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ON ORBITS FOR PAIRS OF OPERATORS ON AN INFINITE-DIMENSIONAL COMPLEX HILBERT SPACE

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Abstract. The results presented in this paper are motivated by some of the results obtained by B. Beauzamy in [1, Chap. III] for a single operator on an infinite-dimensional complex Hilbert space that imply existence of a dense set of vectors with orbits tending strongly to infinity. For the case of invertible operator T, one of B. Beauzamy's results implies that the space actually contains a dense set of vectors for which both the orbits under T and its inverse tend strongly to infinity. We are going to show that this is also true for any suitable pair of operators.

1. INTRODUCTION

Throughout this paper H will denote an infinite-dimensional complex Hilbert space with inner product $\langle \cdot \,|\, \cdot \rangle$ and $B(H)$ the algebra of all bounded linear operators on H. For $T \in B(H)$, with $r(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ we will denote the spectral radius, the spectrum, the point spectrum and the approximate point spectrum of T, respectively. Recall that $\sigma_p(T)$ is the set of all eigenvalues of T, while $\sigma_a(T)$ is the set of all $\lambda \in \sigma(T)$ for which there is a sequence of unit vectors $(x_n)_{n\geq 1}$ such that $\|Tx_n - \lambda x_n\| \to 0$, as $n \to +\infty$; any such sequence is called a sequence of *almost eigenvectors* for λ. Unlike the point spectrum, which can be empty, the approximate point spectrum is nonempty for every $T \in B(H)$. As a mater of fact, $\varnothing \neq \partial\sigma(T) \cup \sigma_p(T) \subseteq \sigma_a(T)$, [2, Prop.VII.6.7].

Orbit of the vector $x \in H$, under the operator T, is the sequence

$$\operatorname{Orb}(T, x) = \left\{x, Tx, T^2x, \dots\right\}.$$

In [1, Thm.III.2.A.1] B. Beauzamy showed that, if $T \in B(H)$ and the circle $\{\lambda \in \mathbb{C} : |\lambda| = r(T)\}$ contains a point in $\sigma(T)$ which is not an eigenvalue for T then, for every positive sequence $(\alpha_n)_{n\geq 1}$ strictly decreasing to 0, in every open ball in H with radius strictly larger then α_1, there is a vector z with $\|T^n z\| \geq \alpha_n r(T)^n$, for all $n \geq 1$. As its proof suggests, this result will remain true if $r(T)$ is replaced with $|\lambda|$, for any $\lambda \in \sigma_a(T)\backslash\sigma_p(T)$. Note that, if $r(T) > 1$ or, in the later case, if $|\lambda| > 1$, then the space will contain a dense set of vectors $z \in H$ with $\operatorname{Orb}(T, z)$ tending strongly to infinity.

B. Beauzamy also stated, almost without a proof, a similar result for the case of invertible operator $T \in B(H)$, [1, Thm.III.2.A.10]: If both the circles $\{\lambda \in \mathbb{C} : |\lambda| = r(T)\}$ and $\{\lambda \in \mathbb{C} : |\lambda| = 1/r(T^{-1})\}$ contain a point in $\sigma(T)$ which is not an eigenvalue for T then, for any two positive sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ strictly decreasing to 0, in every open ball in H with radius strictly larger then $(\alpha_1^2 + \beta_1^2)^{1/2}$, there is a vector z which satisfies simultaneously: $\|T^n z\| \geq \alpha_n r(T)^n$ and $\|T^{-n} z\| \geq \beta_n r(T^{-1})^n$, for all $n \geq 1$. We are going to generalize this result for pairs of operators on H.

2. PRELIMINARY RESULTS

Proposition 2.1. *Let $T \in B(H)$ and $\lambda \in \sigma_a(T)\backslash\sigma_p(T)$.*

(a) *For every sequence of almost eigenvectors $(x_n)_{n\geq 1}$ for λ and every $h \in H$, $\lim\limits_{n\to+\infty} \langle x_n | h\rangle = 0$.*

(b) *If E is any orthonormal basis for H then, there is a sequence $(y_k)_{k\geq 1}$ of almost eigenvectors for λ, such that the sets $\{e \in E : \langle e|y_k\rangle \neq 0\}$, $k \geq 1$ are all finite and pairwise disjoint.*

Proof. The assertion under (a) follows from the Riesz's theorem for representation of bounded linear functional on a Hilbert space and from the fact that, if T is a bounded linear operator on a reflexive Banach space and $\lambda \in \sigma_a(T) \backslash \sigma_p(T)$, then every corresponding sequence of almost eigenvectors for λ tends weakly to 0 [1, Prop.I.1.13].

