Kragujevac J. Math. 30 (2007) 263–275.

AN ERROR ESTIMATE FOR A NUMERICAL SCHEME FOR THE COMPRESSIBLE NAVIER–STOKES SYSTEM

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(Received October 31, 2006)

Abstract. The subject of this paper is an error estimate of the order $h^{1/2}$ in the L^2-norm for an explicit, fully discrete numerical scheme that approximates smooth solutions of the barotropic compressible fluid flow equations in the multidimensional case. Assuming some a-priori estimates for the discrete solution we derive an error estimate using a technique based upon stability results due to Dafermos [3] and DiPerna [5], which were originally formulated for systems of conservation laws.

1. INTRODUCTION

Let $\Omega \subset \mathbf{R}^m$ be an open set and $Q = (0,1)^d$ the unite cube in $\mathbf{R}^d$. Consider the following Cauchy problem

$$\partial_t u + \sum_{i=1}^{d} \partial_i G_i(u) = B[u] \quad \text{in } \mathbf{R}^d \times (0,T), \tag{1}$$

$$u(x,0) = u_0(x), \;\; x \in \mathbf{R}^d, \tag{2}$$

where $G_i : \Omega \to \mathbf{R}^m \, (i = 1,\ldots,d)$ are smooth flux functions and B is an operator. The derivative on $\mathbf{R}^d$ is denoted by ∂ and the partial derivatives by ∂_i. For functions

$F : \Omega \to \mathbf{R}^m$ and $f : \Omega \to \mathbf{R}$ let DF and ∇f be their derivative and gradient, respectively. Finally, the Euclidean norm on $\mathbf{R}^m$ is denoted by $|\cdot|$ and the induced matrix norm on $\mathbf{R}^{m\times m}$ by $\|\cdot\|$.

Assume that the system (1) has a strictly convex entropy η. If the solution $u : \mathbf{R}^d \times [0,T] \to \mathbf{R}^m$ of (1), (2) is smooth, then the additional equality

$$\partial_t \eta(u) + \sum_{i=1}^{d} \partial_i q_i(u) = \nabla\eta(u) \cdot B[u] \quad \text{in } \mathbf{R}^d \times (0,T], \tag{3}$$

holds, with q_i's being the corresponding entropy fluxes. The stability issue for (1), (2) in the hyperbolic setting (i.e. $B \equiv 0$) was explored in Dafermos [3] and DiPerna [5]. If one assumes, for the sake of simplicity, that the smooth solution u of (1), (2) is 1–periodic in the spacial variable, the theory developed there implies the following assertion: if $\bar{u}$ is a 1–periodic admissible weak solution of (1) (with the initial function $\bar{u}_0$), then

$$\int_Q |u(x,t) - \bar{u}(x,t)|^2 \, dx \le c e^{\alpha t} \int_Q |u_0 - \bar{u}_0|^2 dx, \quad t \in [0,T), \tag{4}$$

where c depends on the set where u and $\bar{u}$ take their values, and the constant α depends also on the sup-norm of ∂u.

In an attempt to derive a similar stability result for general B, it turns out that a certain compatibility condition between the conservative hyperbolic part of the system (1) and the operator B is needed. Let us state that condition: If $\mathrm{Dom}(B)$ is the domain of the operator B, then for all $u^1, u^2 \in \mathrm{Dom}(B)$,

$$\int_Q \{B[u^1] - B[u^2]\} \cdot \{\nabla\eta(u^1) - \nabla\eta(u^2)\} \, dx \le C \int_Q |u^1 - u^2|^2 dx, \tag{5}$$

with $t \in [0,T)$. The constant C should depend only on the set where u^1, u^2 take their values (for details see Jovanović [10]). The condition (5) is obviously satisfied if B is a source term. However, it is not quite obvious that (5) also holds for the compressible Navier–Stokes system, with $C = 0$ (see equality (13) below). Hence the possibility of obtaining an inequality of the form (4) for the Navier–Stokes system. We will even go a step further in our generalizations. If we replace the smooth solution $\bar{u}$ in (4) by an arbitrary L^2-function w, we will be in a position to estimate the resulting L^2-distance between u and w by some residual measures (see Definition 1 and Theorem 1 below). Finally, by taking w to be a numerical approximation, we can derive an L^2-error estimate, if we estimate the right-hand-side in the obtained inequality (see (26) below). This is the idea of our approach.

