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ON λ-APPROXIMATIONS FOR ANALYTIC FUNCTIONS ON THE UNIT DISC

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Abstract. In this paper we investigate the asymptotic relation between maximum moduli of a class of functions analytic on the unit disc and their partial sums, i.eẇe formulate the problem of best λ-approximations. We give a solution of the best λ-approximation for analytic functions of rapid growth on the unit disc such as, for example, is the Hardy-Ramanujan generating partition function. Using Ingham Tauberian Theorem we give some interesting applications. Results for functions of medium growth and for entire functions of finite order are also quoted. In growth-measuring an essential role is played by Karamata's class of regularly varying functions.

1. INTRODUCTION

Let $f(z) := \sum_{i=0}^{\infty} a_i z^i$, $|z| < 1$ be an analytic function and $S_n(z) := \sum_{i\leq n} a_i z^i$ its partial sums.

Define also the maximum modulus $M_f(r) := \max |f(z)|_{|z|=r} = |f(re^{i\phi_0}| = |f(z_0)|$; it increases with r and we suppose that $M_f(r) \to \infty$ $(r \to 1^-)$. The problem of maximum moduli of the partial sums of an analytic function defined inside the unit disc is a classical one and has been investigated in many ways. For example, it is well known that the maximum moduli of partial sums of a bounded function need not be bounded, but on the contrary, this is always true (with the same bound) inside the circle $|z| \le 1/2$ (see [8], pp. 236–238).

In general, for a given analytic function $f(z) := \sum_{i=0}^{\infty} a_i z^i$, $|z| < 1$, the moduli of its partial sums $S_n(z) = \sum_{i \le n} a_i z^i$ depend on z and n.

We want to compare $f(z)$ with the partial sums *at the point* z_0 of maximal growth in the following way: determine a real-valued function $n := n(r, \lambda) \to \infty$, $r \to 1^-$; monotone increasing in both variables, such that

$$\frac{S_{n(r,\lambda)}(z_0)}{f(z_0)} = \begin{cases} o(1), & 0 < \lambda < 1 \\ 1 + o(1), & \lambda > 1 \end{cases} \qquad (r \to 1^-). \tag{I}$$

In this sense we are going to find the "shortest" partial sum which is well approximating $f(z_0)$ for r sufficiently close to 1. We call such partial sums best λ-approximating (BLAS). It is evident from (I) that an analogous relation is valid between moduli of BLAS and $M_f(r)$.

Some other questions are related to this one; for a given $n(r, \lambda)$ what can be said about $M_f(r)$ or, how does the ratio $S_{n(r,\lambda)}(z_0)/f(z_0)$ behave when $\lambda \uparrow\downarrow 1$, $r \to 1^-$?

Apart from self-evident role in numerical calculus, the notion of BLAS appears to be very useful in the theory of Hadamard-type convolutions ([7], [8]).

We shall solve the problem of BLAS for a class of analytic functions of rapid growth inside the unit disc.

Of particular importance here is the class of Karamata's regularly varying functions $K_\rho(x)$ i.e., which can be written in the form $K_\rho(x) := x^\rho L(x)$, $\rho \in R$.

Here ρ is the index of regular variation and $L(x)$ is the so-called slowly varying function i.e., positive, measurable and satisfying $L(\lambda x) \sim L(x)$, $\forall \lambda > 0$ $(x \to \infty)$. Some examples of slowly varying functions are:

$$\log^a x,\ \log^b(\log x),\ e^{\frac{\log x}{\log\log x}},\ e^{\log^c x};\quad a, b \in R,\ 0 < c < 1.$$

For further theory of regular variation we recommend [2] and [5]. We quote some facts for latter use:

$$K_\rho(\lambda x) \sim \lambda^\rho K_\rho(x),\ \lambda > 0;\ \log L(x) = o(\log x)\quad (x \to \infty).$$

If $a(x) \sim b(x) \to \infty$ $(x \to \infty)$, then $K_\rho(a(x)) \sim K_\rho(b(x))$ $(x \to \infty)$.