To prove the assertion under (b) fix any corresponding sequence of almost eigenvectors $(x_n)_{n\geq 1}$ for λ. For $h \in H$, let $E(h) = \{e \in E : \langle h|e\rangle \neq 0\}$.

Since $E(h)$ is countable for each $h \in H$, [2, Cor.I.4.9], there is a sequence $(e_k)_{k\geq 1}$ in E such that $\cup_{n\geq 1} E(x_n) \subseteq \{e_k : k \in \mathbb{N}\}$. Then, each x_n can be represented as $x_n = \sum_{k=1}^{+\infty} \langle x_n|e_k\rangle \cdot e_k$, and

$$\left\| x_n - \sum_{k=1}^{m} \langle x_n|e_k\rangle \cdot e_k \right\| \to 0, \ \text{ as } m \to +\infty, \text{ for all } n \geq 1. \tag{1}$$

By (a), applied on $h = e_1$, we can find a positive integer n_1 so that $|\langle x_n|e_1\rangle| < 1/2^3$, for all $n \geq n_1$. Let $n \in \{1, ..., n_1\}$. By (1) there is $s(n) \geq 1$ such that

$$\left\| x_n - \sum_{k=1}^{s(n)} \langle x_n|e_k\rangle \cdot e_k \right\| < 1.$$

Put $z_n = \sum_{k=1}^{s(n)} \langle x_n|e_k\rangle \cdot e_k$.

Suppose that we have found integers $n_0 = 0 < n_1 < \ldots < n_l$ for some $l \geq 1$, with the following property: for all $1 \leq j \leq l$

(i) $|\langle x_n|e_k\rangle| < 1/(j+1)^{j+2}$, for all $1 \leq k \leq j$ and $n \geq n_j$; and

(ii) if $n \in \{n_{j-1}+1, \ldots, n_j\}$ then, there is $s(n) \geq j$ so that the vector z_n, defined with $z_n = \sum_{k=j}^{s(n)} \langle x_n|e_k\rangle \cdot e_k$, satisfies $\|x_n - z_n\| < 1/2^{j-1}$.

Now, applying (a) on each $h \in \{e_1, e_2, \ldots, e_{l+1}\}$, we can find an integer $n_{l+1} > n_l$ so that $|\langle x_n|e_k\rangle| < 1/(l+2)^{l+3}$, for all $1 \leq k \leq l+1$ and $n \geq n_{l+1}$. Let $n \in \{n_l+1, \ldots, n_{l+1}\}$. In the same way as above, we can find $s(n) \geq l+1$ such that $\left\| x_n - \sum_{k=1}^{s(n)} \langle x_n|e_k\rangle \cdot e_k \right\| < 1/2^{l+1}$. Put $z_n = \sum_{k=l+1}^{s(n)} \langle x_n|e_k\rangle \cdot e_k$. Since $n \geq n_l + 1$, applying (i) for $j = l$, we obtain

$$\left\| \sum_{k=1}^{l} \langle x_n|e_k\rangle \cdot e_k \right\| \leq \sum_{k=1}^{l} \frac{1}{(l+1)^{l+2}} = \frac{l}{(l+1)^{l+2}} < \frac{1}{(l+1)^{l+1}} \leq \frac{1}{2^{l+1}} \,.$$

Then

$$\|x_n - z_n\| \le \left\| x_n - \sum_{k=1}^{s(n)} \langle x_n|e_k\rangle \cdot e_k \right\| + \left\| \sum_{k=1}^{l} \langle x_n|e_k\rangle \cdot e_k \right\| < \frac{1}{2^{l+1}} + \frac{1}{2^{l+1}} = \frac{1}{2^l} \; .$$

The previous discussion implies that there is a sequence of integers $n_0 = 0 < n_1 < \ldots < n_j < \ldots$, and a sequence $(z_n)_{n\ge 1}$ in H such that (i) and (ii) hold for every $j \ge 1$. Moreover, the sequence $(z_n)_{n\ge 1}$ has the following properties:

1. $z_n \ne 0$, for all $n \ge 1$ (this follows from (ii) and from the fact that $(x_n)_{n\ge 1}$, as a sequence of almost eigenvectors is with $\|x_n\| = 1$, for every $n \ge 1$) ;

2. $\|z_n\| \to 1$, as $n \to +\infty$;

3. $E(z_n)$ is finite for each $n \ge 1$ (by construction); and

4. for every $n \ge 1$ there is $m > n$ such that $E(z_n) \cap E(z_m) = \varnothing$ (if $j \ge 1$ is such that $n_{j-1} + 1 \le n \le n_j$ and if $z_n = \sum_{k=j}^{s(n)} \langle x_n|e_k\rangle \cdot e_k$, choose any $p \ge s(n) + 1$ and $m \ge n_p$; clearly $p > j$, $m > n_j \ge n$ and $E(z_n) \cap E(z_m) = \varnothing$).

