

Kragujevac J. Math. 30 (2007) 277–292.

MARCHUK IDENTITY-TYPE SECOND ORDER DIFFERENCE SCHEMES OF 2-D AND 3-D ELLIPTIC PROBLEMS WITH INTERSECTED INTERFACES

Ivanka Tr. Angelova and Lubin G. Vulkov

*Center of Applied Mathematics and Informatics, & University of Rouse,
Rousse 7017, Bulgaria*
(e-mails: iangelova@ru.acad.bg, vulkov@ami.ru.acad.bg)

(Received January 16, 2007)

Dedicated to Professor Bosko S. Jovanovic on the occasion of his sixtieth birthday

Abstract. This article presents second-order difference schemes of 2-D and 3-D elliptic problems with intersecting interfaces. The discretization is made using new Marchuk identities. It possesses the typical for the method advantages as conservatism, second-order accuracy even at low smoothness of the differential problem solution. The convergence and accuracy are discussed theoretically and experimentally. Numerical tests show the feasibility of the schemes.

1. INTRODUCTION

We consider the equation

$$\begin{aligned} Lu &:= -\frac{\partial}{\partial x} \left(p(x, y) \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left(q(x, y) \frac{\partial u}{\partial y} \right) + r(x, y)u \\ &= f(x, y) + \delta(x - \xi)K_x(y) + \delta(y - \eta)K_y(x), \quad (x, y) \in \Omega \equiv (0, 1) \times (0, 1), \end{aligned} \quad (1)$$

where $(\xi, \eta) \in \Omega$, $\delta(\cdot)$ is the Dirac-delta function. We assume that the functions p , q , K_x , K_y are piecewise continuous and

$$p(x, y) \geq c_0 > 0, \quad q(x, y) \geq c_0 > 0 \text{ on } \overline{\Omega}. \quad (2)$$

We shall solve (1) subjected with the Dirichlet boundary conditions

$$u = g_j(s) \text{ on } S_j, \quad j = w, s, e, n, \quad (3)$$

where $\partial\Omega = S = S_w \cup S_s \cup S_e \cup S_n$:

$$S_w = \{(x, y) \in S : x = 0\}, \quad S_s = \{(x, y) \in S : y = 0\},$$

$$S_e = \{(x, y) \in S : x = 1\}, \quad S_n = \{(x, y) \in S : y = 1\},$$

$g_j(s)$ are functions defined on S_j .

The equation (1) is equivalent to the following ones:

$$Lu := f(x, y), \quad (x, y) \in \Omega \setminus \Gamma, \quad (4)$$

$$[u]_{\Gamma_x} \equiv u(\xi+, y) - u(\xi-, y) = 0, \quad [u]_{\Gamma_y} \equiv u(x, \eta+) - u(x, \eta-) = 0, \quad (5)$$

$$\left[p(x, y) \frac{\partial u}{\partial x} \right]_{\Gamma_x} = K_x(y), \quad \left[q(x, y) \frac{\partial u}{\partial y} \right]_{\Gamma_y} = K_y(x), \quad (6)$$

where $\Gamma_x = \{(x, y); x = \xi, 0 < y < 1\}$, $\Gamma_y = \{(x, y); 0 < x < 1, y = \eta\}$ and $\Gamma = \Gamma_x \cup \Gamma_y$.

Numerical solutions of second-order elliptic equations are often encountered in modeling of processes in material sciences and fluid dynamics. This explains the great interest in the recent years to numerical methods for such problems [1]-[8], [10], [13], [15], [19], [19], [23].

Han [12] proposed an infinite element method for elliptic interface problems with interface consisting of straight lines. In [8], [13], [23] for problems with continuous fluxes (of type (1),(2) when $K_x \equiv 0$, $K_y \equiv 0$), finite volume schemes based on coupled discretization of fluxes, are derived and investigated, theoretically and numerically. Ill'in [13] derived integro-balance approximations with high order of accuracy (up

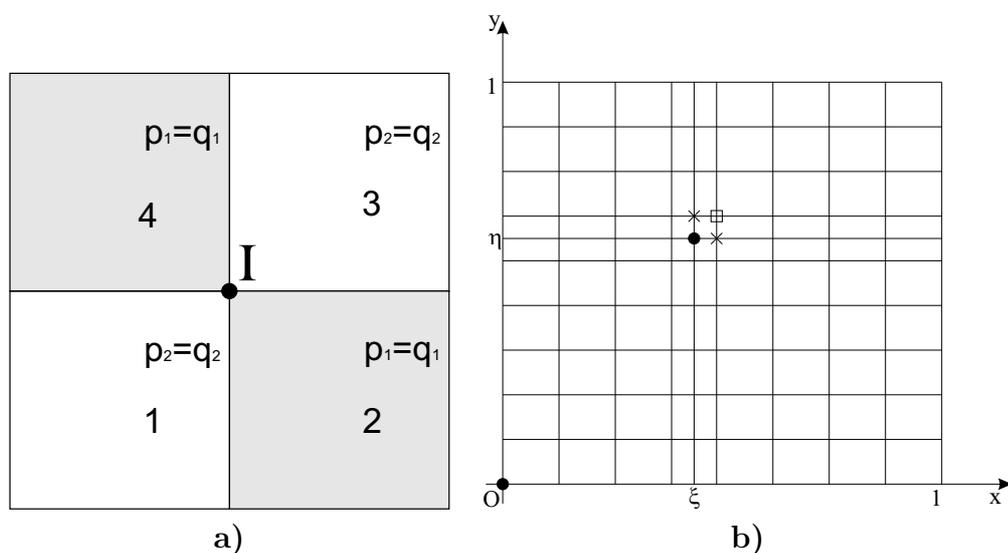


Figure 1: a) The Domain Ω . b) Non-uniform mesh.

to three) for elliptic problems with a line interface. He showed that for the third-order scheme the monotonicity of the difference equations system failed in the case of strongly discontinuous (or called "wild") coefficients $p(x, y)$, $q(x, y)$. At the end of his paper [13], Ill'in remarked that the construction and convergence analysis of approximations to elliptic problems with self-intersected interfaces and Neumann boundary conditions are interesting and difficult open questions.