For what concerns the numerical scheme, we use an explicit finite volume – finite difference scheme on a uniform mesh: the hyperbolic part of the Navier–Stokes system is discretized by a finite volume, and the viscous part by a finite difference scheme. A priori estimates for the numerical solution are assumed. Our main result, the error estimate, is stated in Theorem 2.

The approach of deriving error estimates based upon the Dafermos-DiPerna stability result for hyperbolic problems was pursued in Arvanitis et al. [1], Vila [14], Jovanović, Rohde [11], etc.. Numerical methods for compressible fluid flow are developed in Bristeau et al. [2], Feistauer et al. [6], Fortin et al. [7], Shu et al. [13], etc.. However, a priori error estimates are available only in the one-dimensional case and can be found in the papers by D. Hoff and his collaborators (see [8], [9]).

2. STABILITY RESULT

The Cauchy problem for the compressible, barotropic Navier–Stokes system in $\mathbf{R}^d \times (0,T]$ may be written in the form

$$\begin{aligned} \partial_t\rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho\, v_i) + \textstyle\sum_{j=1}^d \partial_j(\rho\, v_i v_j) + \partial_i p(\rho) = \mu\Delta v_i + (\lambda+\mu)\partial_i(\operatorname{div} v), \end{aligned} \tag{6}$$

$$\rho(x,0) = \rho_0(x),\ v(x,0) = v_0(x),\ \ x \in \mathbf{R}^d, \tag{7}$$

for $i = 1,\dots,d$. Here ρ denotes the density, $v = (v^1,\dots,v^d)^T$ the velocity of the fluid and μ, λ are given parameters with $\mu \geq 0,\ \mu+\lambda \geq 0$. The pressure $p = p(\rho)$ satisfies the conditions $p \in C^2(0,\infty)$ and $p' > 0$. Let $\Omega = \{(\rho,\ v) \in \mathbf{R}^{d+1} : \rho > 0\}$. Suppose that the solution (ρ, v) of (6), (7) satisfies

$$(\rho, v) \ \text{ is a C}^1\text{-function on } \mathbf{R}^d \times [0,T], \tag{8}$$

$$(\rho, v) \text{ is 1–periodic in the spatial variable,} \tag{9}$$

$$\partial^2 v \text{ is continuous on } \mathbf{R}^d \times [0, T], \tag{10}$$

$$(\rho, v) \text{ takes values in a convex, compact set } \ S \subset \Omega. \tag{11}$$

The existence of such solutions in one space-dimension (locally in time) was shown in Kreiss, Lorenz [12].

With the help of the conserved variables

$$m = \rho v, \ u = (\rho, m),$$

our Cauchy problem (6), (7) takes the more convenient form (1), (2) with

$$G(u) = \frac{m \otimes m}{\rho} + p(\rho) I, \ \ G = (G_1, \ldots, G_d),$$
$$B[u] = (0, \mu\Delta v + (\lambda + \mu)\nabla(\mathrm{div} v)),$$
$$u_0 = (\rho_0, m_0).$$

The symbol $\otimes$ is the notation for the tensor product of vectors and I is the identity operator. The conservative hyperbolic part of the compressible Navier–Stokes system possesses the entropy pair

$$\eta(u) = \frac{|m|^2}{2\rho} + \rho\varepsilon(\rho), \ q_i(u) = \frac{|m|^2}{2\rho} + \rho\varepsilon(\rho) + p(\rho)\frac{m_i}{\rho},$$

for $i = 1, \ldots, d$ (see Dafermos [4]). Here it is $\varepsilon(\rho) = \int^\rho \frac{p(s)}{s^2}\, ds$. The gradient for η has the form

$$\nabla\eta(u) = \Big(- \frac{|m|^2}{2\rho^2} + \varepsilon(\rho) + \rho\varepsilon'(\rho), \ \ v\Big). \tag{12}$$

An easy calculation shows that η is a strictly convex function on Ω and uniformly convex on convex, compact subsets of Ω.

