ 4/7,\ 5/8,\ 5/7,\ 4/5\}, \end{aligned}$$

etc.

The Farey tree can be split into two sub-trees ([2], [3]). One of them, denoted by FT^0 ("0-subtree") has the root in 1/3 and contains all rationals from the open interval (0, 1/2); Another, FT^1 ("1-subtree") has the root in 2/3 and contains all rationals from (1/2, 1). It is proved in [3] that elements of FT^0 have continued fractions of the form $[a_0,\ a_1,\ \ldots,\ a_k]$, $a_0 \geq 2$, while elements from FT^1 have the form $[a_0,\ a_1,\ \ldots,\ a_k]$, $a_0 = 1$. The k-th level of FT^i will be denoted by T_k^i, $i = 0, 1$, and it represents "i-half" if of the level T_k.

2. FAREY TREE OPERATOR

Lemma 1. *If in (2) all $a_k \in \mathbb{N}$, and $k \geq 1$, then $r \in (0,\ 1)$.*

Proof. Consider partial "sum" $s_j = \dfrac{1}{a_{k-j}+}\ \cdots\ \dfrac{1}{+\,a_k}$ $(0 \leq j \leq k)$. Obviously, $s_0 = 1/a_k \in (0,\ 1)$ for all $a_k \geq 2$ and $s_0 = 1$ for $a_k = 1$. Then, $s_1 = 1/(a_{k-1} + 1/a_k) \in (0,\ 1)$, and by induction $s_j \in (0,\ 1)$. Thus, $s_k = r \in (0,\ 1)$, except if $k = 0$ (and $a_0 = 1$) which is excluded by supposition. □

Lemma 2. *The simplest difeomorphism*

$$[a_0,\ \ldots,\ a_k] \mapsto [1,\ a_0,\ \ldots,\ a_k], \quad a_k \in \mathbb{N},$$

is given by

$$f : x \to \frac{1}{1+x}\,, \tag{3}$$

and it maps $[0,\ 1]$ *to* $[\frac{1}{2},\ 1]$.

Proof. The proof follows by definition $[1,\, a_0,\, \ldots,\, a_k] = \dfrac{1}{1+[a_0, \ldots, a_k]}$ and by Lemma 1, $x = [a_0,\, \ldots,\, a_k] \in (0,\, 1)$. □

Lemma 3. *The simplest difeomorphism*

$$[a_0,\, a_1,\, \ldots,\, a_k] \;\mapsto\; [a_0+1,\, a_1,\, \ldots,\, a_k]$$

is given by

$$g : x \to \frac{x}{1+x}, \tag{4}$$

and it maps $[0,\, 1]$ *to* $[0,\, \frac{1}{2}]$.

Proof. Let $x = [a_0,\, \ldots,\, a_k] \in (0,\, 1]$. Then

$$[1+a_0,\, a_1,\, \ldots,\, a_k] = \frac{1}{1+a_0+\left(\frac{1}{a_1}+ \;\cdots\; + \frac{1}{a_k}\right)} = \frac{1}{1+\frac{1}{[a_0,\, \ldots,\, a_k]}} = f\left(\frac{1}{x}\right),$$

where f is given by (3). Setting $g(x) = f(1/x)$ for $x \neq 0$ gives (4). □

Note that composition $g \circ f^{-1}$ yields $(g \,\circ\, f^{-1})(x) = g(f^{-1}(x)) = 1-x$. Denote $\sigma(x) = 1-x$. Then, by $\sigma = g \,\circ\, f^{-1}$, it follows $g(x) = (\sigma \,\circ\, f)(x)$. Let $\tilde{\mathbb{Q}}[0,\, 1]$ denotes partition set of $\mathbb{Q}[0,\, 1]$. Consider the following four set valued operators $\tilde{\mathbb{Q}}[0,\, 1] \to \tilde{\mathbb{Q}}[0,\, 1]$:

$$\begin{aligned} F_1 &= \{\sigma \circ f,\, f\}\,, \\ F_2 &= \{\sigma \circ f \circ \sigma,\, f\}\,, \\ F_3 &= \{\sigma \circ f \circ \sigma,\, f \circ \sigma\}\,, \\ F_4 &= \{\sigma \circ f,\, f \circ \sigma\}\,. \end{aligned} \tag{5}$$

