AN ERROR ESTIMATE FOR A NUMERICAL SCHEME FOR THE COMPRESSIONABLE NAVIER–STOKES SYSTEM

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Abstract. The subject of this paper is an error estimate of the order $h^{1/2}$ in the $L^2$-norm for an explicit, fully discrete numerical scheme that approximates smooth solutions of the barotropic compressible fluid flow equations in the multidimensional case. Assuming some a-priori estimates for the discrete solution we derive an error estimate using a technique based upon stability results due to Dafermos [3] and DiPerna [5], which were originally formulated for systems of conservation laws.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^m$ be an open set and $Q = (0,1)^d$ the unite cube in $\mathbb{R}^d$. Consider the following Cauchy problem

$$\partial_t u + \sum_{i=1}^d \partial_i G_i(u) = B[u] \quad \text{in} \quad \mathbb{R}^d \times (0,T),$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^d,$$

where $G_i : \Omega \to \mathbb{R}^m (i = 1, \ldots, d)$ are smooth flux functions and $B$ is an operator. The derivative on $\mathbb{R}^d$ is denoted by $\partial$ and the partial derivatives by $\partial_i$. For functions
$F : \Omega \to \mathbb{R}^m$ and $f : \Omega \to \mathbb{R}$ let $DF$ and $\nabla f$ be their derivative and gradient, respectively. Finally, the Euclidean norm on $\mathbb{R}^m$ is denoted by $\| \cdot \|$ and the induced matrix norm on $\mathbb{R}^{m \times m}$ by $\| \cdot \|$.

Assume that the system (1) has a strictly convex entropy $\eta$. If the solution $u : \mathbb{R}^d \times [0, T] \to \mathbb{R}^m$ of (1), (2) is smooth, then the additional equality

$$
\partial_t \eta(u) + \sum_{i=1}^d \partial_i q_i(u) = \nabla \eta(u) \cdot B[u] \quad \text{in} \quad \mathbb{R}^d \times (0, T),
$$

holds, with $q_i$'s being the corresponding entropy fluxes. The stability issue for (1), (2) in the hyperbolic setting (i.e. $B \equiv 0$) was explored in Dafermos [3] and DiPerna [5]. If one assumes, for the sake of simplicity, that the smooth solution $u$ of (1), (2) is $1$-periodic in the spatial variable, the theory developed there implies the following assertion: if $\bar{u}$ is a $1$-periodic admissible weak solution of (1) (with the initial function $\bar{u}_0$), then

$$
\int_Q |u(x, t) - \bar{u}(x, t)|^2 \, dx \leq ce^{\alpha t} \int_Q |u_0 - \bar{u}_0|^2 \, dx, \quad t \in [0, T),
$$

where $c$ depends on the set where $u$ and $\bar{u}$ take their values, and the constant $\alpha$ depends also on the sup-norm of $\partial u$.

In an attempt to derive a similar stability result for general $B$, it turns out that a certain compatibility condition between the conservative hyperbolic part of the system (1) and the operator $B$ is needed. Let us state that condition: If Dom($B$) is the domain of the operator $B$, then for all $u^1, u^2 \in \text{Dom}(B)$,

$$
\int_Q \{B[u^1] - B[u^2]\} \cdot \{\nabla \eta(u^1) - \nabla \eta(u^2)\} \, dx \leq C \int_Q |u^1 - u^2|^2 \, dx,
$$

with $t \in [0, T)$. The constant $C$ should depend only on the set where $u^1, u^2$ take their values (for details see Jovanović [10]). The condition (5) is obviously satisfied if $B$ is a source term. However, it is not quite obvious that (5) also holds for the compressible Navier–Stokes system, with $C = 0$ (see equality (13) below). Hence the possibility of obtaining an inequality of the form (4) for the Navier–Stokes system. We will even go a step further in our generalizations. If we replace the smooth solution $\bar{u}$ in (4)
by an arbitrary \( L^2 \)-function \( w \), we will be in a position to estimate the resulting \( L^2 \)-distance between \( u \) and \( w \) by some residual measures (see Definition 1 and Theorem 1 below). Finally, by taking \( w \) to be a numerical approximation, we can derive an \( L^2 \)-error estimate, if we estimate the right-hand-side in the obtained inequality (see (26) below). This is the idea of our approach.