Therefore, if $u^1 = (\rho^1, \, \rho^1 v^1)$, $u^2 = (\rho^2, \, \rho^2 v^2)$ satisfy (8)–(11), then

$$\begin{aligned}
&\int_Q \{B[u^1] - B[u^2]\} \cdot \{\nabla\eta(u^1) - \nabla\eta(u^2)\}\, dx \\
&\quad = \ \mu \int_Q (\Delta v^1 - \Delta v^2) \cdot (v^1 - v^2)\, dx \\
&\qquad + \ (\lambda + \mu) \int_Q [\nabla(\mathrm{div}\, v^1) - \nabla(\mathrm{div}\, v^2)] \cdot [v^1 - v^2]\, dx \\
&\quad = \ -\mu \int_Q \|\partial(v^1 - v^2)\|^2 dx - (\lambda + \mu) \int_Q (\mathrm{div}(v^1 - v^2))^2\, dx.
\end{aligned} \tag{13}$$

Now, we introduce the relative entropy by

$$h(a,b) = \eta(b) - \eta(a) - \nabla\eta(a)\cdot(b-a) \quad (a,b\in\Omega). \tag{14}$$

Due to uniform convexity of the entropy η on any convex, compact set $S\subset\Omega$, there are constants $l = l(S) > 0$, $L = L(S) > 0$, such that

$$(\forall a,b\in S) \quad l\,|a-b|^2 \le h(a,b) \le L\,|a-b|^2. \tag{15}$$

Consequently, if $u = (\rho, \rho\, v)$ is the solution of (6), (7) (or equivalently (1), (2)) satisfying (8)–(11), and S is the set from (11), then there exists a constant $\alpha > 0$ depending on S and the sup-norm of $\partial u, \partial^2 v$, such that

$$\begin{aligned} |B[u]\cdot[\nabla\eta(w) - \nabla\eta(u) - \nabla^2\eta(u)(w-u)]| &\le \tfrac{\alpha}{2}h(u,w),\\ \left|\sum_{i=1}^d \nabla^2\eta(u)\partial_i u\cdot[G_i(w) - G_i(u) - DG_i(u)(w-u)]\right| &\le \tfrac{\alpha}{2}h(u,w), \end{aligned} \tag{16}$$

holds on $\mathbf{R}^d\times[0,T]$, for all $w\in S$.

Our aim now is to estimate the L^2-distance between the solution u of (1), (2) and an arbitrary function w by generalizing the inequality (4). In order to do so, we need two auxillary functions, which measure at the extent to wich (1) and (3) are satisfied by w.

Definition 1. *Assume that Assumptions (8) - (11) for the solution u of (1), (2) hold and let $w\in[L^\infty(\mathbf{R}^d\times(0,T))]^d$ be an arbitrary function with values in the set Ω. The weak consistency error $\mu_w : [C^1(\mathbf{R}^d\times[0,T])]^d\to\mathbf{R}$ and the dissipation error $\nu_w : C^1[0,T]\to\mathbf{R}$ are defined by*

$$\begin{aligned} \langle\mu_w,\pi\rangle &= -\int_0^T\int_Q w\cdot\partial_t\pi + G(w)\cdot D\pi + B[u]\cdot\pi\;dxdt - \int_Q u_0(x)\cdot\pi(x,0)\,dx\\ \langle\nu_w,\omega\rangle &= -\int_0^T\int_Q \eta(w)\,\omega' + \nabla\eta(w)\cdot B[u]\,\omega\;dxdt - \int_Q \eta(u_0)\,\omega(0)\,dx. \end{aligned}$$

We are ready to formulate the generalization of (4).