Note that mappings in (5) have the form $F = \{\varphi,\, \psi\}$. Here $\varphi : [0,\ 1] \to [0,\ 1/2]$ is one of two functions: $(\sigma \circ f)(x) = x/(x+1)$ or $(\sigma \circ f \circ \sigma)(x) = (x-1)/(x-2)$. On the other side, $\psi : [0,\ 1] \to [1/2,\ 0]$ is ether $f(x) = 1/(x+1)$ or $(f \circ \sigma)(x) = 1/(2-x)$. Also, the inverse operators to (5) can easily be established provided that the conventions $F_i(\emptyset) = \emptyset$, and

$$F_i(\{r_1, \ldots, r_k\}) = \{\varphi(r_1), \ldots,\, \varphi(r_k),\, \psi(r_1), \ldots,\, \psi(r_k)\}, \tag{6}$$

are adopted. Namely, $F_i^{-1}(\emptyset) = \emptyset$, and

$$\begin{aligned} &F_i^{-1}(\{s_1, \ldots, s_k, s_{k+1}, \ldots, s_{2k}\}) = \\ &\{\varphi^{-1}(s_1), \ldots,\, \varphi^{-1}(s_k),\, \psi^{-1}(s_{k+1}), \ldots,\, \psi^{-1}(s_{2k})\}. \end{aligned} \tag{7}$$

Theorem 1. *Neighbor levels of FT map one to another by any of the operators (5) or their inverses. More precisely,* $F_i(T_n) = T_{n+1}$ *and* $F_i^{-1}(T_{n+1}) = T_n$, $n = 0, 1, \ldots$.

Proof. Consider the operator F_1. Suppose that $r_k = [a_0,\, \ldots,\, a_k] \in T_n$. By definition, and convention (6), $F_1(\{r_k\}) = \{g(r_k),\, f(r_k)\}$, and by use Lemma 2 and 3 we conclude that

$$F_1(\{r_k\}) = F_1(\{[a_0,\, \ldots,\, a_k]\}) = \{g([a_0,\, \ldots,\, a_k]),\, f([a_0,\, \ldots,\, a_k])\}$$

$$= \{[a_0 + 1,\, a_1,\, \ldots,\, a_k],\, [1,\, a_0,\, \ldots,\, a_k]\}.$$

Then, $F_1(\{r_k\}) = \{r_p,\, r_q\}$, and by Definition 1, $r_p, r_q \in T_{n+1}$. Since any of 2^n rationals from T_n has one-to-one unique expansion in continued fraction, the operator F_1 applying on them produces 2^n expansions that define exactly 2^{n+1} new rationals from the upper level T_{n+1}. Since the mapping σ just reverse the order, i.e., $F_2(x) = \{g(1-x),\, f(x)\}$, $F_3(x) = \{g(1-x),\, f(1-x)\}$ and $F_4(x) = \{g(x),\, f(1-x)\}$, we conclude that $F_i(T_n) = T_{n+1}$ $(n = 0,\, 1,\, \ldots)$ for $i = 2, 3$ and 4. The inverse mapping F_1^{-1} (7) maps $T_{n+1} \to T_n$. The similar reasoning applies to other F_i's from (5). □

The immediate consequence of Theorem 1 is that $(F_i \circ F_j)(T_n) = T_{n+2}$, for any $i,\, j \in \{1,\, 2,\, 3,\, 4\}$. This leads to the main result of this note.

Theorem 2. *Let G be any composition of mappings in (5)*

$$G_m = F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_m} \;\; i_j \in \{1,\, 2,\, 3,\, 4\},\; m \in \mathbb{N}. \tag{8}$$

Then,

$$G_m(\{1/2\}) = T_m. \tag{9}$$

Proof. Now, note that difeomorphism f given by (3) is bijective mapping of the level T_{n-1} onto T_n^1 "1-half" of the level T_n. For any $r = [a_0,\, a_1,\, \ldots,\, a_k] \in T_{n-1}$, $a_i \geq 1$, $i = 0,\, 1,\, \ldots,\, k-1$, $a_k \geq 2$, and by Definition 1, $\sum a_i = n + 1$. Also, $f(r) = [1, a_0,\, a_1,\, \ldots,\, a_k]$, with the sum of partial quotients $1 + \sum a_i = n + 2$. Therefore, $f(r) \in T_n$. Further, the first partial quotient of $f(r)$ is 1 making it the member of the "1-subtree"or $f(r) \in T_n^1$. Similar reasoning may be applied on g given by (4), which maps T_{n-1} onto T_n^0 bijectively. Suppose that $r \in T_{n-1}$. Then, by Definition 1, $r = [a_0, a_1, \ldots, a_k]$, where $a_i \geq 1$, $i = 1, 2, \ldots, k-1$, $a_k \geq 2$, and $\sum a_i = n+1$. Since $g(r) = g([a_0, a_1, \ldots, a_k]) = [a_0 + 1, a_1, \ldots, a_k]$, then the sum of partial quotients is $1 + \sum a_i = n+2$, so $g(r) \in T_n$. Further, the first coefficient in the continued fraction expansion of $g(r)$ is ≥ 2, which guarantees that $g(r)$ belongs to the "0-subtree". This gives $g(r) \in T_n^0$.