By 4. we can find a subsequence $(z_{n_k})_{k\ge 1}$ of $(z_n)_{n\ge 1}$ so that the sets $E(z_{n_k})$, $k \ge 1$ are pairwise disjoint. Put $y_k = z_{n_k} / \|z_{n_k}\|$, $k = 1, 2, \ldots$. Obviously, the sets $\{e \in E : \langle e|y_k\rangle \ne 0\}$, $k \ge 1$ are all finite and pairwise disjoint, and

$$\|x_{n_k} - y_k\| \le \|x_{n_k} - z_{n_k}\| + \|z_{n_k} - y_k\| = \|x_{n_k} - z_{n_k}\| + \|z_{n_k}\| \cdot \left|1 - \|z_{n_k}\|^{-1}\right| \to 0,$$

as $k \to +\infty$. This will imply that $\|Ty_k - \lambda y_k\| \to 0$, as $k \to +\infty$, and so $(y_k)_{k\ge 1}$ would be the required sequence of almost eigenvectors for λ. □

Lemma 2.2. *If $(u_n)_{n\ge 1}$ is a sequence in H which tends weakly to 0 and $A \in B(H)$ then, for every $u \in H$ and every $\delta > 0$:*

(*a*) $\limsup_{n\to+\infty} \|A(u + \delta u_n)\|^2 \ge \|Au\|^2$*; and*

(*b*) *if* $\|Au_n\| \to \alpha$ *as* $n \to +\infty$*, then* $\|A(u + \delta u_n)\|^2 \to \|Au\|^2 + \alpha^2\delta^2$.

Proof. For each $n \ge 1$

$$\|A(u + \delta u_n)\|^2 = \|Au\|^2 + 2\delta \operatorname{Re} \langle u_n|A^*Au\rangle + \|Au_n\|^2 , \tag{2}$$

where A^* denotes the Hilbert space adjoint of A .

Since $(u_n)_{n\geq 1}$ tends weakly to 0, $\langle u_n|h\rangle \to 0$, as $n\to +\infty$, for all $h\in H$. Thus, by the continuity of the function $\lambda \mapsto \operatorname{Re}\lambda$, we have that $\operatorname{Re}\langle u_n|A^*Au\rangle \to 0$, as $n\to+\infty$, for all $u\in H$. Now, by (2) we obtain that

$$\begin{aligned}\limsup_{n\to+\infty}\|A(u+\delta u_n)\|^2 &\geq \limsup_{n\to+\infty}(\|Au\|^2+2\delta\operatorname{Re}\langle u_n|A^*Au\rangle)\\ &= \lim_{n\to+\infty}(\|Au\|^2+2\delta\operatorname{Re}\langle u_n|A^*Au\rangle)=\|Au\|^2\ ,\end{aligned}$$

which proves (a) and, if $\|Au_n\|\to\alpha$ then, by taking limits in (2), we obtain (b) . □

3. MAIN RESULTS

Theorem 3.1. *If $T,S\in B(H)$, $\lambda\in\sigma_a(T)\backslash\sigma_p(T)$ and $\mu\in\sigma_a(S)\backslash\sigma_p(S)$ then, for any two sequences of positive numbers $(\alpha_n)_{n\geq1}$ and $(\beta_n)_{n\geq1}$ strictly decreasing to 0, in every open ball in H with radius strictly larger then $(\alpha_1^2+\beta_1^2)^{1/2}$ there is a vector $z\in H$ such that $\|T^nz\|\geq\alpha_n|\lambda|^n$ and $\|S^nz\|\geq\beta_n|\mu|^n$, for all $n\geq1$.*

Proof. By the assumptions in the theorem and Proposition 2.1 we can find sequences of unit vectors $(x_n)_{n\geq1}$ and $(y_n)_{n\geq1}$ with the following properties:

(a) $(x_n)_{n\geq1}$ and $(y_n)_{n\geq1}$ tend weakly to 0;

(b) $\|Tx_n-\lambda x_n\|\to0$, $\|Sy_n-\mu y_n\|\to0$ as $n\to+\infty$;

(c) $\langle x_n|x_m\rangle=0=\langle y_n|y_m\rangle$ whenever $n\neq m$; and

(d) $\langle x_n|y_m\rangle=0$ for all $n\geq1$ and $m\geq1$.