Theorem 1. *Assume that $u = (\rho, \rho\, v)$ is the solution of (1), (2) (or equivalently (6), (7)) which satisfies (8)–(11) and S is the set from (11). Then*

$$l\int_0^T\int_Q e^{-\alpha t}|u-w|^2\,dxdt \le \langle\nu_w,\theta\rangle - \langle\mu_w,\psi\rangle. \tag{17}$$

Here it is $\theta(t) = e^{-\alpha t}(T-t)$ and $\psi = \theta\nabla\eta(u)$ for α given by (16). The constant $l > 0$ is defined in (15).

Proof. Let $\omega \in C^1[0,T]$ be such that $\omega(.,T) = 0$ and let $\pi = \omega\nabla\eta(u)$. From (1), (3) and the definition of the measures $\mu_w,\ \nu_w$ it follows that

$$
\begin{aligned}
-\int_0^T\int_Q h(u,w)\,\omega'\,dxdt &= -\int_0^T\int_Q \eta(w)\,\omega'\,dxdt - \int_Q \eta(u_0)\,\omega(0)\,dx \\
&- \int_0^T\int_Q \omega\, B[u]\cdot\nabla\eta(u)\,dxdt \\
&+ \int_0^T\int_Q \{\partial_t[\omega\,\nabla\eta(u)] - \omega\,\nabla^2\eta(u)\partial_t u\}\cdot\{w-u\}\,dxdt \\
&= \langle\nu_w,\omega\rangle - \langle\mu_w,\pi\rangle + \int_0^T\int_Q B[u]\cdot\nabla\eta(w)\,\omega - \sum_{i=1}^d G_i(w)\cdot\partial_i\pi\ dxdt \\
&-2\int_0^T\int_Q B[u]\cdot\pi\ dxdt + \int_0^T\int_Q \omega\sum_{i=1}^d \nabla^2\eta(u)DG_i(u)\partial_i u\cdot(w-u)\,dxdt \\
&- \int_0^T\int_Q \omega\nabla^2\eta(u)B[u]\cdot(w-u)\,dxdt + \int_0^T\int_Q \partial_t u\cdot\pi\,dxdt.
\end{aligned}
$$

Using the symmetry of the operators $\nabla^2\eta,\ \nabla^2\eta DG_i$ (see [4]) and the equality

$$
\int_0^T\int_Q \partial_t u\cdot\pi\,dxdt = \int_0^T\int_Q \omega\sum_{i=1}^d \nabla^2\eta(u)\partial_i u\cdot G_i(u) + B[u]\cdot\pi\ dxdt,
$$

we obtain

$$
\begin{aligned}
-\int_0^T\int_Q h(u,w)\,\omega'\,dxdt &= \langle\nu_w,\omega\rangle - \langle\mu_w,\pi\rangle \\
&+ \int_0^T\int_Q \omega\, B[u]\cdot[\nabla\eta(w) - \nabla\eta(u) - \nabla^2\eta(u)(w-u)]\,dxdt \\
&- \int_0^T\int_Q \omega\sum_{i=1}^d \nabla^2\eta(u)\partial_i u\cdot[G_i(w) - G_i(u) - DG_i(u)(w-u)]\,dxdt.
\end{aligned}
$$

Plugging $\omega = \theta$ in the last equality, one concludes, thanks to (15) and the relations

$$\begin{aligned}
&\int_0^T \int_Q \theta\, B[u] \cdot [\nabla\eta(w) - \nabla\eta(u) - \nabla^2\eta(u)(w-u)]\, dxdt \\
&\qquad \le \frac{\alpha}{2} \int_0^T \int_Q \theta\, h(u,w)\, dxdt \\
&\qquad - \int_0^T \int_Q \theta \sum_{i=1}^{d} \nabla^2\eta(u)\partial_i u \cdot [G_i(w) - G_i(u) - DG_i(u)(w-u)]\, dxdt \\
&\qquad \le \frac{\alpha}{2} \int_0^T \int_Q \theta\, h(u,w)\, dxdt,
\end{aligned}$$