If $L_1(x)$, $L_2(x)$ are slowly varying functions, then $L_1(x) + L_2(x)$; $L_1(x)L_2(x)$; $L_1(x)/L_2(x)$; $(L_1(x))^a$, $a \in R$, are also slowly varying.

Analogously to Valiron's proximate order (cf[2], [8]) in the theory of entire functions, we are using here Karamata's class for measuring the growth of a given analytic function on the unit disc.

2. RESULTS ON FUNCTIONS OF RAPID GROWTH

Let $f(z)$, $S_n(z)$, $M_f(r)$, $n(r,\lambda)$, $K_\rho(x)$, z_0 be defined as above. Throughout the paper we suppose that λ is a fixed positive number $\neq 1$ and r is sufficiently close to 1^-.

Theorem 1. *If* $\log M_f(r) \sim K_\rho(\frac{1}{1-r})$, $\rho > 0$ $(r \to 1^-)$ *and*

$$n(r,\lambda) \sim \frac{C_\rho(\lambda)}{1-r} \log M_f(r) \quad (r \to 1^-),$$

where

$$C_\rho(\lambda) := \begin{cases} \rho\lambda^\rho, & \rho > 1, \\ \lambda^2, & \rho = 1, \\ \rho\lambda, & 0 < \rho < 1, \end{cases} \tag{1}$$

then (I) *holds; i.e.,* $S_{n(r,\lambda)}(z_0)$ *is the best* λ*-approximating partial sum.*

Proof. A simple implementation of Cauchy Integral formula gives:

$$\frac{1}{2\pi i}\int_D f(w)\frac{(z_0/w)^{n+1}}{w-z_0}dw = \begin{cases} -S_n(z_0), & z_0 \notin \mathrm{int}D, \\ f(z_0) - S_n(z_0), & z_0 \in \mathrm{int}D. \end{cases} \tag{2}$$

Let the contour D be a circle $w = Re^{i\phi}$, where $R = R(r,\lambda) := 1 - \frac{1}{\lambda}(1-r)$. Since $|z_0| = r > R$ for $0 < \lambda < 1$; $r < R$ for $\lambda > 1$; from (2) follows

$$I := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(Re^{i\phi})}{f(re^{i\phi_0})} \frac{(\frac{r}{R}e^{i(\phi_0-\phi)})^n}{\frac{R}{r}e^{i(\phi-\phi_0)} - 1} d\phi = \begin{cases} -\frac{S_n(z_0)}{f(z_0)}, & 0 < \lambda < 1, \\ 1 - \frac{S_n(z_0)}{f(z_0)}, & \lambda > 1. \end{cases} \tag{3}$$

Since $|f(z_0)| = M_f(r)$, estimating integral on the left side of (3), we get

$$I \leq \frac{M_f(R)}{M_f(r)} \frac{e^{n\log(r/R)}}{|R/r - 1|}. \tag{4}$$

But, when $r \to 1^-$ we have

$$\log M_f(R) \sim K_\rho\Big(\frac{1}{1-R}\Big) = K_\rho\Big(\frac{\lambda}{1-r}\Big) \sim \lambda^\rho K_\rho\Big(\frac{1}{1-r}\Big) \sim \lambda^\rho \log M_f(r)$$
$$\Big|\frac{R}{r}-1\Big| > (1-r)\Big|1-\frac{1}{\lambda}\Big|; \ \ \log\frac{r}{R} \sim (1-r)\Big(\frac{1}{\lambda}-1\Big); \ \ \log\frac{1}{1-r} = o(\log M_f(r)).$$

Putting this in (4) with $n = n(r,\lambda) = \frac{C_\rho(\lambda)}{1-r}\log M_f(r)(1+o(1))$, we obtain for $r \to 1^-$

$$|I| \le \frac{\lambda}{|\lambda-1|}\exp(\log M_f(r)(\lambda^\rho - 1 + C_\rho(\lambda)(\frac{1}{\lambda}-1)+o(1)) = \frac{\lambda}{|\lambda-1|}M_f(r)^{-B_\rho(\lambda)}.$$