For what concerns the numerical scheme, we use an explicit finite volume – finite difference scheme on a uniform mesh: the hyperbolic part of the Navier–Stokes system is discretized by a finite volume, and the viscous part by a finite difference scheme. A priori estimates for the numerical solution are assumed. Our main result, the error estimate, is stated in Theorem 2.

The approach of deriving error estimates based upon the Dafermos-DiPerna stability result for hyperbolic problems was pursued in Arvanitis et al. [1], Vila [14], Jovanović, Rohde [11], etc.. Numerical methods for compressible fluid flow are developed in Bristeau et al. [2], Feistauer et al. [6], Fortin et al. [7], Shu et al. [13], etc.. However, a priori error estimates are available only in the one-dimensional case and can be found in the papers by D. Hoff and his collaborators (see [8], [9]).

2. STABILITY RESULT

The Cauchy problem for the compressible, barotropic Navier–Stokes system in \( \mathbb{R}^d \times (0, T] \) may be written in the form

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div}(\rho v) &= 0, \\
\frac{\partial v_i}{\partial t} + \sum_{j=1}^d \frac{\partial}{\partial x_j} (\rho v_i v_j) + \frac{\partial p}{\partial x_i} &= \mu \Delta v_i + (\lambda + \mu) \frac{\partial}{\partial x_i} \text{div} v, \\
\rho(x, 0) &= \rho_0(x), \quad v(x, 0) = v_0(x), \quad x \in \mathbb{R}^d,
\end{align*}
\]

(6)

(7)

for \( i = 1, \ldots, d \). Here \( \rho \) denotes the density, \( v = (v^1, \ldots, v^d)^T \) the velocity of the fluid and \( \mu, \lambda \) are given parameters with \( \mu \geq 0, \mu + \lambda \geq 0 \). The pressure \( p = p(\rho) \) satisfies the conditions \( p \in C^2(0, \infty) \) and \( p' > 0 \). Let \( \Omega = \{ (\rho, v) \in \mathbb{R}^{d+1} : \rho > 0 \} \). Suppose that the solution \( (\rho, v) \) of (6), (7) satisfies

\[
(\rho, v) \text{ is a } C^1 \text{-function on } \mathbb{R}^d \times [0, T],
\]

(8)
$(\rho, v)$ is 1-periodic in the spatial variable, 
\begin{equation}
\partial^2 v \text{ is continuous on } \mathbb{R}^d \times [0, T],
\end{equation}
$(\rho, v)$ takes values in a convex, compact set $S \subset \Omega$.

The existence of such solutions in one space-dimension (locally in time) was shown in Kreiss, Lorenz [12].

With the help of the conserved variables $m = \rho v$, $u = (\rho, m)$, our Cauchy problem (6), (7) takes the more convenient form (1), (2) with
\begin{align*}
G(u) &= \frac{m \otimes m}{\rho} + p(\rho) I, \quad G = (G_1, \ldots, G_d), \\
B[u] &= (0, \mu \Delta v + (\lambda + \mu) \nabla (\text{div} v)), \\
u_0 &= (r_0, m_0).
\end{align*}
The symbol $\otimes$ is the notation for the tensor product of vectors and $I$ is the identity operator. The conservative hyperbolic part of the compressible Navier–Stokes system possesses the entropy pair
\begin{align*}
\eta(u) &= \frac{|m|^2}{2\rho} + \rho \varepsilon(\rho), \\
q_i(u) &= \frac{|m|^2}{2\rho} + \rho \varepsilon(\rho) + p(\rho) \frac{m_i}{\rho},
\end{align*}
for $i = 1, \ldots, d$ (see Dafermos [4]). Here it is $\varepsilon(\rho) = \int_{\mathbb{R}^d} \frac{p(s)}{s^2} \, ds$. The gradient for $\eta$ has the form
\begin{equation}
\nabla \eta(u) = \left( - \frac{|m|^2}{2\rho^2} + \varepsilon(\rho) + \rho \varepsilon'(\rho), \ v \right).
\end{equation}
An easy calculation shows that $\eta$ is a strictly convex function on $\Omega$ and uniformly convex on convex, compact subsets of $\Omega$.