Since λ , as an element of the set $\sigma_a(T)\backslash\sigma_p(T)\subseteq\sigma(T)$, is with $|\lambda|\leq r(T)\leq\|T\|$, for every integer $k\geq1$

$$\begin{aligned}\left\|T^kx_n-\lambda^kx_n\right\| &= \left\|(T^{k-1}+\lambda T^{k-2}+\ldots+\lambda^{k-2}T+\lambda^{k-1})(Tx_n-\lambda x_n)\right\|\\ &\leq k\|T\|^{k-1}\|Tx_n-\lambda x_n\|\,.\end{aligned}$$

and so $\left\|T^kx_n-\lambda^kx_n\right\|\to0$, as $n\to+\infty$. In the same way, $\left\|S^ky_n-\mu^ky_n\right\|\to0$, as $n\to+\infty$, and consequently, since x_n's and y_n's are unit vectors,

$$\left\|T^k x_n\right\| \to |\lambda|^k \quad \text{and} \quad \left\|S^k y_n\right\| \to |\mu|^k \quad \text{as } n \to +\infty, \quad \text{for all } k \geq 1. \tag{3}$$

Now, fix $x \in H$ and $\varepsilon > 0$. Let $A_k = (\alpha_k^2 - \alpha_{k+1}^2)^{1/2}$ and $B_k = (\beta_k^2 - \beta_{k+1}^2)^{1/2}$, $k \geq 1$.

First, we look at the sequence $(x+(1+\varepsilon)A_1 x_n)_{n\geq 1}$. By (a), (3) and Lemma 2.2.(b)

$$\lim_{n\to+\infty} \|T(x + (1+\varepsilon)A_1 x_n)\|^2 = \|Tx\|^2 + (1+\varepsilon)^2 A_1^2 |\lambda|^2 > A_1^2 |\lambda|^2 ,$$

which allows us to find a positive integer n_1 such that the vector $z_1' = x+(1+\varepsilon)A_1 x_{n_1}$ is with $\|Tz_1'\| > A_1 |\lambda|$.

In the same way, we obtain that

$$\lim_{n\to+\infty} \|S(z_1' + (1+\varepsilon)B_1 y_n)\|^2 = \|Sz_1'\|^2 + (1+\varepsilon)^2 B_1^2 |\mu|^2 > B_1^2 |\mu|^2 .$$

So, we can find an integer $m_1' > n_1$ such that $\|S(z_1' + (1+\varepsilon)B_1 y_n)\| > B_1^2 |\mu|^2$ for all $n \geq m_1'$. Now we look at the sequence $(T(z_1' + (1+\varepsilon)B_1 y_n))_{n\geq 1}$. By Lemma 2.2.$(a)$

$$\sup_{n\geq m_1'} \|Tz_1' + (1+\varepsilon)B_1 y_n\|^2 \geq \limsup_{n\to+\infty} \|Tz_1' + (1+\varepsilon)B_1 y_n\|^2 \geq \|Tz_1'\|^2 > A_1^2 |\lambda|^2 .$$

This implies that, there is an integer $m_1 \geq m_1'$ such that the vector

$$z_1 = z_1' + (1+\varepsilon)B_1 y_{m_1} = x + (1+\varepsilon)(A_1 x_{n_1} + B_1 y_{m_1})$$

is with $\|Tz_1\| > A_1 |\lambda|$ and $\|Sz_1\| > B_1 |\mu|$.

Suppose that we have found integers $0 < n_1 < m_1 < \ldots < n_{l-1} < m_{l-1}$, for some $l \geq 2$, such that the vectors $z_k = x + (1+\varepsilon)\left(A_1 x_{n_1} + B_1 y_{m_1} + \ldots + A_k x_{n_k} + B_k y_{m_k}\right)$, $1 \leq k \leq l-1$ satisfy both (4) and (5) bellow.

$$\left\|T^j z_k\right\| > (A_j^2 + \ldots + A_k^2)^{1/2} |\lambda|^j , \quad \text{for all } 1 \leq j \leq k \leq l-1, \tag{4}$$

$$\left\|S^j z_k\right\| > (B_j^2 + \ldots + B_k^2)^{1/2} |\mu|^j , \quad \text{for all } 1 \leq j \leq k \leq l-1. \tag{5}$$

Now, we start with the sequence $(z_{l-1} + (1+\varepsilon)A_l x_n)_{n\geq 1}$. Applying again (a), (3) and Lemma 2.2.(b), we have

$$\lim_{n\to+\infty} \left\|T^j(z_{l-1} + (1+\varepsilon)A_l x_n)\right\|^2 = \left\|T^j z_{l-1}\right\|^2 + (1+\varepsilon)^2 A_l^2 |\lambda|^{2j} , \quad \text{for all } 1 \leq j \leq l.$$