The famous Marchuk integral identity is a balance equation over a finite number of control volumes [8]. Therefore, it is a modification of the finite volume method (FVM) which starts in the works of Samarskii [22]. In the recent years the FVM which is capable of producing accurate approximations of general triangular and quadrilateral grids [7]. FEM formulation of the approximation obtained by Marchuk identity was derived by Agoshkov [21].

In this paper we present a modification of the Marchuk identity method for construction of second-order difference approximation of problem (4)-(6). The paper is organized as follows. In Section 2, we present a discretization based on new Marchuk identities [21]. The convergence is discussed at assumptions for regularity of the differential problem solution. Section 4 contains a generalization of the 2-D results to the 3-D case. Numerical experiments in the last section support our claims.

2. CONSTRUCTION OF THE DIFFERENCE SCHEME

Let define the meshes:

$$\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_k, \quad \omega = \bar{\omega} \cap \Omega, \quad \sigma = \bar{\omega} \setminus \omega, \quad \text{where}$$

$$\bar{\omega}_h = \{x_0 = 0, x_i = x_{i-1} + h_i, i = 1, \dots, N_1 - 1, x_{N_1} = x_{N_1-1} + h = \xi, x_{N_1+1} = \xi + h, x_i = x_{i-1} + h_i, i = N_1 + 2, \dots, N, x_N = 1\},$$

$$\bar{\omega}_k = \{y_0 = 0, y_j = y_{j-1} + k_j, j = 1, \dots, M_1 - 1, y_{M_1} = y_{M_1-1} + h = \eta, y_{M_1+1} = \eta + h, y_j = y_{j-1} + k_j, j = M_1 + 2, \dots, M, y_M = 1\}.$$

Let V be a discrete function defined on $\bar{\omega}_{hk}$. By $\|V\|_{\bar{\omega}_{hk}} = \max_{\bar{\omega}_{hk}} |V_{ij}|$, we denote the discrete maximum norm on $\bar{\omega}_{hk}$. The finite-difference operators are defined in standard manner by $U(x, y)$:

$$U_{\bar{x}} = U_{\bar{x},i} = (U(x_i, y_j) - U(x_{i-1}, y_j))/h_i, \quad U_x = U_{x,i} = U_{\bar{x},i+1},$$

$$U_{\bar{y}} = U_{\bar{y},j} = (U(x_i, y_j) - U(x_i, y_{j-1}))/k_j, \quad U_y = U_{y,j} = U_{\bar{y},j+1},$$

$$U_{\hat{x}} = U_{\hat{x},i} = (U(x_{i+1}, y_j) - U(x_i, y_j))/\bar{h}_i, \quad \bar{h}_i = \frac{1}{2}(h_i + h_{i+1}), \quad \bar{h}_0 = \frac{h_1}{2}, \quad \bar{h}_N = \frac{h_N}{2},$$

$$U_{\hat{y}} = U_{\hat{y},j} = (U(x_i, y_{j+1}) - U(x_i, y_j))/\bar{k}_j, \quad \bar{k}_j = \frac{1}{2}(k_j + k_{j+1}), \quad \bar{k}_0 = \frac{k_1}{2}, \quad \bar{k}_M = \frac{k_M}{2},$$

$$U_{\bar{x}\bar{x}} = U_{\bar{x}\bar{x},i} = \frac{1}{\bar{h}_i}(U_{x,i} - U_{\bar{x},i}), \quad U_{\bar{y}\bar{y}} = U_{\bar{y}\bar{y},j} = \frac{1}{\bar{k}_j}(U_{y,j} - U_{\bar{y},j}).$$

Here V_{ij} is any discrete function. Note that when it is clear that $u(x, y)$ is a continuous function, we shall sometimes use the notation $u_{ij} := u(x_i, y_j)$, while when it is clear that V_{ij} is a discrete function, we shall sometimes use the notation $V(x_i, y_j) := V_{ij}$.

Let $g(x, y)$ is a piecewise continuous function define in Ω .

$$g_{\bar{x}} = g_{\bar{x},i}(y) = \frac{h_i g(x_i-, y) + h_{i+1} g(x_i+, y)}{h_i + h_{i+1}},$$

$$g_{\bar{y}} = g_{\bar{y},j}(x) = \frac{k_j g(x, y_j-) + k_{j+1} g(x, y_j+)}{k_j + k_{j+1}},$$

$$g_{\bar{x}\bar{y}} = g_{\bar{x}\bar{y},ij} = (g_{\bar{x},i})_{\bar{y},j} = (g_{\bar{y},j})_{\bar{x},i}.$$

If $h_i = h_{i+1}$, then $g_{\bar{x}} = \{g\}_{x_i}$. If $k_j = k_{j+1}$, then $g_{\bar{y}} = \{g\}_{y_j}$.

First we explain our idea for construction of the difference scheme to 1-d elliptic problem. Setting in (1) $p(x, y) = p(x)$, $q(x, y) = 0$, $r(x, y) = r(x)$, $f(x, y) = f(x)$, we have:

$$w(x) = p(x) \frac{du}{dx}, \quad -\frac{dw}{dx} + r(x)u = f(x), \quad x \in (0, \xi) \cup (\xi, 1), \quad (7)$$

$$[u]_{\xi} = 0, \quad [w]_{\xi} = K. \quad (8)$$

Integrating the left equality in (7) first on (x_{i-1}, x_i) and next on (x_i, x_{i+1}) and subtracting the results, we get

$$\begin{aligned} & -\frac{1}{\bar{h}_i} \left(\frac{u_{i+1} - u_i}{\xi_i(x_{i+})} - \frac{u_i - u_{i-1}}{\xi_i(x_{i-})} \right) \\ &= -\frac{1}{\bar{h}_i} [w]_{x_i} - \frac{1}{\bar{h}_i} \left(\int_{x_{i-1}}^{x_i} \frac{dw}{dx} \frac{\xi_i(x)}{\xi_i(x_{i-})} dx + \int_{x_i}^{x_{i+1}} \frac{dw}{dx} \frac{\xi_i(x)}{\xi_i(x_{i+})} dx \right), \end{aligned} \quad (9)$$

where

$$\xi_i(x) = \begin{cases} \int_{x_{i-1}}^x \frac{dt}{p(t)}, & x_{i-1} \leq x < x_i, \\ \int_x^{x_{i+1}} \frac{dt}{p(t)}, & x_i < x \leq x_{i+1}, \\ 0, & x \notin [x_{i-1}, x_{i+1}]. \end{cases}$$