Therefore, if $u^1 = (\rho^1, \rho^1 v^1)$, $u^2 = (\rho^2, \rho^2 v^2)$ satisfy (8)–(11), then
\begin{align*}
\int_Q \{B[u^1] - B[u^2]\} \cdot \{\nabla \eta(u^1) - \nabla \eta(u^2)\} \, dx \\
= \mu \int_Q (\Delta v^1 - \Delta v^2) \cdot (v^1 - v^2) \, dx \\
+ (\lambda + \mu) \int_Q [\nabla (\text{div} v^1) - \nabla (\text{div} v^2)] \cdot [v^1 - v^2] \, dx \\
= -\mu \int_Q \|\partial (v^1 - v^2)\|^2 \, dx - (\lambda + \mu) \int_Q (\text{div} (v^1 - v^2))^2 \, dx.
\end{align*}
Now, we introduce the relative entropy by

\[ h(a, b) = \eta(b) - \eta(a) - \nabla \eta(a) \cdot (b - a) \quad (a, b \in \Omega). \tag{14} \]

Due to uniform convexity of the entropy \( \eta \) on any convex, compact set \( S \subset \Omega \), there
are constants \( l = l(S) > 0, L = L(S) > 0 \), such that

\[ (\forall a, b \in S) \quad l |a - b|^2 \leq h(a, b) \leq L |a - b|^2. \tag{15} \]

Consequently, if \( u = (\rho, \rho v) \) is the solution of (6), (7) (or equivalently (1), (2)) satisfying (8)–(11), and \( S \) is the set from (11), then there exists a constant \( \alpha > 0 \) depending on \( S \) and the sup-norm of \( \partial u, \partial^2 v \), such that

\[ \left| B[u] \cdot [\nabla \eta(w) - \nabla \eta(u) - \nabla^2 \eta(u)(w - u)] \right| \leq \frac{\alpha}{2} h(u, w), \]

\[ \left| \sum_{i=1}^{d} \nabla^2 \eta(u) \partial_i u \cdot [G_i(w) - G_i(u) - DG_i(u)(w - u)] \right| \leq \frac{\alpha}{2} h(u, w), \tag{16} \]

holds on \( \mathbb{R}^d \times [0, T] \), for all \( w \in S \).

Our aim now is to estimate the \( L^2 \)-distance between the solution \( u \) of (1), (2) and
an arbitrary function \( w \) by generalizing the inequality (4). In order to do so, we need
two auxiliary functions, which measure at the extent to which (1) and (3) are satisfied
by \( w \).

**Definition 1.** Assume that Assumptions (8) - (11) for the solution \( u \) of (1), (2) hold and let \( w \in [L^\infty(\mathbb{R}^d \times (0, T))]^d \) be an arbitrary function with values in the set \( \Omega \). The weak consistency error \( \mu_w : [C^1(\mathbb{R}^d \times [0, T])]^d \to \mathbb{R} \) and the dissipation error \( \nu_w : C^1[0, T] \to \mathbb{R} \) are defined by

\[ \langle \mu_w, \pi \rangle = - \int_0^T \int_Q w \cdot \partial_t \pi + G(w) \cdot D\pi + B[u] \cdot \pi \ dxdt - \int_Q u_0(x) \cdot \pi(x, 0) \ dx \]

\[ \langle \nu_w, \omega \rangle = - \int_0^T \int_Q \eta(w) \omega' + \nabla \eta(w) \cdot B[u] \omega \ dxdt - \int_Q \eta(u_0) \omega(0) \ dx. \]

We are ready to formulate the generalization of (4).