By (4) for $1 \leq j \leq k = l-1$, and $\left\|T^l z_{l-1}\right\|^2 + (1+\varepsilon)^2 A_l^2 \left|\lambda\right|^{2l} > A_l^2 \left|\lambda\right|^{2l}$, for $j = l$,

$$\lim_{n\to+\infty} \left\|T^j(z_{l-1} + (1+\varepsilon)A_l x_n)\right\|^2 > (A_j^2 + \ldots + A_l^2)\left|\lambda\right|^{2j} \text{ for all } 1 \leq j \leq l.$$

So, we can find an integer $n_l' > m_{l-1}$ such that

$$\left\|T^j(z_{l-1} + (1+\varepsilon)A_l x_n)\right\|^2 > (A_j^2 + \ldots + A_l^2)\left|\lambda\right|^{2j}, \text{ for all } 1 \leq j \leq l \text{ and } n > n_l'. \quad (6)$$

On the other hand, by (5), (a) and Lemma 2.2.(a)

$$\limsup_{n\to+\infty} \left\|S^j(z_{l-1} + (1+\varepsilon)A_l x_n)\right\|^2 \geq \left\|S^j z_{l-1}\right\|^2 > (B_j^2 + \ldots + B_{l-1}^2)\left|\mu\right|^{2j},$$

for all $1 \leq j \leq l-1$. This implies that there are strictly increasing sequences of positive integers $(N_j(n))_{n\geq 1}$, $1 \leq j \leq l-1$ such that

$$\left\|S^j(z_{l-1} + (1+\varepsilon)A_l x_{N_1(\ldots(N_j(n))\ldots)})\right\| > (B_j^2 + \ldots + B_{l-1}^2)^{1/2}\left|\mu\right|^j, \quad (7)$$

for all $1 \leq j \leq l-1$ and $n \geq 1$.

By (6) and (7) we can find a positive integer n_0 so that $n_l = N_1(\ldots(N_{l-1}(n_0))\ldots) \geq n_l' > m_{l-1}$ and the vector $z_l' = z_{l-1} + (1+\varepsilon)A_l x_{n_l}$ satisfies both (8) an (9) bellow

$$\left\|T^j z_l'\right\| > (A_j^2 + \ldots + A_l^2)^{1/2}\left|\lambda\right|^j, \text{ for all } 1 \leq j \leq l, \quad (8)$$

$$\left\|S^j z_l'\right\| > (B_j^2 + \ldots + B_{l-1}^2)^{1/2}\left|\mu\right|^j, \text{ for all } 1 \leq j \leq l-1. \quad (9)$$

Now, we look at the sequence $(z_l' + (1+\varepsilon)B_l y_n)_{n\geq 1}$. In the same way as before, we obtain that

$$\lim_{n\to+\infty} \left\|S^j(z_l' + (1+\varepsilon)B_l y_n)\right\|^2 = \left\|S^j z_l'\right\|^2 + (1+\varepsilon)^2 B_l^2 \left|\mu\right|^{2j},$$

which, together with (9) will imply that

$$\lim_{n\to+\infty} \left\|S^j(z_l' + (1+\varepsilon)B_l y_n)\right\|^2 > (B_j^2 + \ldots + B_l^2)\left|\mu\right|^{2j}, \text{ for all } 1 \leq j \leq l,$$

and that (by (8) and Lemma 2.2.(a))

$$\begin{aligned}\limsup_{n\to+\infty} \left\|T^j(z_l' + (1+\varepsilon)B_l y_n)\right\|^2 &\geq \left\|T^j z_l'\right\|^2 \\ &> (A_j^2 + \ldots + A_l^2)\left|\lambda\right|^{2j}, \text{ for all } 1 \leq j \leq l.\end{aligned}$$