An application of the trapezoid numerical integration with accuracy $O(\bar{h}_i^2)$ leads to the equality:

$$\frac{h_i}{h_i + h_{i+1}} X_1 + \frac{h_{i+1}}{h_i + h_{i+1}} X_2 = \frac{1}{\bar{h}_i} \left(\frac{u_{i+1} - u_i}{\xi_i(x_{i+})} - \frac{u_i - u_{i-1}}{\xi_i(x_{i-})} \right) - \frac{1}{\bar{h}_i} [w]_{x_i} + O(\bar{h}_i^2),$$

where

$$X_1 = \frac{dw}{dx}(x_{i-}) = r(x_{i-})u_i - f(x_{i-}), \quad X_2 = \frac{dw}{dx}(x_{i+}) = r(x_{i+})u_i - f(x_{i+}).$$

Considering the last three relations as a system of three algebraic equations for two unknowns X_1 , X_2 , we find:

$$\begin{aligned} & -\frac{1}{\bar{h}_i} \left(\frac{u_{i+1} - u_i}{\xi_i(x_{i+})} - \frac{u_i - u_{i-1}}{\xi_i(x_{i-})} \right) + \frac{h_i r(x_{i-}) + h_{i+1} r(x_{i+})}{h_i + h_{i+1}} u_i \\ &= \frac{h_i f(x_{i-}) + h_{i+1} f(x_{i+})}{h_i + h_{i+1}} - \frac{1}{\bar{h}_i} [w]_{x_i} + O(\bar{h}_i^2), \end{aligned}$$

where $[w]_{x_i} = 0$ for $x_i \neq \xi$ and $[w]_{\xi} = K$. From here we get the difference scheme

$$-\left(\frac{1}{\xi(x)} U_{\bar{x}} \right)_{\hat{x}} + r_{\hat{x}} u = f_{\hat{x}} - \frac{1}{\bar{h}_i} [w]_{x_i}. \quad (10)$$

Now we turn to the two-dimensional case. For $1 < i < N - 1$, $1 < j < M - 1$ we introduce the functions

$$\xi_i(x, y) = \begin{cases} \int_{x_{i-1}}^x \frac{dt}{p(t, y)}, & x_{i-1} \leq x < x_i, \\ \int_x^{x_{i+1}} \frac{dt}{p(t, y)}, & x_i < x \leq x_{i+1}, \\ 0, & x \notin [x_{i-1}, x_{i+1}], \end{cases} \quad \eta_j(x, y) = \begin{cases} \int_{y_{j-1}}^y \frac{ds}{q(x, s)}, & y_{j-1} \leq y < y_j, \\ \int_y^{y_{j+1}} \frac{ds}{q(x, s)}, & y_j < y \leq y_{j+1}, \\ 0, & y \notin [y_{j-1}, y_{j+1}] \end{cases}$$

and the flows $w_1(x, y) = p(x, y) \frac{\partial u}{\partial x}$, $w_2(x, y) = q(x, y) \frac{\partial u}{\partial y}$.

Also, we will use the functions:

$$\begin{aligned} \varphi_i(y) &= \frac{1}{\bar{h}_i} \left(\frac{u_{i+1}(y) - u_i(y)}{\xi_i(x_{i+}, y)} - \frac{u_i(y) - u_{i-1}(y)}{\xi_i(x_{i-}, y)} \right) = \\ & \frac{1}{\bar{h}_i} [w_1]_{x_i} + \frac{1}{\bar{h}_i} \left(\int_{x_{i-1}}^{x_i} \frac{dw_1}{dx} \frac{\xi_i(x, y)}{\xi_i(x_{i-}, y)} dx + \int_{x_i}^{x_{i+1}} \frac{dw_1}{dx} \frac{\xi_i(x, y)}{\xi_i(x_{i+}, y)} dx \right), \quad i = 1, \dots, N - 1, \\ \psi_j(x) &= \frac{1}{\bar{k}_j} \left(\frac{u_{j+1}(x) - u_j(x)}{\eta_j(x, y_{j+})} - \frac{u_j(x) - u_{j-1}(x)}{\eta_j(x, y_{j-})} \right) = \\ & \frac{1}{\bar{k}_j} [w_2]_{y_j} + \frac{1}{\bar{k}_j} \left(\int_{y_{j-1}}^{y_j} \frac{dw_2}{dy} \frac{\eta_j(x, y)}{\eta_j(x, y_{j-})} dy + \int_{y_j}^{y_{j+1}} \frac{dw_2}{dy} \frac{\eta_j(x, y)}{\eta_j(x, y_{j+})} dy \right), \quad j = 1, \dots, M - 1. \end{aligned}$$