It is easy to check that $B_\rho(\lambda) := 1-\lambda^\rho + \frac{C_\rho(\lambda)}{\lambda}(\lambda-1)+o(1)$ is strictly positive for each fixed positive $\lambda \ne 1$ and r sufficiently close to 1. □

Therefore, Theorem 1 is proved and moreover we have a good estimation for the o terms in (I), i.e.,

Theorem 2. *Under the conditions of Theorem 1 we have*

$$\frac{S_{n(r,\lambda)}(z_0)}{f(z_0)} = \begin{cases} A_\lambda \ e^{\log M_f(r)(-B_\rho(\lambda))}, & 0<\lambda<1, \\ 1 + A_\lambda e^{\log M_f(r)(-B_\rho(\lambda))}, & \lambda > 1. \end{cases} \qquad (r \to 1^-) \tag{5}$$

with $|A_\lambda| \le \frac{\lambda}{|\lambda-1|}$.

3. SUPPLEMENTARIES

Functions of rapid growth on the unit disc naturally arise from Laplace-Stieltjes transforms of the so-called partition functions (cf. [1], [3], [4]). The main tool in dealing with the partition problem is the now classical Ingham Tauberian Theorem (cf. [3]):

Let

$$\widehat{A}(x) := \int_0^\infty e^{-ux}dA(u), \quad x = s+it, \quad s>0,$$

and $A(u)$ satisfy

1) $A(0)=0$; 2) $A(u)$ is non-decreasing for sufficiently large u;

3) $\widehat{A}(x) \sim C(M/x)^{m\rho-1/2}e^{(M/x)^\rho/\rho}$, $(C, M, \rho \in R^+,\ m \in R)$, uniformly for $x \to 0$ in each angle of the form $t \le \Delta s$, $0<\Delta<\infty$.

Then

$$A(u) \sim C\sqrt{\frac{1-\theta}{2\pi}}(uM)^{m\theta-1/2}e^{(uM)^\theta/\theta}, \quad \theta = \frac{\rho}{1+\rho}, \quad (u \to \infty).$$

We use this Theorem in the following way: let, as before, $f(z) := \sum a_n z^n$, $|z| < 1$, and suppose that the coefficients a_n are non-negative, $a_0 := 0$. Then, denoting by $A(u) := \sum_{n \le u} a_n$, we obtain $A(0) = 0$, $A(u)$ non-decreasing and its LS transform $\widehat{A}(x) := \int_0^\infty e^{-ux} dA(u) = f(e^{-x})$, $\mathrm{Re} x > 0$.

On the other hand, for $x = s + it$,

$$|f(e^{-x})| = \Big|\sum a_n e^{-nx}\Big| \le \sum a_n e^{-ns} = f(e^{-s}), \quad s > 0;$$

i.e., for $z = e^{-x}$, $z_0 = e^{-s}$, $M_f(e^{-s}) = f(e^{-s})$.

Since $1 - e^{-s} \sim s$, $s \to 0^+$, the condition from the Theorem 1 turns out to be $\log M_f(e^{-s}) \sim K_\rho(\frac{1}{s}) = (\frac{1}{s})^\rho L(\frac{1}{s})$ and $n(e^{-s}, \lambda) \sim \frac{C_\rho(\lambda)}{s} \log M_f(e^{-s})$ $s \to 0^+$.

By the assumption 3) of Ingham's Theorem we have that

$$\log M_f(e^{-s}) = \log \widehat{A}(s) \sim \frac{1}{\rho}\Big(\frac{M}{s}\Big)^\rho.$$

It is easy to derive from Ingham's Theorem that, for nondecreasing a_n (cf[3]),

$$a_n \sim CM\sqrt{\frac{1-\theta}{2\pi}}(Mn)^{(m+1)\theta-3/2}e^{\frac{1}{\theta}(Mn)^\theta}, \quad n \to \infty.$$

This, along with the Theorem 1 (with $L(1/s) := M^\rho/\rho$), gives the next BLAS proposition for Ingham's class of functions:

Proposition 1. *For any $M, \rho \in R^+$, $m \in R$, $\theta = \rho/(1+\rho)$, $n(e^{-s}, \lambda) := C_\rho(\lambda)\frac{1}{\rho s}(M/s)^\rho$,*

$$\begin{aligned}\frac{1}{f(e^{-s})}\sum_{n \le n(e^{-s},\lambda)} a_n e^{-ns} &:= s^{m\rho-1/2}e^{-\frac{1}{\rho}(M/s)^\rho}\sum_{n \le n(e^{-s},\lambda)} n^{(m+1)\theta-3/2}e^{\frac{1}{\theta}(Mn)^\theta - ns} \\ &\sim \begin{cases} 0, & 0 < \lambda < 1, \\ \sqrt{2\pi(1+\rho)}M^{\theta(m\rho-1)}, & \lambda > 1. \end{cases} \quad s \to 0^+\end{aligned}$$

The famous Hardy–Ramanujan partition problem is connected with functions of rapid growth, too. Namely, let $p(n)$, $n \in N$, denote the number of solutions of the Diophantine equation $n = 1x_1 + 2x_2 + \cdots + mx_m + \cdots$ in non-negative integers x_i.

A very interesting story about efforts to find an exact asymptotic formula for $p(n)$ is given in [4].

Let $q(s)$ be the generating function for $p(n)$ i.e., $q(s) := \sum_n p(n)e^{-ns}$.

Since $q(s) \sim \sqrt{\frac{s}{2\pi}} e^{\frac{\pi^2}{6}\frac{1}{s}}$, $s \to 0$ (cf[1], [3]), by applying the Theorem 2 with $\log M_q(s) \sim \frac{\pi^2}{6s}$, $\rho = 1$, $B_1(\lambda) = (\lambda - 1)^2 + o(1)$, we obtain a BLAS formula for partitions $p(n)$:

Proposition 2.

$$\frac{e^{-\frac{\pi^2}{6s}}}{\sqrt{s}} \sum_{n \le \lambda \frac{\pi^2}{6s^2}} p(n)e^{-ns} = \begin{cases} A_\lambda \; e^{-\frac{\pi^2}{6s}((1-\sqrt{\lambda})^2 + o(1)))}, & 0 < \lambda < 1, \\ \frac{1}{\sqrt{2\pi}} + A_\lambda \; e^{-\frac{\pi^2}{6s}((\sqrt{\lambda}-1)^2 + o(1)))}, & \lambda > 1. \end{cases} \qquad s \to 0^+$$

with $A_\lambda \le \frac{\sqrt{\lambda}}{|\sqrt{\lambda}-1|}$.

4. RESULTS ON FUNCTIONS OF MEDIUM GROWTH

Let $f(z)$, $S_n(z)$, $M_f(r)$, $n(r, l)$, $K_\rho(x)$, z_0 be defined as above. We have (cf [6]),

Theorem 3. *If* $\ln M_f(r) \sim K_\rho(\log(\frac{1}{1-r}))$, $\rho > 0$, $(r \to 1^-)$ *then we can take*

$$n(r, \lambda) \sim (\frac{1}{1-r})^\lambda \quad (r \to 1^-),$$

independently of $K_\rho(\cdot)$.

In the case $\rho = 0$, the form of $n(r, \lambda)$ is drastically changed as the next example shows.

It is not difficult to check that for

$$\log(\frac{1}{1-r}) = \sum_{k=1}^{\infty} \frac{r^k}{k}, \qquad r \in [0, 1),$$

the considered function $n(r, \lambda)$ is

$$n(r, \lambda) = \exp[\log(1/(1-r))e^{-(\log\log(1/(1-r)))^{1-\lambda}}].$$

5. RESULTS FOR ENTIRE FUNCTIONS OF FINITE ORDER

For a given entire function $f(z) := \sum_{k=0}^{\infty} a_k z^k$ we define, as usual, its partial sums $S_n(z) := \sum_{k\le n} a_k z^k$ and maximum moduli $M_f(r) := \max |f(z)|_{|z|=r} = |f(re^{i\phi_0})| = |f(z_0)|$. The order ρ of $f(z)$ is $\rho := \limsup_{r\to\infty} \log\log M_f(r)/\log r$.