But then we can find integer $m_l > n_l$ so that the vector

$$z_l = z_l' + (1+\varepsilon)B_l y_{m_l} = x + (1+\varepsilon)(A_1 x_{n_1} + B_1 y_{m_1} + ... + A_l x_{n_l} + B_l y_{m_l}) \qquad (10)$$

satisfies

$$\left\|T^j z_l\right\| > (A_j^2 + \ldots + A_l^2)^{1/2} |\lambda|^j = (\alpha_j^2 - \alpha_{l+1}^2)^{1/2} |\lambda|^j \qquad (11)$$

and

$$\left\|S^j z_l\right\| > (B_j^2 + \ldots + B_l^2)^{1/2} |\mu|^j = (\beta_j^2 - \beta_{l+1}^2)^{1/2} |\mu|^j , \qquad (12)$$

for all $1 \leq j \leq l$. Moreover , by (c) and (d),

$$\begin{aligned} \|z_l - x\|^2 &= (1+\varepsilon)^2 \|A_1 x_{n_1} + B_1 y_{m_1} + \ldots + A_k x_{n_l} + B_k y_{m_l}\|^2 \\ &= (1+\varepsilon)^2 (A_1^2 + B_1^2 + \ldots + A_l^2 + B_l^2) < (1+2\varepsilon)^2 (\alpha_1^2 + \beta_1^2 - \alpha_{l+1}^2 - \beta_{l+1}^2), \end{aligned}$$

i.e.

$$\|z_l - x\| < (1+2\varepsilon)(\alpha_1^2 + \beta_1^2 - \alpha_{l+1}^2 - \beta_{l+1}^2)^{1/2}. \qquad (13)$$

Thus, by induction, we obtain that there are positive integers $n_1 < m_1 < \ldots < n_l < m_l < \ldots$ such that the sequence $(z_l)_{l \geq 1}$ given with (10) satisfies (11) – (13), for all $l \geq 1$ and $1 \leq j \leq l$. The sequence $(z_l)_{l \geq 1}$ is a Cauchy sequence: since $\alpha_l \to 0$ and $\beta_l \to 0$, for every positive integers l and k with $l > k$

$$\begin{aligned} \|z_l - z_k\|^2 &= (1+\varepsilon)^2 \left\|A_{k+1} x_{n_{k+1}} + B_{k+1} y_{m_{k+1}} + ... + A_l x_{n_l} + B_l y_{m_l}\right\|^2 \\ &= (1+\varepsilon)^2 (A_{k+1}^2 + B_{k+1}^2 + \ldots + A_l^2 + B_l^2) \\ &= (1+\varepsilon)^2 (\alpha_{k+1}^2 + \beta_{k+1}^2 - \alpha_{l+1}^2 - \beta_{l+1}^2) \to 0, \quad \text{when } \ k, l \to +\infty. \end{aligned}$$

Since H is Hilbert space, there is $z \in H$ such that

$$z = \lim_{l \to +\infty} z_l = x + (1+\varepsilon) \sum_{i=1}^{+\infty} (A_i x_{n_i} + B_i y_{m_i}).$$

This vector is with the desired properties:

1. $\|z - x\| = \lim\limits_{l \to +\infty} \|z_l - x\| < (1+2\varepsilon)\,(\alpha_1^2 + \beta_1^2)^{1/2}$, (by (13));

and, for all $n \geq 1$

2. $\|T^n z\| = \lim_{l\to+\infty} \|T^n z_l\| \geq \lim_{l\to+\infty} (\alpha_n^2 - \alpha_{l+1}^2)^{1/2} |\lambda|^n = \alpha_n |\lambda|^n$ (by (11) and $\alpha_l \to 0$);

3. $\|S^n z\| = \lim_{l\to+\infty} \|S^n z_k\| \geq \lim_{l\to+\infty} (\beta_n^2 - \beta_{l+1}^2)^{1/2} |\mu|^n = \beta_n |\mu|^n$ (by (12) and $\beta_l \to 0$).

which completes the proof. □

Corollary 3.2. *If the sets $\sigma_a(T)\backslash\sigma_p(T)$ and $\sigma_a(S)\backslash\sigma_p(S)$ both have a nonempty intersection with the domain $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ then, there is a dense set of vectors $z \in H$ such that both the orbits $\mathrm{Orb}(T, z)$ and $\mathrm{Orb}(S, z)$ tend strongly to infinity.*

We turn now to the original Beauzamy's result on invertible operator $T \in B(H)$. For its spectrum we have

$$\sigma(T) \subseteq \left\{\lambda \in \mathbb{C} : \frac{1}{r(T^{-1})} \leq |\lambda| \leq r(T)\right\}. \tag{14}$$

If both the circles $\{\lambda \in C : |\lambda| = r(T)\}$ and $\{\lambda \in C : |\lambda| = 1/r(T^{-1})\}$ contain a point in $\sigma(T)$ which is not an eigenvalue for T then, there are points $\lambda, \mu \in \sigma(T)\backslash\sigma_p(T)$ with $|\lambda| = r(T)$ and $|\mu| = 1/r(T^{-1})$. By (14), this points must be contained in the boundary $\partial\sigma(T)$, and consequently $\lambda, \mu \in \sigma_a(T)\backslash\sigma_p(T)$. It is easy to verify that

$$\alpha \in \sigma_p(T) \text{ if and only if } \alpha^{-1} \in \sigma_p(T^{-1}) \tag{15}$$

and

$$\alpha \in \sigma_a(T) \text{ if and only if } \alpha^{-1} \in \sigma_a(T^{-1}). \tag{16}$$