Following the procedure described for the 1-d case, we find

$$\varphi_i(y_j \pm) = \frac{1}{\bar{h}_i} [w_1]_{x_i}(y_j \pm) + \frac{1}{\bar{h}_i} \left(\frac{h_i}{2} \frac{dw_1}{dx}(x_{i-}, y_j \pm) + \frac{h_{i+1}}{2} \frac{dw_1}{dx}(x_{i+}, y_j \pm) \right) + O(\bar{h}_i^2), \tag{11}$$

$$\psi_j(x_i \pm) = \frac{1}{\bar{k}_j} [w_2]_{y_j}(x_i \pm) + \frac{1}{\bar{k}_j} \left(\frac{k_j}{2} \frac{dw_2}{dy}(x_i \pm, y_{j-}) + \frac{k_{j+1}}{2} \frac{dw_2}{dy}(x_i \pm, y_{j+}) \right) + O(\bar{k}_j^2). \tag{12}$$

Setting

$$\begin{aligned} \frac{dw_1}{dx}(x_{i-}, y_{j-}) &= X_1, \quad \frac{dw_2}{dy}(x_{i-}, y_{j-}) = Y_1, \quad \frac{dw_1}{dx}(x_{i+}, y_{j-}) = X_2, \quad \frac{dw_2}{dy}(x_{i+}, y_{j-}) = Y_2, \\ \frac{dw_1}{dx}(x_{i+}, y_{j+}) &= X_3, \quad \frac{dw_2}{dy}(x_{i+}, y_{j+}) = Y_3, \quad \frac{dw_1}{dx}(x_{i-}, y_{j+}) = X_4, \quad \frac{dw_2}{dy}(x_{i-}, y_{j+}) = Y_4. \end{aligned}$$

we obtain from (1), (11-12) the linear system of algebraic equations for the unknowns $X_i, Y_i, i = 1, 2, 3, 4$:

$$X_1 + Y_1 = (ru - f)(x_{i-}, y_{j-}), \quad X_2 + Y_2 = (ru - f)(x_{i+}, y_{j-}),$$

$$\begin{aligned}
X_2 + Y_3 &= (ru - f)(x_{i+}, y_{j+}), \quad X_4 + Y_4 = (ru - f)(x_{i-}, y_{j+}), \\
h_i X_1 + h_{i+1} X_2 &= -2[w_1]_{x_i}(y_{j-}) + (h_i + h_{i+1})\varphi_i(y_{j-}) + O(\bar{h}_i^3), \\
h_i X_4 + h_{i+1} X_3 &= -2[w_1]_{x_i}(y_{j+}) + (h_i + h_{i+1})\varphi_i(y_{j+}) + O(\bar{h}_i^3), \\
k_j Y_1 + k_{j+1} Y_4 &= -2[w_2]_{y_j}(x_{i-}) + (k_j + k_{j+1})\psi_j(x_{i-}) + O(\bar{k}_j^3), \\
k_j Y_2 + k_{j+1} Y_3 &= -2[w_2]_{y_j}(x_{i+}) + (k_j + k_{j+1})\psi_j(x_{i+}) + O(\bar{k}_j^3).
\end{aligned}$$

Eliminating $X_i, Y_i, i = 1, 2, 3, 4$ we find the approximation in the mesh point (x_i, y_j) of the equation (1) :

$$\begin{aligned}
& -\frac{1}{2\bar{h}_i\bar{k}_j} \left[\left(\frac{k_j}{\xi_i(x_{i+}, y_{j-})} + \frac{k_{j+1}}{\xi_i(x_{i+}, y_{j+})} \right) (u_{i+1j} - u_{ij}) \right. \\
& \quad - \left(\frac{k_j}{\xi_i(x_{i-}, y_{j-})} + \frac{k_{j+1}}{\xi_i(x_{i-}, y_{j+})} \right) (u_{ij} - u_{i-1j}) \\
& \quad + \left(\frac{h_i}{\eta_j(x_{i-}, y_{j+})} + \frac{h_{i+1}}{\eta_j(x_{i+}, y_{j+})} \right) (u_{ij+1} - u_{ij}) \\
& \quad \left. - \left(\frac{h_i}{\eta_j(x_{i-}, y_{j-})} + \frac{h_{i+1}}{\eta_j(x_{i+}, y_{j-})} \right) (u_{ij} - u_{ij-1}) \right] \\
& + \frac{h_i k_j r(x_{i-}, y_{j-}) + h_{i+1} k_j r(x_{i+}, y_{j-}) + h_{i+1} k_{j+1} r(x_{i+}, y_{j+}) + h_i k_{j+1} r(x_{i-}, y_{j+})}{(h_i + h_{i+1})(k_j + k_{j+1})} u_{ij} \\
& = \frac{h_i k_j f(x_{i-}, y_{j-}) + h_{i+1} k_j f(x_{i+}, y_{j-}) + h_{i+1} k_{j+1} f(x_{i+}, y_{j+}) + h_i k_{j+1} f(x_{i-}, y_{j+})}{(h_i + h_{i+1})(k_j + k_{j+1})} \\
& - \frac{k_j [w_1]_{x_i}(y_{j-}) + k_{j+1} [w_1]_{x_i}(y_{j+})}{\bar{h}_i(k_j + k_{j+1})} - \frac{h_i [w_2]_{y_j}(x_{i-}) + h_{i+1} [w_2]_{y_j}(x_{i+})}{\bar{k}_j(h_i + h_{i+1})} + O(\bar{h}_i^2 + \bar{k}_j^2).
\end{aligned}$$

From here we get the difference scheme

$$-\left(\frac{1}{\xi(x, y)} U_{\bar{x}} \right)_{\hat{x}\hat{y}} - \left(\frac{1}{\eta(x, y)} U_{\bar{y}} \right)_{\hat{y}\hat{x}} + r_{\hat{x}\hat{y}} u = f_{\hat{x}\hat{y}} - \frac{1}{\bar{h}_i} ([w_1]_{x_i})_{\hat{y}} - \frac{1}{\bar{k}_j} ([w_2]_{y_j})_{\hat{x}}, \quad (13)$$

where $[w_1]_{x_i} = 0$ for $x_i \neq \xi$ and $[w_1]_{\xi} = K_x(y)$, $[w_2]_{y_j} = 0$ for $y_j \neq \eta$ and $[w_2]_{\eta} = K_y(x)$.