In [6], we gave a notion of best λ-approximating (BLAS) partial sums for functions analytic on the unit disc. This can be easily reformulated for entire functions (analytic on the whole complex plane) as:

If there is an integer-valued function $n := n(r,\lambda) \to \infty$ $(r \to \infty)$ such that

$$\frac{S_{n(r,\lambda)}(z_0)}{f(z_0)} = \begin{cases} o(1), & 0<\lambda<1; \\ 1+o(1), & \lambda>1; \end{cases} \qquad (r\to\infty) \tag{I}$$

we call $S_{n(r,\lambda)}(z_0)$ the best λ-approximating partial sum (BLAS).

In order to study entire functions of order zero we shall consider a subclass of R_0 i.e. de Haan's class Π_l,

$$h(x)\in\Pi_l \iff \frac{h(tx)-h(x)}{l(x)} \sim \log t, \quad \forall t>0; \quad (x\to\infty) \tag{0.2}$$

where $l(x) \in R_0$ is called the auxiliary function and we can take $h(x) = l(x) + \int_1^x l(t)/t dt$ [2, pp. 160–165].

We are going to apply our BLAS results to entire functions with non-negative coefficients i.e., to determine the asymptotic behavior of Hadamard-type convolutions $T_f(r) := \sum n^\alpha l_n a_n r^n$, where (l_n) are slowly varying sequences; therefore improving our results from [7].

Let $f(z)$, $M_f(r)$, $n(r,\lambda)$, ρ, z_0 be defined as above. Then we have the following

Theorem 4. *If* $\log M_f(r) \in R_\rho$, $\rho>0$, *and*

$$n(r,\lambda) \sim \lambda\rho \log M_f(r). \qquad (r\to\infty) \tag{1}$$

Then

$$\frac{S_{n(r,\lambda)}(z_0)}{f(z_0)} = \begin{cases} \epsilon_1(r,\lambda), & 0<\lambda<1; \\ 1+\epsilon_2(r,\lambda), & \lambda>1, \end{cases}$$

with

$$|\epsilon_i(r,\lambda)| \le \frac{1}{|\lambda^{1/\rho}-1|} M_f(r)^{-(\lambda\log\lambda-\lambda+1+o(1))}, \ i=1,2 \qquad (r\to\infty).$$

In the case of entire functions of order zero, we shall treat the subclass whose logarithm of the maximum modulus belongs to de Haan's class Π_l with unbounded auxiliary function $l \in R_0$.

Hence,

Theorem 5. *If* $\log M_f(r) \in \Pi_l$ *with auxiliary function* $R_0 \ni l(r) \to \infty$ *(*$r\to\infty$*), and* $n(r,\lambda) \sim \lambda l(r)$ *(*$r\to\infty$*), then*

$$\frac{S_{n(r,\lambda)}(z_0)}{f(z_0)} = \begin{cases} \mu_1(r,\lambda), & 0<\lambda<1 \\ 1+\mu_2(r,\lambda), & \lambda>1 \end{cases}$$

with

$$|\mu_i| \le \frac{1}{|\lambda-1|} e^{-l(r)((\lambda-1)\log\lambda+o(1))}, \quad i=1,2; \quad (r\to\infty).$$

Now, we give an application of our BLAS results. For a given entire function $f(r) := \sum_n a_n r^n$ with non-negative coefficients, there is a classical problem of estimating asymptotic behavior of Hadamard-type convolutions $T_f(r) := \sum_n b_n a_n r^n$ $(r\to\infty)$.

Theorem 6. *Let an entire function* $f(r) := \sum_n a_n r^n$*,* $a_n \ge 0$*, of order* $\rho > 0$*, satisfy* $\log f(r) \in R_\rho$*. Then*

$$T_f(r) := \sum_n c_n a_n r^n \sim \rho^\alpha \ c_{[\log f(r)]} \ f(r) \qquad (r\to\infty),$$

for any regularly varying sequence (c_n) *of arbitrary index* $\alpha \in R$*.*

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