**Theorem 1.** Assume that \( u = (\rho, \rho v) \) is the solution of (1), (2) (or equivalently (6), (7)) which satisfies (8)–(11) and \( S \) is the set from (11). Then

\[ l \int_0^T \int_Q e^{-\alpha t} |u - w|^2 \ dx dt \leq \langle \nu_w, \theta \rangle - \langle \mu_w, \psi \rangle. \tag{17} \]
Here it is $\theta(t) = e^{-\alpha t}(T-t)$ and $\psi = \theta\nabla \eta(u)$ for $\alpha$ given by (16). The constant $l > 0$ is defined in (15).

**Proof.** Let $\omega \in C^1[0, T]$ be such that $\omega(., T) = 0$ and let $\pi = \omega \nabla \eta(u)$. From (1), (3) and the definition of the measures $\mu_w, \nu_w$ it follows that

\[
- \int_0^T \int_Q h(u, w) \omega' \, dxdt = - \int_0^T \int_Q \eta(w) \omega' \, dxdt - \int_Q \eta(u_0) \omega(0) \, dx
- \int_0^T \int_Q \omega B[u] \cdot \nabla \eta(u) \, dxdt
+ \int_0^T \int_Q \{ \partial_t [\omega \nabla \eta(u)] - \omega \nabla^2 \eta(u) \partial_t u \} \cdot \{ w - u \} \, dxdt
= \langle \nu_w, \omega \rangle - \langle \mu_w, \pi \rangle + \int_0^T \int_Q B[u] \cdot \nabla \eta(w) \omega - \sum_{i=1}^d G_i(w) \cdot \partial_i \pi \, dxdt
- 2 \int_0^T \int_Q B[u] \cdot \pi \, dxdt + \int_0^T \int_Q \omega \sum_{i=1}^d \nabla^2 \eta(u) DG_i(u) \partial_i u \cdot (w - u) \, dxdt
- \int_0^T \int_Q \omega \nabla^2 \eta(u) B[u] \cdot (w - u) \, dxdt + \int_0^T \int_Q \partial_t u \cdot \pi \, dxdt.
\]

Using the symmetry of the operators $\nabla^2 \eta$, $\nabla^2 \eta DG_i$ (see [4]) and the equality

\[
\int_0^T \int_Q \partial_t u \cdot \pi \, dxdt = \int_0^T \int_Q \omega \sum_{i=1}^d \nabla^2 \eta(u) \partial_i u \cdot G_i(u) + B[u] \cdot \pi \, dxdt,
\]

we obtain

\[
- \int_0^T \int_Q h(u, w) \omega' \, dxdt = \langle \nu_w, \omega \rangle - \langle \mu_w, \pi \rangle
+ \int_0^T \int_Q \omega B[u] \cdot [\nabla \eta(w) - \nabla \eta(u) - \nabla^2 \eta(u)(w - u)] \, dxdt
- \int_0^T \int_Q \omega \sum_{i=1}^d \nabla^2 \eta(u) \partial_i u \cdot [G_i(w) - G_i(u) - DG_i(u)(w - u)] \, dxdt.
\]
Plugging $\omega = \theta$ in the last equality, one concludes, thanks to (15) and the relations

$$\int_0^T \int_Q \theta B[u] \cdot [\nabla \eta(w) - \nabla \eta(u) - \nabla^2 \eta(u)(w - u)] \, dx \, dt$$

$$\leq \frac{\alpha}{2} \int_0^T \int_Q \theta h(u, w) \, dx \, dt$$

$$- \int_0^T \int_Q \theta \sum_{i=1}^d \nabla^2 \eta(u) \partial_i u \cdot [G_i(w) - G_i(u) - DG_i(u)(w - u)] \, dx \, dt$$

$$\leq \frac{\alpha}{2} \int_0^T \int_Q \theta h(u, w) \, dx \, dt,$$

that (17) holds.