3. CONVERGENCE

Let introduce the scalar products and the corresponding norms:

$$\begin{aligned}
 (U, V)_{\bar{\omega}} &= \sum_{i=0}^N \sum_{j=0}^M \bar{h}_i \bar{k}_j (UV)_{\bar{x}_i \bar{y}_j}, \quad \|U\|_0^2 = (U, U)_{\bar{\omega}}, \\
 (U, V)_{\omega} &= \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} \bar{h}_i \bar{k}_j (UV)_{\bar{x}_i \bar{y}_j}, \\
 (U, V)_{\omega_1^+ \times \omega_2} &= \sum_{i=1}^N \sum_{j=0}^M h_i \bar{k}_j (UV)_{\bar{y}_j, i}, \quad \|U_{\bar{x}}\|_0^2 = (U_{\bar{x}}, U_{\bar{x}})_{\omega_1^+ \times \omega_2}, \\
 (U, V)_{\bar{\omega}_1 \times \omega_2^+} &= \sum_{i=0}^N \sum_{j=1}^M \bar{h}_i k_j (UV)_{\bar{x}_i, j}, \quad \|U_{\bar{y}}\|_0^2 = (U_{\bar{y}}, U_{\bar{y}})_{\bar{\omega}_1 \times \omega_2^+}, \\
 (U, V)_{\omega_1^+ \times \omega_2^+} &= \sum_{i=1}^N \sum_{j=1}^M h_i k_j (UV)_{i, j}, \\
 \|\nabla U\|_0^2 &= \|U_{\bar{x}}\|_0^2 + \|U_{\bar{y}}\|_0^2, \quad \|U\|_1^2 = \|U\|_0^2 + \|\nabla U\|_0^2,
 \end{aligned}$$

where $\bar{\omega}_1 = \bar{\omega}_h$, $\bar{\omega}_2 = \bar{\omega}_k$, $\bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2$, $\sigma = \partial\Omega \cap \bar{\omega}$.

It is well known that the rate of convergence of the difference schemes essentially depends on the smoothness of the differential problem solution [9]. A corresponding treatment for problem (1)-(3) is beyond the scope of this paper and we will explain this on the following example of R.B. Kellog [16]. Let $\Omega_i = \{x \in \Omega : \frac{(i-1)\pi}{2} < \varphi < \frac{i\pi}{2}, \quad 1 \leq i \leq 4\}$, see Fig.1, where (r, φ) are polar coordinates. We consider (4)-(6), (1)-(3) in case of piecewise constant coefficients, Fig.1, and zero singular sources $K_x \equiv K_y \equiv 0$ with zero Dirichlet conditions $g_j \equiv 0, j = w, s, e, n$. Then every weak solution $u \in W_2^1(\Omega)$ of (4)-(6), (1)-(3) admits the following asymptotic expansion near the intersection interface point I [17].

$$\eta^I u = \eta^I u_{reg} + \eta^I \sum_{\alpha \in (0,1)} C_\alpha r^\alpha v_\alpha(\varphi)$$

Here, η^I is a cut-off function with respect I (see Fig. 1.a) and $\eta^I u_{reg}|_{\Omega_i} \in W^{2,2}(\Omega_i)$. The sum extends over the eigenvalues α of related eigenvalue problem and functions

v_α are the corresponding eigenfunctions. The regularity of u is now given by the smallest eigenvalue α_{min} in the interval $(0, 1)$ [17] : for every $\varepsilon > 0$

$$\eta^I u|_{\Omega_i} \in W^{1+\alpha_{min}-\varepsilon, 2}(\Omega_i).$$

For the special domain Ω introduced above and choice $p_1 = p_3 = 1$ and $p_2 = p_4 = \rho > 0$, it is shown [17] that $\alpha \in (0, 1)$ is exponent for the expansion [17] if and only if

$$\cos(\alpha\pi) = 1 - \frac{8\rho}{(1+\rho)^2}.$$

It follows from this relation that $\alpha_{min} \rightarrow 0$ as $\rho \rightarrow 0$ or $\rho \rightarrow \infty$ and therefore the regularity of a weak solution can be arbitrary low, i.e $u|_{\Omega_i}$ is from $W_2^{1+\varepsilon}$, $\varepsilon > 0$ small. This example shows that one can not guarantee a regularity of weak solution in a space $W_2^{1+m}(\Omega_i)$ for fixed $m > 0$ without any further assumptions on the coefficients p , q , r , and the right hand side K_x , K_y , f and the boundary condition function g . Results in this direction for the case of one line interface, i.e. $K_y \equiv 0$ are obtained in [11].

Theorem 1. *Suppose that for the solution of the problem (4)-(6) $u \in C(\bar{\Omega}) \cap (\bigcap_{s=1}^4 C^{4+\alpha}(\bar{\Omega}_s))$, $\frac{\partial u}{\partial x} \in C^{3+\alpha}(\bar{\Omega}_1 \cup \bar{\Omega}_4) \cap C^{3+\alpha}(\bar{\Omega}_2 \cup \bar{\Omega}_3)$, $\frac{\partial u}{\partial y} \in C^{3+\alpha}(\bar{\Omega}_1 \cup \bar{\Omega}_2) \cap C^{3+\alpha}(\bar{\Omega}_3 \cup \bar{\Omega}_4)$. Then the truncation error of the scheme (13) is of order one. The problem*

$$\Lambda U = -(pU_{\bar{x}})_{\hat{x}\hat{y}} - (qU_{\bar{y}})_{\hat{y}\hat{x}} + r_{\hat{x}\hat{y}}U = \varphi, U|_\sigma = 0 \quad (14)$$

has unique solution that satisfies the estimate

$$\|U\|_1 \leq C\|\varphi\|_{-1},$$

where

$$\|\Psi\|_{-1} = \sup_{v|_\gamma=0} \frac{|(\Psi, v)_w|}{\|v\|_1}.$$

Theorem 2. *Suppose that the assumptions for smoothness in Theorem 1 are fulfilled. Then for the error $z_{ij} = U_{ij} - u(x_i, y_j)$ of the difference scheme (13) the estimate holds*