that (17) holds. $\square$

3. ERROR ESTIMATE

Let $\mathcal{T}_h$ be the uniform triangulation of $\mathbf{R}^d$ consisting of open cubes of the size h parallel to $(0,\, h)^d$ and let $\overline{\mathcal{T}}_h = \mathcal{T}_h \cap Q$. For a given $K \in \mathcal{T}_h$ the set $\mathcal{E}(K)$ contains all edges of K. If $e \in \mathcal{E}(K)$, then K_e is the neighbouring cell to K with the common edge e, while $K_i^{\pm} := K \pm he_i$ are special neighbouring cells, with $\{e_i \ : \ i = 1, \ldots, d\}$ to be the canonical basis of $\mathbf{R}^d$. If $e \in \mathcal{E}(K)$, then by $n_{K,e}$ we denote the outward unit normal on e and $e_K^{i\pm} \in \mathcal{E}(K)$ are the edges with the property $n_{K,e_K^{i\pm}} = \pm e_i$.

The mesh with respect to t is uniform as well: $t^n = n\Delta t$ ($n \in \mathbf{N} \cup \{0\}$). Here $\Delta t > 0$ is such that there is an $N \in \mathbf{N}$ with $N\Delta t = T$. With this number, we define the set $\mathcal{N} = \{0, 1, \ldots, N-1\}$.

For discretization of the Cauchy problem for the compressible Navier–Stokes system (1), (2) we use the following finite volume – finite difference scheme:

$$\begin{aligned}
u_K^{n+1} &= u_K^n - \frac{\Delta t}{|K|} \sum_{e \in \mathcal{E}(K)} |e|\, g_{K,e}^n(u_K^n, u_{K_e}^n) + \Delta t\, B_h[u_K^n], \\
u_K^0 &= \frac{1}{|K|} \int_K u_0(x)\, dx.
\end{aligned} \tag{18}$$

Here, $u_K^n = (\rho_K^n,\, m_K^n)^T$ and $v_K^n := m_K^n/\rho_K^n$. From the iterations u_K^n we define the piecewise constant approximation $u_h : \mathbf{R}^d \times [0,T] \to \mathbf{R}^{d+1}$ of u by

$$u_h(x,t) = u_K^n \quad \text{for } \ x \in K,\ \ t \in [t^n, t^{n+1}), \tag{19}$$

where $K \in \mathcal{T}_h$ and $n \in \mathcal{N}$.

For the numerical flux $g^n_{K,e}$ in (18) we suppose the usual consistency and conservation properties.

- For all $n \in \mathcal{N},\ K \in \mathcal{T}_h,\ e \in \mathcal{E}(K)$ we have

$$g^n_{K,e}(v,v) = \sum_{i=1}^{d} n^i_{K,e} G_i(v) \quad (v \in \Omega), \tag{20}$$

where $n_{K,e} = (n^1_{K,e}, \dots, n^d_{K,e})^T$ is the unit outward normal to $e \in \mathcal{E}(K)$.

- For all $n \in \mathcal{N},\ K \in \mathcal{T}_h,\ e \in \mathcal{E}(K)$ we have

$$g^n_{K,e}(v,w) = -g^n_{K_e,e}(w,v) \quad (v,w \in \Omega). \tag{21}$$

We also suppose that the numerical flux is locally Lipschitz continuous.

The finite difference operator B_h is defined by

$$B_h[u_h] = (0,\ \mu\Delta_h v_h + (\mu+\nu)\nabla_{\bar{h}}(\mathrm{div}_h v_h)\,)^T,$$

where

$$\Delta_h v_h = \sum_{i=1}^{d} (v_h)_{x_i\bar{x}_i}, \quad \mathrm{div}_h v_h = \sum_{i=1}^{d} (v^i_h)_{x_i} \quad (\nabla_{\bar{h}} f)_i = f_{\bar{x}_i}.$$

Thereby, for $f : \mathbf{R}^d \to \mathbf{R}$ a piecewise constant function on $\mathcal{T}_h$, we define the forward and backward finite differences in a classical way:

$$f_{x_i}(x) = \frac{1}{h}(f_{K_i^+} - f_K), \quad f_{\bar{x}_i}(x) = \frac{1}{h}(f_K - f_{K_i^-}), \quad (x \in K).$$