3. ERROR ESTIMATE

Let $T_h$ be the uniform triangulation of $\mathbb{R}^d$ consisting of open cubes of the size $h$ parallel to $(0, h)^d$ and let $\overline{T}_h = T_h \cap Q$. For a given $K \in \mathcal{T}_h$ the set $\mathcal{E}(K)$ contains all edges of $K$. If $e \in \mathcal{E}(K)$, then $K_e$ is the neighbouring cell to $K$ with the common edge $e$, while $K^\pm_e := K \pm he_i$ are special neighbouring cells, with $\{ e_i : i = 1, \ldots, d \}$ to be the canonical basis of $\mathbb{R}^d$. If $e \in \mathcal{E}(K)$, then by $n_{K,e}$ we denote the outward unit normal on $e$ and $e^\pm_{K,e} \in \mathcal{E}(K)$ are the edges with the property $n_{K,e_{K,e}} = \pm e_i$.

The mesh with respect to $t$ is uniform as well: $t^n = n \Delta t$ ($n \in \mathbb{N} \cup \{0\}$). Here $\Delta t > 0$ is such that there is an $N \in \mathbb{N}$ with $N \Delta t = T$. With this number, we define the set $\mathcal{N} = \{0, 1, \ldots, N - 1\}$.

For discretization of the Cauchy problem for the compressible Navier–Stokes system (1), (2) we use the following finite volume – finite difference scheme:

$$u_{K}^{n+1} = u_{K}^n - \frac{\Delta t}{|K|} \sum_{e \in \mathcal{E}(K)} |e| g_{K,e}^n (u_{K,e}^n, u_{K,e}^n) + \Delta t B_h[u_{K}^n],$$

$$u_{K}^0 = \frac{1}{|K|} \int_K u_0(x) \, dx. \quad (18)$$

Here, $u_{K}^n = (\rho_{K}^n, m_{K}^n)^T$ and $v_{K}^n := m_{K}^n / \rho_{K}^n$. From the iterations $u_{K}^n$ we define the piecewise constant approximation $u_h : \mathbb{R}^d \times [0, T] \to \mathbb{R}^{d+1}$ of $u$ by

$$u_h(x, t) = u_{K}^n \quad \text{for} \quad x \in K, \ t \in [t^n, t^{n+1}), \quad (19)$$
where $K \in T_h$ and $n \in \mathcal{N}$.

For the numerical flux $g_{K,e}^n$ in (18) we suppose the usual consistency and conservation properties.

- For all $n \in \mathcal{N}$, $K \in T_h$, $e \in \mathcal{E}(K)$ we have

$$g_{K,e}^n(v,v) = \sum_{i=1}^d n_{K,e}^i G_i(v) \quad (v \in \Omega),$$

(20)

where $n_{K,e} = (n_{K,e}^1, \ldots, n_{K,e}^d)^T$ is the unit outward normal to $e \in \mathcal{E}(K)$.

- For all $n \in \mathcal{N}$, $K \in T_h$, $e \in \mathcal{E}(K)$ we have

$$g_{K,e}^n(v,w) = -g_{K,e}^n(w,v) \quad (v, w \in \Omega).$$

(21)

We also suppose that the numerical flux is locally Lipschitz continuous.

The finite difference operator $B_h$ is defined by

$$B_h[u_h] = (0, \mu \Delta_h v_h + (\mu + \nu) \nabla_h (\text{div}_h v_h))^T,$$

where

$$\Delta_h v_h = \sum_{i=1}^d (v_h)_{x_i}, \quad \text{div}_h v_h = \sum_{i=1}^d (v^i_h)_{x_i}, \quad (\nabla_h f)_i = f_{x_i}.$$

Thereby, for $f : \mathbb{R}^d \rightarrow \mathbb{R}$ a piecewise constant function on $T_h$, we define the forward and backward finite differences in a classical way:

$$f_{x_i}(x) = \frac{1}{h} (f_{K_i^+} - f_K), \quad f_{\bar{x}_i}(x) = \frac{1}{h} (f_K - f_{K_i^-}), \quad (x \in K).$$