$$\|z\|_1 \leq C(\|h^2\|_0 + \|k^2\|_0). \quad (15)$$

4. 3D INTERFACE PROBLEM

The results, obtained in the previous sections for 2-D problems can be generalized to the 3-D case. Now we turn to the three-dimensional case. We consider the equation

$$\begin{aligned} Lu &:= -\frac{\partial}{\partial x} \left(p(x, y, z) \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left(q(x, y, z) \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial z} \left(r(x, y, z) \frac{\partial u}{\partial z} \right) + s(x, y, z) \\ &= f(x, y, z) + \delta(x - \xi) K_{x,z}(y) + \delta(y - \eta) K_y(x, z) \\ &+ \delta(z - \zeta) K_z(x, y), \quad (x, y, z) \in \Omega \equiv (0, 1)^3, \end{aligned} \quad (16)$$

where $(\xi, \eta, \zeta) \in \Omega$, $\delta(\cdot)$ is the Dirac-delta function. We assume that the functions p , q , K_x , K_y , K_z are piecewise continuous and

$$p(x, y, z), q(x, y, z), r(x, y, z) \geq c_0 > 0, s(x, y, z) \geq c_0 > 0 \text{ on } \bar{\Omega}. \quad (17)$$

We shall solve (16) subjected with the Dirichlet boundary conditions

$$u|_{\partial\Omega} = g(x, y, z). \quad (18)$$

The equation (16) is equivalent to the following ones:

$$Lu := f(x, y, z), \quad (x, y, z) \in \Omega \setminus \Gamma, \quad (19)$$

$$[u]_{\Gamma_x} \equiv u(\xi+, y, z) - u(\xi-, y, z) = 0,$$

$$[u]_{\Gamma_y} \equiv u(x, \eta+, z) - u(x, \eta-, z) = 0,$$

$$[u]_{\Gamma_z} \equiv u(x, y, \zeta+) - u(x, y, \zeta-, z) = 0,$$

$$\left[p(x, y, z) \frac{\partial u}{\partial x} \right]_{\Gamma_x} = K_x(y, z),$$

$$\left[q(x, y, z) \frac{\partial u}{\partial y} \right]_{\Gamma_y} = K_y(x, z),$$

$$\left[r(x, y, z) \frac{\partial u}{\partial z} \right]_{\Gamma_z} = K_z(x, y),$$

where $\Gamma_x = \{(x, y, z); x = \xi, 0 < y < 1, 0 < z < 1\}$, $\Gamma_y = \{(x, y, z); 0 < x < 1, y = \eta, 0 < z < 1\}$, $\Gamma_z = \{(x, y, z); 0 < x < 1, 0 < y < 1, z = \zeta\}$, $\Gamma = \Gamma_x \cup \Gamma_y \cup \Gamma_z$.

We discretize on the mesh

$$\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_k \times \bar{\omega}_t, \quad \omega = \bar{\omega} \cap \Omega, \quad \sigma = \bar{\omega} \setminus \omega,$$

where $\bar{\omega}_h, \bar{\omega}_k$ were defined in Section 2 and

$$\bar{\omega}_t = \{z_0 = 0, z_l = z_{l-1} + t_l, l = 1, \dots, P_1 - 1, z_{P_1} = z_{P_1-1} + h = \zeta, z_{P_1+1} = \zeta + h, z_l = z_{l-1} + t_l, l = P_1 + 2, \dots, P, z_N = 1\};$$

$$U_{\bar{z}} = U_{\bar{z},l} = (U(x_i, y_j, z_l) - U(x_i, y_j, z_{l-1}))/t_l, \quad U_z = U_{z,l} = U_{\bar{z},l+1},$$

$$U_{\bar{z}\bar{z}} = U_{\bar{z}\bar{z},l} = \frac{1}{\bar{t}_l}(U_{z,l} - U_{\bar{z},l}), \quad \bar{t}_l = \frac{1}{2}(t_l + t_{l+1}), \quad \bar{t}_0 = \frac{t_0}{2}, \quad \bar{t}_P = \frac{t_P}{2},$$

$$g_{\bar{z}} = g_{\bar{z}_l}(x, y) = \frac{t_l g(x, y, z_{l-1}) + t_{l+1} g(x, y, z_{l+1})}{t_l + t_{l+1}}.$$

For $1 < i < N - 1$, $1 < j < M - 1$, $1 < l < P - 1$ we introduce the functions

$$\xi_i(x, y, z) = \begin{cases} \int_{x_{i-1}}^x \frac{dt}{p(t, y, z)}, & x_{i-1} \leq x < x_i, \\ \int_x^{x_{i+1}} \frac{dt}{p(t, y, z)}, & x_i < x \leq x_{i+1}, \\ 0, & x \notin [x_{i-1}, x_{i+1}], \end{cases}$$

$$\eta_j(x, y, z) = \begin{cases} \int_{y_{j-1}}^y \frac{ds}{q(x, s, z)}, & y_{j-1} \leq y < y_j, \\ \int_y^{y_{j+1}} \frac{ds}{q(x, s, z)}, & y_j < y \leq y_{j+1}, \\ 0, & y \notin [y_{j-1}, y_{j+1}], \end{cases}$$

$$\zeta_l(x, y, z) = \begin{cases} \int_{z_{l-1}}^z \frac{dt}{r(x, y, t)}, & z_{l-1} \leq z < z_l, \\ \int_z^{z_{l+1}} \frac{dt}{r(x, y, t)}, & z_l < z \leq z_{l+1}, \\ 0, & z \notin [z_{l-1}, z_{l+1}], \end{cases}$$

and the flows $w_1(x, y, z) = p(x, y, z) \frac{\partial u}{\partial x}$, $w_2(x, y, z) = q(x, y, z) \frac{\partial u}{\partial y}$, $w_3(x, y, z) = r(x, y, z) \frac{\partial u}{\partial z}$.