(To obtain (16), apply the equality $T^{-1} - \alpha^{-1} = -\alpha^{-1}T^{-1}(T - \alpha)$.) This will imply that $\mu^{-1} \in \sigma_a(T^{-1})\backslash\sigma_p(T^{-1})$. Now, since $|\mu^{-1}| = r(T^{-1})$, by applying Theorem 3.1 on T and $S = T^{-1}$, we obtain that in every open ball in H of radius strictly larger then $(\alpha_1^2 + \beta_1^2)^{1/2}$, there is a vector z which satisfies simultaneously: $\|T^n z\| \geq \alpha_n r(T)^n$ and $\|T^{-n} z\| \geq \beta_n r(T^{-1})^n$, for all $n \geq 1$. Note that, if both $r(T) > 1$ and $r(T^{-1}) > 1$, the previous discussion will imply that there is a dense set of vectors $z \in H$ such that both the orbits $\mathrm{Orb}(T, z)$ and $\mathrm{Orb}(T^{-1}, z)$ tend strongly to infinity. Moreover, by (15), (16), Theorem 3.1 and Corollary 3.2 we have the following results.

Theorem 3.3. *If $T \in B(H)$ is invertible operator and $\lambda \in \sigma_a(T)\backslash\sigma_p(T)$ then, for any two sequences of positive numbers $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ strictly decreasing to*

0, *in every open ball in* H *with radius strictly larger then* $(\alpha_1^2+\beta_1^2)^{1/2}$ *there is a vector* $z \in H$ *such that* $\|T^n z\| \geq \alpha_n |\lambda|^n$ *and* $\|T^{-n} z\| \geq \beta_n / |\lambda|^n$ *for all* $n \geq 1$.

Corollary 3.4. *If* $T \in B(H)$ *is invertible operator and the domains* $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ *and* $\{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\}$ *both have nonempty intersection with* $\sigma_a(T) \backslash \sigma_p(T)$ *then, there is a dense set of vectors* $z \in H$ *such that both the orbits* $\mathrm{Orb}(T, z)$ *and* $\mathrm{Orb}(T^{-1}, z)$ *tend strongly to infinity.*

4. ON ORBITS UNDER T AND $f(T)$

In this section Ω will denote a nonempty open subset of the complex plane whose boundary $\partial\Omega$ consists of finite number of rectifiable Jordan curves, oriented in the positive sense and $\mathrm{Hol}(\Omega)$ the set of all holomorphic functions on some open neighborhood of the closure of Ω. We assume that the reader is familiar with the basics of the theory of the functional calculus for operators (see for example [1], [2] and [6]).

Theorem 4.1. *If* $T \in B(H)$, $\sigma(T) \subset \Omega$ *and* $f \in \mathrm{Hol}(\Omega)$, *then*

(*a*) $\sigma(f(T)) = f(\sigma(T))$;

(*b*) $f(\sigma_p(T)) \subseteq \sigma_p(f(T))$ *and* $f(\sigma_a(T)) \subseteq \sigma_a(f(T))$;

(*c*) *if* f *is nonconstant function on each of the components of* Ω, *then* $f(\sigma_p(T)) = \sigma_p(f(T))$ *and* $f(\sigma_a(T)) = \sigma_a(f(T))$.

Proof. The assertion under (*a*) is the well known Spectral Mapping Theorem. The assertions on the point spectrum in (*b*) and (*c*) are parts of [6, Thm. 10.33]. We follow the same lines of the proof of this theorem to prove the assertions on the approximate point spectrum.

(*b*): Let $\lambda_0 \in \sigma(T)$. Then, there is $g \in \mathrm{Hol}(\Omega)$ such that $f(\lambda) - f(\lambda_0) = g(\lambda)(\lambda - \lambda_0)$, and consequently, $f(T) - f(\lambda_0) = g(T)(T - \lambda_0)$. So, if $\lambda_0 \in \sigma_a(T)$ and $(x_n)_{n\geq 1}$ is any corresponding sequence of almost eigenvectors for λ_0, then

$$\|f(T)x_n - f(\lambda_0)x_n\| = \|g(T)(Tx_n - \lambda_0 x_n)\| \leq \|g(T)\| \cdot \|Tx_n - \lambda_0 x_n\| \to 0,$$

as $n \to +\infty$, which implies that $\|f(T)x_n - f(\lambda_0)x_n\| \to 0$, as $n \to +\infty$, which implies $f(\lambda_0) \in \sigma_a(f(T))$.