For the discrete solution $u_h$ we make the following assumptions:

$$u_h = (\rho_h, m_h) : S \subset (0, \infty) \times \mathbb{R}^d, \quad S \text{ is from (11)}$$

(22)

$$\sum_{n \in \mathcal{N}} \sum_{K \in T_h} |K| |u_{K,n+1}^n - u_{K,n}^n|^2 \leq C \Delta t$$

(23)

$$\sum_{n \in \mathcal{N}} \sum_{K \in T_h} \sum_{e \in \mathcal{E}(K)} |e| |u_{K,n}^n - u_{K,e}^n|^2 \leq C$$

(24)
Here, \( C \) denotes a generic constant that does not depend on \( h \). \(|K| = h^d\), \(|e| = h^{d-1}\) are measures of \( K \) and \( e \), respectively. From (22) and (24) it follows that

\[
\sum_{n \in N} \sum_{K \in T_h} \sum_{e \in \mathcal{E}(K)} |e| |v^n_K - v^n_{K_e}|^2 \leq C. \tag{25}
\]

If we replace \( w \) by \( u_h \) in (17), we obtain

\[
I \int_0^T \int_Q e^{-\alpha t}|u - u_h|^2 \, dx \, dt \leq \langle \nu_{u_h}, \theta \rangle - \langle \mu_{u_h}, \psi \rangle. \tag{26}
\]

Therefore, it only remains to estimate the right-hand side in order to obtain the \( L^2 \)-error estimate. For that purpose we need several technical lemmas.

**Lemma 1.**

\[
\langle \nu_{u_h}, \theta \rangle - \langle \mu_{u_h}, \psi \rangle \leq L + R + \int_0^T \int_Q \theta(t) B[u] \cdot [\nabla \eta(u) - \nabla \eta(u_h)] \, dx \, dt + C h^2,
\]

where

\[
L = \sum_{n \in N} \sum_{K \in T_h} \int_K \theta(n+1)[\eta(u_{K}^{n+1}) - \eta(u_{K}^{n}) - \nabla \eta(u(x, t^{n+1})) \cdot (u_{K}^{n+1} - u_{K}^{n})] \, dx,
\]

\[
R = \frac{1}{2} \sum_{n \in N} \sum_{K \in T_h} \sum_{e \in \mathcal{E}(K)} \left[ \sum_{i=1}^d n_{K,e}^i (G_i(u_{K}^{n}) - G_i(u_{K_e}^{n})) \right] \cdot \int_{t^n}^{t^{n+1}} \int_e \psi \, d\sigma \, dt.
\]

**Proof.** The same as the proof of Lemma 4.2 from [11]. \( \square \)

**Lemma 2.** \( R \leq C(h + \Delta t) + R_3 \), where

\[
R_3 = \frac{1}{2} \sum_{n \in N} \sum_{K \in T_h} \sum_{e \in \mathcal{E}(K)} \left[ \sum_{i=1}^d n_{K,e}^i (G_i(u_{K}^{n}) - G_i(u_{K_e}^{n})) \right] \cdot \int_{t^n}^{t^{n+1}} \theta \int_e \left[ \nabla \eta(u) - \frac{1}{2} (\nabla \eta(u_{K}^{n}) + \nabla \eta(u_{K_e}^{n})) \right] \, d\sigma \, dt.
\]

**Proof.** Since

\[
0 = \int_0^T \int_Q \sum_{i=1}^d q_i(u_h) \partial_i \theta \, dx \, dt
\]

\[
= \frac{1}{2} \sum_{n \in N} \sum_{K \in T_h} \sum_{e \in \mathcal{E}(K)} |e| \left[ \sum_{i=1}^d n_{K,e}^i (q_i(u_{K}^{n}) - q_i(u_{K_e}^{n})) \right] \cdot \int_{t^n}^{t^{n+1}} \int_e \theta \, d\sigma \, dt,
\]