By similar argument and in the previous proof, we have

$$(F_i \circ F_j)(T_{n-1}) = T_{n+1},$$

for any $i, j \in \{1, 2, 3, 4\}$, and therefore

$$G_2(T_0) = (F_i \circ F_j)(T_0) = (F_i \circ F_j)(\{1/2\}) = T_2.$$

Thus, (9) follows by induction. □

Definition 2. *The operator $r \mapsto H_m(r)$, given by*

$$H_m(r) = \bigcup_{n=1}^{m} G_n(\{r\}), \quad r \in \mathbb{Q}[0, 1] \tag{10}$$

will be called partial Farey tree operator.

The following two statements are consequences of Theorem 2:

Corollary 1. *If in (10) $m \to \infty$, the Farey tree (without root) is obtained as an image of a single rational number, $r = 1/2$. Namely,*

$$H_\infty(1/2) = \bigcup_{n=1}^{\infty} G_n(\{1/2\}) = FT \setminus \{1/2\}.$$

By using $F^{(n)}$ to denote n-th auto-iteration of the operator F, with usual convention that $F^{(0)}$ is identity, we may state a short definition of the Farey tree, given as Corollary of Theorem 2.

Corollary 2. $FT = \bigcup_{n=1}^{\infty} F^{(n)}(\{1/2\})$, *where $F \in \{F_1, F_2, F_3, F_4\}$.*

Here, we restricted ourselves on the first operator in (5). Of course, others are applicable as well.

3. BRANCH OPERATOR

Schroeder in [4] gives the simple algorithm for indexing rationals from Farey tree. In fact, for any $n \in \mathbb{N}$,

$$r_n = [a_0,\, a_1,\, \ldots,\, a_k],\ a_j \in \mathbb{N},$$

the sequence of partial quotients $(a_0,\, a_1,\, \ldots,\, a_k)$ represents cardinal numbers of subsets of successive units or zeros in the sequence $(b_0,\, b_1,\, \ldots,\, b_{m-1},\, b_m,\, b_m)$, where $(b_0,\, b_1,\, \ldots,\, b_m)$, $b_i \in \{0,\, 1\}$, $b_0 = 1$ is the sequence of binary digits making the binary expansion of n.

Let α and β represent the following simple mappings of rationals from $(0,\, 1)$:

$$\begin{array}{l} \alpha : [a_0,\, a_1,\, \ldots,\, a_n] \;\mapsto\; [a_0,\, a_1,\, \ldots,\, a_n - 1,\, 2], \\ \beta : [a_0,\, a_1,\, \ldots,\, a_n] \;\mapsto\; [a_0,\, a_1,\, \ldots,\, a_n + 1] \end{array} . \tag{11}$$

In [1], the following statement is proven.

Lemma 4. *The "children" of the Farey tree element r_k $(k = 1,\, 2,\, \ldots)$ are*

$$\begin{array}{lll} r_{2k+1} = \alpha(r_k), & r_{2k} = \beta(r_k), & \textit{if } k \textit{ is even;} \\ r_{2k+1} = \beta(r_k), & r_{2k} = \alpha(r_k), & \textit{if } k \textit{ is odd.} \end{array}$$

It is known ([2], [3]) that k-th rational r_k from FT belongs to the level $T_{\lfloor \log_2 k \rfloor}$, where $\lfloor x \rfloor$ stands for "entire part of x". Having in mind that $\lfloor \log_2(2k) \rfloor = \lfloor \log_2(2k+1) \rfloor = n+1$, it is clear that $r_k \in T_n$, implies $r_{2k},\, r_{2k+1} \in T_{n+1}$. For instance,

$$\begin{array}{ll} 1/2 = [2] \mapsto \{[3],\, [1,\, 2]\}, & 1/3 = [3] \mapsto \{[4],\, [2,\, 2]\}, \\ 2/3 = [1,\, 2] \mapsto \{[1,\, 1,\, 2],\, [1,\, 3]\}, & 1/4 = [4] \mapsto \{[5], [3,\, 2]\}, \\ 2/5 = [2,\, 2] \mapsto \{[2, 1, 2], [2, 3]\}, & 3/5 = [1, 1, 2] \mapsto \{[1, 1, 3], [1, 1, 1, 2]\}, \\ 3/4 = [1, 3] \mapsto \{[1, 2, 2], [1, 4]\}, & \end{array}$$

etc.

Upon the Lemma 4, we construct the following operator $\Phi : [0,\, 1] \to [0,\, 1]^2$:

$$\Phi(r_k) = \frac{1+(-1)^k}{2}\{\alpha(r_k),\, \beta(r_k)\} \;+\; \frac{1+(-1)^k}{2}\{\beta(r_k),\, \alpha(r_k)\},$$
$$r_k \in T_{\lfloor \log_2 k \rfloor},\; k \in \mathbb{N}. \tag{12}$$

Let n denote the n-th iteration of the type $\Phi^2 = \Phi \cap \Phi$, $\Phi^3 = (\Phi \cap \Phi) \cap (\Phi \cap \Phi)$, etc. If BR_k denotes the branch of FT emanating from the element r_k, then operator (12) may be used for defining this branch, in accordance with the next theorem.