For the discrete solution u_h we make the following assumptions:

$$u_h = (\rho_h,\, m_h) \subset S \subset (0,\,\infty) \times \mathbf{R}^d, \quad S \text{ is from (11)} \tag{22}$$

$$\sum_{n\in\mathcal{N}} \sum_{K\in\overline{\mathcal{T}}_h} |K|\,|u_K^{n+1} - u_K^n|^2 \le C\Delta t \tag{23}$$

$$\sum_{n\in\mathcal{N}} \sum_{K\in\overline{\mathcal{T}}_h} \sum_{e\in\mathcal{E}(K)} |e|\,|u_K^n - u_{K_e}^n|^2 \le C \tag{24}$$

Here, C denotes a generic constant that does not depend on h. $|K| = h^d$, $|e| = h^{d-1}$ are measures of K and e, respectively. From (22) and (24) it follows that

$$\sum_{n\in\mathcal{N}} \sum_{K\in\overline{\mathcal{T}}_h} \sum_{e\in\mathcal{E}(K)} |e|\,|v_K^n - v_{K_e}^n|^2 \le C. \tag{25}$$

If we replace w by u_h in (17), we obtain

$$l\int_0^T \int_Q e^{-\alpha t}|u-u_h|^2\,dxdt \le \langle \nu_{u_h}, \theta\rangle - \langle \mu_{u_h}, \psi\rangle. \tag{26}$$

Therefore, it only remains to estimate the right-hand side in order to obtain the L^2-error estimate. For that purpose we need several technical lemmas.

Lemma 1.

$$\langle \nu_{u_h}, \theta\rangle - \langle \mu_{u_h}, \psi\rangle \le L + R + \int_0^T \int_Q \theta(t)B[u]\cdot[\nabla\eta(u) - \nabla\eta(u_h)]\,dxdt + Ch^2,$$

where

$$\begin{aligned} L &= \sum_{n\in\mathcal{N}} \sum_{K\in\overline{\mathcal{T}}_h} \int_K \theta(t^{n+1})[\eta(u_K^{n+1}) - \eta(u_K^n) - \nabla\eta(u(x,t^{n+1}))\cdot(u_K^{n+1}-u_K^n)]\,dx, \\ R &= \frac{1}{2}\sum_{n\in\mathcal{N}} \sum_{K\in\overline{\mathcal{T}}_h} \sum_{e\in\mathcal{E}(K)} \Big[\sum_{i=1}^d n_{K,e}^i(G_i(u_K^n) - G_i(u_{K_e}^n))\Big]\cdot\int_{t^n}^{t^{n+1}}\int_e \psi\,d\sigma dt. \end{aligned}$$

Proof. The same as the proof of Lemma 4.2 from [11]. □

Lemma 2. $R \le C(h+\Delta t) + R_3,$ *where*

$$\begin{aligned} R_3 &= \frac{1}{2}\sum_{n\in\mathcal{N}} \sum_{K\in\overline{\mathcal{T}}_h} \sum_{e\in\mathcal{E}(K)} \Big[\sum_{i=1}^d n_{K,e}^i(G_i(u_K^n) - G_i(u_{K_e}^n))\Big] \\ &\quad \cdot\int_{t^n}^{t^{n+1}} \theta \int_e \Big[\nabla\eta(u) - \frac{1}{2}(\nabla\eta(u_K^n) + \nabla\eta(u_{K_e}^n))\Big]\,d\sigma dt. \end{aligned}$$

Proof. Since

$$\begin{aligned} 0 &= \int_0^T \int_Q \sum_{i=1}^d q_i(u_h)\partial_i\theta\,dxdt \\ &= \frac{1}{2}\sum_{n\in\mathcal{N}} \sum_{K\in\overline{\mathcal{T}}_h} \sum_{e\in\mathcal{E}(K)} |e|\Big[\sum_{i=1}^d n_{K,e}^i(q_i(u_K^n) - q_i(u_{K_e}^n))\Big]\cdot\int_{t^n}^{t^{n+1}}\int_e \theta\,d\sigma dt, \end{aligned}$$