we have that \( R = R_1 + R_2 + R_3 \), where
\[ R_1 = -\frac{1}{2} \sum_{n \in \mathcal{N}} \sum_{K \in T_h} \sum_{e \in \mathcal{E}(K)} |e| \sum_{i=1}^d n_{K,e}^i (g_i(u^n_K) - g_i(u^n_{K_e})) \cdot (G_i(u^n_K) - G_i(u^n_{K_e})) \int_{t^n}^{t^{n+1}} \theta \, dt, \]

\[ R_2 = \frac{1}{2} \sum_{n \in \mathcal{N}} \sum_{K \in T_h} \sum_{e \in \mathcal{E}(K)} |e| \sum_{i=1}^d n_{K,e}^i \left[ \frac{1}{2} (\nabla \eta(u^n_K) + \nabla \eta(u^n_{K_e})) - \nabla \eta(u^n_{K_e}) \right] \cdot [G_i(u^n_K) - G_i(u^n_{K_e})] \int_{t^n}^{t^{n+1}} \theta \, dt, \]

\[ R_3 = \frac{1}{2} \sum_{n \in \mathcal{N}} \sum_{K \in T_h} \sum_{e \in \mathcal{E}(K)} \left[ \sum_{i=1}^d n_{K,e}^i (g_i(u^n_K) - G_i(u^n_{K,e})) \right] \cdot [G_i(u^n_K) - G_i(u^n_{K_e})] \int_{t^n}^{t^{n+1}} \theta \, dt, \]

From \( q_i(b) - q_i(a) - \nabla \eta(a) \cdot (G_i(b) - G_i(a)) \leq C|a-b|^2 \) for \( a, b \in S \) and (24) we conclude that \( R_1 \leq C \Delta t \). Similarly, from (24) it follows that \( R_2 \leq C \Delta t \). 

**Lemma 3.** \( L \leq C(h + \Delta t) + \frac{1}{2} \int_0^T \int_Q e^{-\alpha t} |u - u_h|^2 \, dx \, dt \)

\[ + \sum_{n \in \mathcal{N}} \sum_{K \in T_h} \int_{t^n}^{t^{n+1}} \int_K \theta (\nabla \eta(u^n_K) - \nabla \eta(u)) \cdot B_h[u_h] \, dx \, dt + P_3, \]

where \( P_3 = \frac{1}{2} \sum_{n \in \mathcal{N}} \sum_{K \in T_h} \sum_{e \in \mathcal{E}(K)} \left[ g_{K,e}(u^n_K, u^n_{K_e}) - g_{K,e}(u^n_K, u^n_{K_e}) \right] \cdot \int_{t^n}^{t^{n+1}} \theta \int_e [\nabla \eta(u) - \frac{1}{2} (\nabla \eta(u^n_K) + \nabla \eta(u^n_{K_e}))] \, d\sigma \, dt. \)

**Proof.** Similar to the proof of Lemma 4.3 from [11].

**Lemma 4.** \( \int_0^T \int_Q \theta (\nabla \eta(u) - \nabla \eta(u_h)) \cdot (B[u] - B_h[u_h]) \leq \)

\[ C h - \mu \int_0^T \int_Q \theta ||\partial v - \partial_h v_h||^2 \, dx \, dt - (\mu + \nu) \int_0^T \int_Q \theta (\text{div} v - \text{div}_h v_h)^2 \, dx \, dt. \)

**Proof.** From (12) and the definitions of \( B \) and \( B_h \) it follows that

\[ \int_Q (\nabla \eta(u) - \nabla \eta(u_h)) \cdot (B[u] - B_h[u_h]) \, d \]