Theorem 3. *The branch of Farey tree, emanating from the element r_k is given by*

$$\lim_{n\to\infty} \bigcup_{i=0}^{n} \Phi^i(r_k) = BR_k,\; k \in \mathbb{N}.$$

Proof. By Lemma 4 and relation (12), for even k, one has $\Phi(r_k) = \{r_{2k+1}, r_{2k}\}$. By induction,

$$\Phi^i(r_k) = \left\{r_{2^i(k+1)-1},\, r_{2^i(k+1)-2},\, \ldots,\, r_{2^i(k+1)-2^i}\right\}.$$

If $r_k \in T_n$, then $n = \lfloor \log_2 k \rfloor$, so the elements of $\Phi^i(r_k)$ belongs to T_{n+i}, since

$$\begin{aligned} \lfloor \log_2(2^i(k+1)-1) \rfloor &= \lfloor \log_2(2^i(k+1)-2) \rfloor \\ &= \lfloor \log_2(2^i(k+1)-2^i) = i + \lfloor \log_2 k \rfloor = i + n. \end{aligned}$$

On the other hand, $\Phi^i(r_k)$ contains all 2^i descents of element r_k that belong to T_{n+i}, and, consequently $\bigcup_{i=0}^{n} \Phi^i(r_k)$ contains all $2^{i+1}-2$ descents of element r_k up to the level T_{n+i}. When $n \to \infty$,

$$\begin{aligned} \bigcup_{i=0}^{n} \Phi^i(r_k) &= \{\{r_k\},\, \{r_{2k+1},\, r_{2k}\}, \\ &\qquad \left\{r_{4(k+1)-1},\, r_{4(k+1)-2},\, r_{4(k+1)-3},\, r_{4(k+1)-4}\right\},\, \ldots\right\} = \mathrm{BR}_k. \end{aligned}$$

□

Since BR_1 is the whole Farey tree, one has

Corollary 3. $FT = \lim_{n\to\infty} \bigcup_{i=0}^{n} \Phi^i(1/2)$.

It is interesting to compare Corollaries 2 and 3. Both give Farey tree in the process of forming an infinite set level by level. But in the case of operators F, in spite of simple mappings f and g, levels are forming disorderly. On the contrary, the operator can not be described by simple functions, yet levels of descents are forming orderly. To be convinced of the complexity of operator Φ, it is enough to examine mappings α and β, given by (11), for the case when rational number r has a simple continued fraction expansion, say

$$r = [x,\, y,\, z] = \frac{1}{x+}\frac{1}{y+}\frac{1}{z} = \frac{1+yz}{x+z+xyz},\; x,\, y,\, z \in \mathbb{N},\; z \geq 2.$$

Now, according to (11), $\alpha(r) = [x,\, y,\, z-1,\, 2] = \dfrac{2-y+2yz}{-1+2x+2z-xy+2xyz}$ and $\beta(r) = [x,\, y,\, z{+}1] = \dfrac{1+y+yz}{1+x+z+xy+xyz}$. Both $\alpha(r)$ and $\beta(r)$ are rational functions of x, y and z that can not be expressed by some simple mapping of $(1+yz)/(x+z+xyz)$.

4. CONCLUSION

The authors aim was to demonstrate that one so complex structure as the Farey tree, can be defined by iterating some simple mappings such as f and g given by (3) and (4) respectively. There are four combinations of these mappings that give the same result, the Farey tree as an image of its root, 1/2. But, the sequence of rational numbers, obtained in each iteration is not ordered naturally. On the other hand, two other mappings α and β, given by (11) although very complicated still yield ordered levels in building the Farey tree.

Acknowledgements: The authors thanks to the referees for their highly valuable suggestions.

References

[1] P. Cvitanović, *Circle maps: irrationally winding*, Number Theory and Physics (C. Itzykson, P. Moussa and M. Waldschmidt, eds.), Les Houches (1989) Spring School, Springer, New York (1992).

[2] Lj. M. Kocić, R. L. Stefanovska, *A property of Farey Tree*, Lect. Notes in Mathematics, **3401** (2005), 346–352.

[3] Lj. M. Kocić, R. L. Stefanovska, *Two Algorithms for Farey Tree*, Mathematical Buletin (Skopje), **29** (**LV**) (2005), 89–98.

[4] M. R. Schroeder, Fractals, *Chaos, Power Laws: Minutes from an Infinite Paradise*, Freeman and Co., New York (1991).