we have that $R = R_1 + R_2 + R_3$, where

$$\begin{aligned} R_1 &= -\frac{1}{2}\sum_{n\in\mathcal{N}}\sum_{K\in\overline{\mathcal{T}}_h}\sum_{e\in\mathcal{E}(K)}|e|\sum_{i=1}^{d} n^i_{K,e}[q_i(u^n_K)-q_i(u^n_{K_e})-\nabla\eta(u^n_{K_e})\cdot \\ &\quad \cdot(G_i(u^n_K)-G_i(u^n_{K_e}))\Big]\textstyle\int_{t^n}^{t^{n+1}}\theta\,dt, \end{aligned}$$

$$\begin{aligned} R_2 &= \frac{1}{2}\sum_{n\in\mathcal{N}}\sum_{K\in\overline{\mathcal{T}}_h}\sum_{e\in\mathcal{E}(K)}|e|\sum_{i=1}^{d} n^i_{K,e}[\frac{1}{2}(\nabla\eta(u^n_K)+\nabla\eta(u^n_{K_e}))-\nabla\eta(u^n_{K_e})]\cdot \\ &\quad \cdot[G_i(u^n_K)-G_i(u^n_{K_e})]\textstyle\int_{t^n}^{t^{n+1}}\theta\,dt, \end{aligned}$$

$$\begin{aligned} R_3 &= \frac{1}{2}\sum_{n\in\mathcal{N}}\sum_{K\in\overline{\mathcal{T}}_h}\sum_{e\in\mathcal{E}(K)}\Big[\sum_{i=1}^{d} n^i_{K,e}(G_i(u^n_K)-G_i(u^n_{K_e}))\Big]\cdot \\ &\quad \cdot\textstyle\int_{t^n}^{t^{n+1}}\theta\int_e[\nabla\eta(u)-\frac{1}{2}(\nabla\eta(u^n_K)+\nabla\eta(u^n_{K_e}))]\,d\sigma dt. \end{aligned}$$

From $q_i(b)-q_i(a)-\nabla\eta(a)\cdot(G_i(b)-G_i(a))\le C|a-b|^2$ for $a,b\in S$ and (24) we conclude that $R_1\le C\Delta t$. Similarly, from (24) it follows that $R_2\le C\Delta t$. □

Lemma 3.
$$\begin{aligned} L &\le C(h+\Delta t)+\frac{1}{2}l\int_0^T\int_Q e^{-\alpha t}|u-u_h|^2dxdt \\ &\quad +\sum_{n\in\mathcal{N}}\sum_{K\in\overline{\mathcal{T}}_h}\int_{t^n}^{t^{n+1}}\int_K\theta(\nabla\eta(u^n_K)-\nabla\eta(u))\cdot B_h[u_h]\,dxdt+P_3, \end{aligned}$$

where

$$\begin{aligned} P_3 &= \frac{1}{2}\sum_{n\in\mathcal{N}}\sum_{K\in\overline{\mathcal{T}}_h}\sum_{e\in\mathcal{E}(K)}\Big[g_{K,e}(u^n_K,u^n_{K_e})-g_{K,e}(u^n_K,u^n_K)\Big]\cdot \\ &\quad \cdot\textstyle\int_{t^n}^{t^{n+1}}\theta\int_e[\nabla\eta(u)-\frac{1}{2}(\nabla\eta(u^n_K)+\nabla\eta(u^n_{K_e}))]\,d\sigma dt. \end{aligned}$$

Proof. Similar to the proof of Lemma 4.3 from [11]. □

Lemma 4. $\displaystyle\int_0^T\int_Q\theta(\nabla\eta(u)-\nabla\eta(u_h))\cdot(B[u]-B_h[u_h])\le$

$$Ch-\mu\int_0^T\int_Q\theta\|\partial v-\partial_h v_h\|^2dxdt\ -(\mu+\nu)\int_0^T\int_Q\theta(\mathrm{div}v-\mathrm{div}_h v_h)^2dxdt.$$