\[ = \mu \int_Q (v - v_h) \cdot (\Delta v - \Delta v_h) \, dx + (\mu + \nu) \int_Q (v - v_h) \cdot (\nabla (\text{div} v) - \nabla_h (\text{div}_h v_h)) \, dx. \]
Partial integration yields
\[
\int_{\Omega} (\mathbf{v} - \mathbf{v}_h) \cdot (\Delta \mathbf{v} - \Delta \mathbf{v}_h) \, dx = - \int_{\Omega} \| \partial \mathbf{v} - \partial_h \mathbf{v}_h \|^2 \, dx + \sum_{K \in T_h} \sum_{j=1}^d \varphi_{K,j}(\mathbf{v}) \cdot (\mathbf{v}_h)_{x_j}(x),
\]
where \( x \in K \) in the sum above, and
\[
\varphi_{K,j}(\mathbf{v}) = h \int_{e^K_j} \frac{\partial \mathbf{v}}{\partial x_j} \, d\sigma + \frac{1}{h} \left( \int_{K_j^+} \mathbf{v} \, dx - \int_K \mathbf{v} \, dx \right) - 2 \int_K \frac{\partial \mathbf{v}}{\partial x_j} \, dx.
\]
Similarly,
\[
\int_{\Omega} (\mathbf{v} - \mathbf{v}_h) \cdot (\nabla (\text{div} \mathbf{v}) - \nabla_h (\text{div} \mathbf{v}_h)) \, dx =
\]
\[
- \int_{\Omega} (\text{div} \mathbf{v} - \text{div}_h \mathbf{v}_h)^2 \, dx + \sum_{K \in T_h} \sum_{j=1}^d \bar{\varphi}_{K,j}(\mathbf{v}) \cdot (\mathbf{v}_h)_{x_j}(x),
\]
where \( x \in K \) and
\[
\bar{\varphi}_{K,j}(\mathbf{v}) = \sum_{i=1}^d \frac{1}{h} \left( \int_{K_i^+} \mathbf{v}^i \, dx - \int_K \mathbf{v}^i \, dx \right) (\mathbf{v}_h^i)_{x_j} + \int_{e^K_j} \text{div} \mathbf{v} \, d\sigma - 2 \int_K \text{div} \mathbf{v} \, dx.
\]
Let \( | \cdot |_{2,A} \) denote the Sobolev 2-seminorm on an open set \( A \subset \mathbb{R}^d \). Employing the Bramble-Hilbert lemma, one concludes that
\[
|\varphi_{K,j}(\mathbf{v})| \leq Ch^{\frac{\alpha+1}{2}}|v|_{2,K_{x_j}^+}, \quad |\bar{\varphi}_{K,j}(\mathbf{v})| \leq Ch^{\frac{\alpha+1}{2}}|v|_{2,K_{x_j}^+},
\]
which, together with (25), implies the assertion of the Lemma.

Thus we deduce the main result of the paper.

**Theorem 2.** Let \( u = (\rho, m, \dot{m} = \rho \dot{v} \) be the solution of (6), (7) (or equivalently of (1), (2)) satisfying (8)–(11). As the numerical scheme for the underlying system let us consider (18). If the numerical solution \( u_h \) given by (19) satisfies (22)–(24), then the following a-priori error estimate
\[
\frac{l}{2} \int_0^T \int_{\Omega} e^{-\alpha t} |u - u_h|^2 \, dx \, dt + \mu \int_0^T \int_{\Omega} \| \partial \mathbf{v} - \partial_h \mathbf{v}_h \|^2 \, dx \, dt
\]
\[
+ (\mu + \nu) \int_0^T \int_{\Omega} (\text{div} \mathbf{v} - \text{div}_h \mathbf{v}_h)^2 \, dx \, dt \leq C(h + \Delta t),
\]
holds, where \( C \) does not depend on the mesh and \( \alpha, l \) are given in (16), (15), respectively.
Proof. Applying Lemmas 1 – 3, we obtain

\[
\frac{l}{2} \int_0^T \int_Q e^{-\alpha t}|u - u_h|^2 \, dx \, dt \leq C(h + \Delta t) + P_3 + R_3
\]

\[
+ \int_0^T \int_Q \theta(\nabla \eta(u) - \nabla \eta(u_h)) \cdot (B[u] - B_h[u_h]) \, dx \, dt.
\]

Due to \( P_3 + R_3 = 0 \) (see the proof of Theorem 4.4 in [11]) and Lemma 4, we conclude that the assertion holds.

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References