$$\begin{aligned}
 & - \left(\frac{1}{\xi(x, y, z)} U_{\bar{x}} \right)_{\hat{x}\hat{y}\hat{z}} - \left(\frac{1}{\eta(x, y, z)} U_{\bar{y}} \right)_{\hat{y}\hat{x}\hat{z}} - \left(\frac{1}{\zeta(x, y, z)} U_{\bar{z}} \right)_{\hat{z}\hat{x}\hat{y}} + r_{\hat{x}\hat{y}\hat{z}} u \\
 & = f_{\hat{x}\hat{y}\hat{z}} - \frac{1}{\bar{h}_i} ([w_1]_{x_i})_{\hat{y}\hat{z}} - \frac{1}{\bar{k}_j} ([w_2]_{y_j})_{\hat{x}\hat{z}} - \frac{1}{\bar{t}_l} ([w_3]_{z_l})_{\hat{x}\hat{y}}.
 \end{aligned}$$

where $[w_1]_{x_i} = 0$ for $x_i \neq \xi$ and $[w_1]_{\xi} = K_x(y, z)$, $[w_2]_{y_j} = 0$ for $y_j \neq \eta$ and $[w_2]_{\eta} = K_y(x, z)$, $[w_3]_{z_l} = 0$ for $z_l \neq \zeta$ and $[w_3]_{\zeta} = K_z(x, y)$.

5. NUMERICAL TESTS

In this section, we test the difference schemes derived in sections 2, 4 for 2-D and 3-D problems, respectively. In the 2-D case on each subdomain Ω_s , $s = 1, 2, 3, 4$ we seek an exact solution in the form

$$\begin{aligned}
 u(x, y) = & u_1 \left(\frac{b-x}{b-a} \right)^v \left(\frac{d-y}{d-c} \right)^w + u_2 \left(\frac{x-a}{b-a} \right)^v \left(\frac{d-y}{d-c} \right)^w \\
 & + u_3 \left(\frac{x-a}{b-a} \right)^v \left(\frac{y-c}{d-c} \right)^w + u_4 \left(\frac{b-x}{b-a} \right)^v \left(\frac{y-c}{d-c} \right)^w,
 \end{aligned}$$

where, for example on Ω_1 (the same procedure for $\Omega_2, \Omega_3, \Omega_4$) we take $a = 0$, $b = \xi$, $c = 0$, $d = \eta$, $u_1 = u(0, 0)$, $u_2 = u(\xi, 0)$, $u_3 = u(\xi, \eta)$, $u_4 = u(0, \eta)$ are given real numbers and v, w, m, n are integers (see Tables 1-4). The Dirichlet boundary condition, and the functions f, K_x, K_y are taken from the exact solution.

The maximum error order over all grid points,

$$\|E_N\|_{\infty} = \max_{i,j} \{ |u(x_i, y_j) - U_{ij}| \}$$

is presented, where U_{ij} is the computed approximation at the grid point (x_i, y_j) . For our schemes we give $\|T_N\|_{\infty}$, the infinity norm of the local truncation error over all grid points. We also display the ratios of the successive errors

$$\text{ratio}_1 = \log_2 \frac{\|T_N\|_{\infty}}{\|T_{2N}\|_{\infty}}, \quad \text{ratio}_2 = \log_2 \frac{\|E_N\|_{\infty}}{\|E_{2N}\|_{\infty}}.$$

The computational results are displayed in Tables 1,2.

In the 3-D case on each subdomain Ω_s , $s = 1, \dots, 8$ we seek an exact solution in the form

$$\begin{aligned} u(x, y, z) = & \left(u_1 \left(\frac{b-x}{b-a} \right)^v \left(\frac{d-y}{d-c} \right)^w + u_2 \left(\frac{x-a}{b-a} \right)^v \left(\frac{d-y}{d-c} \right)^w \right. \\ & + u_3 \left(\frac{x-a}{b-a} \right)^v \left(\frac{y-c}{d-c} \right)^w + u_4 \left(\frac{b-x}{b-a} \right)^v \left(\frac{y-c}{d-c} \right)^w \left. \right) \left(\frac{f-z}{f-e} \right)^\rho \\ & + \left(u_5 \left(\frac{b-x}{b-a} \right)^v \left(\frac{d-y}{d-c} \right)^w + u_6 \left(\frac{x-a}{b-a} \right)^v \left(\frac{d-y}{d-c} \right)^w \right. \\ & \left. + u_7 \left(\frac{x-a}{b-a} \right)^v \left(\frac{y-c}{d-c} \right)^w + u_8 \left(\frac{b-x}{b-a} \right)^v \left(\frac{y-c}{d-c} \right)^w \right) \left(\frac{z-e}{f-e} \right)^\rho. \end{aligned}$$

The computational results are similar to those of the 2-D case, Table3.

Table 1: Equal coefficients $p_s = q_s = 1$; $r_s = 2$, $s = 1, 2, 3, 4$, $\xi = 0.5628$, $\eta = 0.4067$, $v = w = 4$.

$N = M$	$N_1 = M_1$	$\ T_N\ _\infty$	$ratio_1$	$\ E_N\ _\infty$	$ratio_2$
8	4	8.0760	-	0.0589	-
16	8	4.3459	0.8940	0.0137	2.1023
32	16	2.2499	0.9498	0.0033	2.0554
64	32	1.1442	0.9755	8.0853e-4	2.0300
128	64	0.5769	0.9879	1.9988e-4	2.0162
256	128	0.2897	0.9940	4.9674e-5	2.0086

Table 2: Large contrasts coefficients $p_1 = q_1 = p_3 = q_3 = 10^{-2}$, $p_2 = q_2 = p_4 = q_4 = 10^4$, $r_1 = r_3 = 2 \cdot 10^{-2}$, $r_2 = r_4 = 2 \cdot 10^4$, $\xi = 0.5628$, $\eta = 0.4067$, $v = w = 4$.