(c): Now, let $\mu_0 \in \sigma(f(T))$. Then, by (a), the set $Z(f - \mu_0)$ of all zeros of the function $f - \mu_0$ has a nonempty intersection with $\sigma(T)$. Since f is nonconstant on each component of Ω and $\sigma(T)$ is a compact subset of Ω, the set $Z(f - \mu_0) \cap \sigma(T)$ is finite [7, Thm. 10.18]. Let $Z(f - \mu_0) \cap \sigma(T) = \{\xi_1, \dots, \xi_m\}$. Without lost of generality, we may assume that each zero is of order 1. So, we can find $g \in \mathrm{Hol}(\Omega)$ such that $g(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$ and $f(\lambda) - \mu_0 = g(\lambda)(\lambda - \xi_1) \dots (\lambda - \xi_m)$. Then $g(T)$ is invertible operator and

$$f(T) - \mu_0 = g(T)(T - \xi_1) \dots (T - \xi_m). \tag{17}$$

Let $\mu_0 \in \sigma_a(f(T))$ and $(y_n)_{n\geq 1}$ is any corresponding sequence of almost eigenvector for μ_0. Let us assume that none of the points $\xi_1, \dots, \xi_m$ belongs to $\sigma_a(T)$. Then, there are positive constants $c_1, \dots, c_m$ such that $\|(T - \xi_j)x\| \geq c_j \|x\|$, for all $x \in H$ and $1 \leq j \leq m$ [2, Prop.VII.6.4] and, since $g(T)$ is invertible operator, a constant $c > 0$ such that $\|g(T)x\| \geq c \|x\|$ for all $x \in H$. This, together with (17) will give

$$\|f(T)y_n - \mu_0 y_n\| = \|g(T)(T - \xi_1) \dots (T - \xi_m)y_n\| \geq c c_1 \dots c_m, \text{ for all } n \geq 1,$$

which contradicts $\|f(T)y_n - \mu_0 y_n\| \to 0$, as $n \to +\infty$. So, there must be an integer $1 \leq j \leq m$ such that $\xi_j \in \sigma_a(T)$ and, consequently $\mu_0 = f(\xi_j) \in f(\sigma_a(T))$. □

If, in addition to the hypotheses of Theorem 4.1.(c), we assume that the function f is injective then, $f(\sigma_a(T) \backslash \sigma_p(T)) = f(\sigma_a(T)) \backslash f(\sigma_p(T)) = \sigma_a(f(T)) \backslash \sigma_p(f(T))$. Now, applying Theorem 3.1 on T and $S = f(T)$ we obtain the following result.

Theorem 4.2. *If $T \in B(H)$, $\lambda \in \sigma_a(T) \backslash \sigma_p(T)$ and $f \in \mathrm{Hol}(\Omega)$ is an injective and nonconstant function on each of the components of $\Omega \supset \sigma(T)$ then, for any two sequences of positive numbers $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ strictly decreasing to 0, in every open ball in H with radius strictly larger then $(\alpha_1^2 + \beta_1^2)^{1/2}$ there is a vector $z \in H$ such that $\|T^n z\| \geq \alpha_n |\lambda|^n$ and $\|f(T)^n z\| \geq \beta_n |f(\lambda)|^n$, for all $n \geq 1$.*

Remark 4.3 In view of this result, Theorem 3.3 can be derived as a consequence of Theorem 4.2. for the case $f(\lambda) = \lambda^{-1}$ and $T^{-1} = f(T)$. One only need to observe that $\Omega = \{\lambda \in C : m < |\lambda| < M\} \supset \sigma(T)$ for any $0 < m < 1/r(T^{-1})$ and $M > r(T)$, and that $f(\lambda) = \lambda^{-1}$ is injective, holomorphic and nonconstant function on Ω.

By Corollary 3.2 and Theorem 4.2 we also have the following

Corollary 4.4. *If $T \in B(H)$, $\lambda \in \sigma_a(T)\backslash\sigma_p(T)$ is with $|\lambda| > 1$ and $f \in \mathrm{Hol}(\Omega)$ is injective and nonconstant function on each of the components of $\Omega \supset \sigma(T)$ such that $|f(\lambda)| > 1$ then, there is a dense set of vectors $z \in H$ such that both the orbits $\mathrm{Orb}(T, z)$ and $\mathrm{Orb}(f(T), z)$ tend strongly to infinity.*

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