Proof. From (12) and the definitions of B and B_h it follows that

$$\int_Q(\nabla\eta(u)-\nabla\eta(u_h))\cdot(B[u]-B_h[u_h])\,d$$

$$=\mu\int_Q(v-v_h)\cdot(\Delta v-\Delta v_h)\,dx+(\mu+\nu)\int_Q(v-v_h)\cdot(\nabla(\mathrm{div}v)-\nabla_{\bar h}(\mathrm{div}_h v_h))\,dx.$$

Partial integration yields

$$\int_Q (v - v_h) \cdot (\Delta v - \Delta v_h)\, dx = -\int_Q \|\partial v - \partial_h v_h\|^2 dx + \sum_{K \in \overline{\mathcal{T}}_h} \sum_{j=1}^{d} \varphi_{K,j}(v) \cdot (v_h)_{x_j}(x),$$

where $x \in K$ in the sum above, and

$$\varphi_{K,j}(v) = h \int_{e_K^{j+}} \frac{\partial v}{\partial x_j}\, d\sigma + \frac{1}{h}\Big(\int_{K_j^+} v\, dx - \int_K v\, dx \Big) - 2 \int_K \frac{\partial v}{\partial x_j}\, dx.$$

Similarly,

$$\int_Q (v - v_h) \cdot (\nabla(\mathrm{div} v) - \nabla_{\bar{h}}(\mathrm{div}_h v_h))\, dx =$$

$$-\int_Q (\mathrm{div} v - \mathrm{div}_h v_h)^2 dx + \sum_{K \in \overline{\mathcal{T}}_h} \sum_{j=1}^{d} \bar{\varphi}_{K,j}(v) \cdot (v_h^j)_{x_j}(x),$$

where $x \in K$ and

$$\bar{\varphi}_{K,j}(v) = \sum_{i=1}^{d} \frac{1}{h} \Big(\int_{K_i^+} v^i\, dx - \int_K v^i\, dx \Big) (v_h^j)_{x_j} + \int_{e_K^{j+}} \mathrm{div}\, v\, d\sigma - 2 \int_K \mathrm{div}\, v\, dx.$$

Let $|\cdot|_{2,A}$ denote the Sobolev 2-seminorm on an open set $A \subset \mathbf{R}^d$. Employing the Bramble-Hilbert lemma, one concludes that

$$|\varphi_{K,j}(v)| \le C h^{\frac{d}{2}+1} |v|_{2,K\cup K_j^+}, \quad |\bar{\varphi}_{K,j}(v)| \le C h^{\frac{d}{2}+1} |v|_{2,K\cup K_j^+},$$

which, together with (25), implies the assertion of the Lemma. □

Thus we deduce the main result of the paper.

Theorem 2. *Let $u = (\rho, m)$, $m = \rho v$ be the solution of (6), (7) (or equivalently of (1), (2)) satisfying (8)–(11). As the numerical scheme for the underlying system let us consider (18). If the numerical solution u_h given by (19) satisfies (22)–(24), then the following a-priori error estimate*

$$\frac{l}{2} \int_0^T \int_Q e^{-\alpha t} |u - u_h|^2 dx dt + \mu \int_0^T \int_Q \|\partial v - \partial_h v_h\|^2 dx dt$$

$$+(\mu + \nu) \int_0^T \int_Q (\mathrm{div} v - \mathrm{div}_h v_h)^2 dx dt \le C(h + \Delta t),$$

holds, where C does not depend on the mesh and α, l are given in (16), (15), respectively.

Proof. Applying Lemmas 1 – 3, we obtain

$$\frac{l}{2}\int_0^T\int_Q e^{-\alpha t}|u-u_h|^2 dxdt \le C(h+\Delta t)+P_3+R_3$$

$$+\int_0^T\int_Q \theta(\nabla\eta(u)-\nabla\eta(u_h))\cdot(B[u]-B_h[u_h])\,dxdt.$$

Due to $P_3+R_3=0$ (see the proof of Theorem 4.4 in [11]) and Lemma 4, we conclude that the assertion holds. □

Acknowledgements: The author was supported by the Republika Srpska's Ministry of Science and Technology (Project # 2005: Konveksnost, varijacione nejednakosti, parcijalne diferencijalne jednačine).

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