$N = M$	$N_1 = M_1$	$\ T_N\ _\infty$	$ratio_1$	$\ E_N\ _\infty$	$ratio_2$
8	4	4.0380e+4	-	0.0575	-
16	8	2.1729e+4	0.8940	0.0134	2.1021
32	16	1.1249e+4	0.9498	0.0032	2.0525
64	32	5.7209e+3	0.9755	7.9143e-4	2.0283
128	64	2.8845e+3	0.9879	1.9574e-4	2.0155
256	128	1.4483e+3	0.9940	4.8713e-5	2.0066

Table 3: Large contrasts coefficients $p_{2i-1} = q_{2i-1} = 1$, $p_{2i} = q_{2i} = 10$, $r_{2i-1} = 1$, $r_{2i} = 2$, $i = 1, 2, 3, 4$; $\xi = 0.5628$, $\eta = 0.4067$, $\zeta = .62$, $v = w = 4$.

$N = M = L$	$N_1 = M_1 = L_1$	$\ T_N\ _\infty$	$ratio_1$	$\ E_N\ _\infty$	$ratio_2$
4	2	76.4365	-	0.0660	-
8	4	34.4407	1.1501	0.0192	1.7814
16	8	15.8447	1.1201	0.0050	1.9411
32	16	7.5245	1.0743	0.0013	1.9434

Acknowledgements: This research is supported by Bulgarian National Fund of Science under Project HS-MI-106/2005.

References

- [1] I. Angelova, *Error estimates of high-order difference schemes for elliptic equations with intersecting interfaces*, Serdica, (in press)
- [2] I. Angelova, L. G. Vulkov. *Singularly perturbed differential equations with discontinuous coefficients and concentrated factors*, Appl. Math. Comput., V. **158** (2004), 683–701.
- [3] I. Angelova, L. Vulkov. *Hight order difference schemes based on new Marchuck integral identities for one-dimensional interface problems*, J. Numer. Math., Vol. **13**, No. **1** (2005), 1–18.
- [4] I. Angelova, L. Vulkov, *High-order difference schemes for elliptic problems with intersecting interfaces*, Appl. Math. Comput. (in press)
- [5] I. Braianov, L. Vulkov, *Numerical solution of a reaction-diffusion elliptic interface problem with strong anisotropy*, Computing, **71** (2003), 153–173.

- [6] J. E. Castillo, J. M. Hyman, M. Yu. Shashkov, S. Sleinberg. *The sensitivity and accuracy of fourth order finite difference schemes on nonuniform grids in one dimension*, J. Computers & Math. Applic, **30** (1998), 175–202.
- [7] Z. Chen, J. Zou, *Finite element methods and their convergence for elliptic and parabolic interface problems*, Numer. Math., **79** (1998), 175–202.
- [8] R. Ewing, O. Iliev, R. Lazarov, *A modified finite volume approximation of second order elliptic equations with discontinuous coefficients*, SIAM J. Sci. Comput., **23**, N4 (2001), 1334–1350.
- [9] B. Jovanović, *Finite difference method for boundary value problems with weak solutions*, Possebna izdanja Mat. Instituta **16**, Belgrade (1993).
- [10] B. S. Jovanović, L. G. Vulkov, *Finite difference approximation of an elliptic interface problem with variable coefficients*, Lect. Notes Comput. Sci., **3401** (2005), 46–55.
- [11] B. Jovanović, *Regularity and a priori estimates for solutions of an elliptic problem with a singular source* (submitted).
- [12] H. Han, *The numerical-solution of interface problems by infinite element method*, Numer. Math., **39** (1982), 39–50.
- [13] V. P. Il'in, *Balance approximations with high accuracy for the Poisson's equation*, Siberian J. of Math., **37**, N1 (1996), 151–169 (in Russian).
- [14] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston (1985).
- [15] J. D. Kandilarov, L. G. Vulkov, *The immersed interface method for two-dimensional heat-diffusion equations with singular own sources*, Appl. Numer. Math, Available online 7 (August 2006).

- [16] R. B. Kellog, *On the Poisson equation with intersecting interfaces*, Appl. Anal., **4** (1975), 101–129.
- [17] D. Knees, *Regularity results for quasilinear elliptic systems of power-law growth in nonsmooth domains - Boundary, transmission and crack problems*, PhD Tesis, Institute fur Angew. Anal. and Num. Simulation, Univ. Stuttgart (2005).
- [18] O. A. Ladyzhenskaya, N. N. Ural'tseva, *Linear and Quasilinear Equations of Elliptic Type*, Nauka, Moscow (1973) (in Russian).
- [19] R. J. Leveque, Z. L. Li, *The immersed interface method for elliptic equations with discontinuous coefficients and singular sources*. SIAM J. Numer. Anal., **31** (1994), 1019–1044.
- [20] Z. Li, *An overview of the immersed interface method and its applications*, Taiwanese J. Math., **7** (2003) , N1, 1–49.
- [21] G. I. Marchuk, V. I. Agoshkov, *Introduction in Projection-Mesh Methods*. Moscow: Nauka, (1981) (Russian).
- [22] A. A. Samarskiĭ, *Theory of Difference Schemes*, Nauka, Moscow (1989) (in Russian); English transl.: *The Theory of Difference Schemes*, Marcel Dekker, Inc., New York (2001).
- [23] A. A. Samarskiĭ, V. Andreev, *Difference Methods for Elliptic Equations*, Nauka, Moscow (1976) (in Russian).
- [24] V. V. Smelov, *On generalized solution of two-dimensional elliptic problem with piecewise constant coefficients based on splitting of a differential and using specific basis functions*, Siberian J. of Numer. Math. **1** (2003), 59–72.
- [25] G. Strang, G. J. Fix. *An analysis of the finite element method*, Prentice-Kall, Englewood Cliffs, New York (1973).