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PROBABILITY LOGICS WITH VECTOR–VALUED MEASURES

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Abstract. Probability logic introduced by this paper is based on probability logic $L_{\mathbb{A}P}$. Measure ranges in probability models will not be linearly ordered, more precisely, measures will be vector–valued, having ranges $\mathbb{Q}^n \cap [0,1]^n$. Axioms and rules of inference are adjusted to determine these types of measures. The completeness theorem for the introduced logic is proved.

INTRODUCTION

The development of the model theory of first-order logic has brought up the need for the study of logics with a stronger expressive power. This allows us to incorporate into the realm of logic certain common mathematical structures and concepts that have been left out of first-order logic due to its limited scope.

Probability logics are logics adequate for the study of structures arising in Probability Theory. Probability logics (see [2, 3, 6]) are formed by adding probability quantifiers $P\vec{x} \geq r$ to first order logic, where $(P\vec{x} \geq r)\varphi(\vec{x})$ means $\{\vec{x} \mid \varphi(\vec{x})\}$ has probability $\geq r$. Logic in this paper is extension of logic $L_{\mathbb{A}P}$, is formed by using

n-typles of reals and probability quantifiers $P\vec{x} \geq (r_1, \ldots, r_n)$. Measures are vector valued and their ranges are $[0,1]^n \cap \mathbb{Q}^n$, are not linearly ordered.

Loeb's construction is expanded to $^*B$-valued measures where it is B Banach space (see [4, 5, 8]).

The axiom system was given and the completeness theorem for the logic whose models have vector–valued measures was proved, which resulted in positive solution of Keisler‘s problem for the vector space $\mathbb{R}^n$ (see [3]).

1. LOEB COMPLETION OF INTERNAL VECTOR–VALUED MEASURES

First, we will show that the important Loeb's construction can also be expended to internal spaces of forms $\langle X, \mathcal{A}, \mu, {}^*B\rangle$, where $\langle B, \|\cdot\|\rangle$ is a Banach space and μ is an internal finitely additive, $^*B$-valued measure defined on A.

Definition 1.1. *Let $\langle X, \mathcal{A}, \mu\rangle$ be an internal measure space with internal, finitely-additive, finite measure μ. A subset $A \subset X$ is Loeb measurable if for every standard $\epsilon > 0$ there exist $B, C \in \mathcal{A}$ such that $B \subset A \subset C$ and $\mathrm{st}\big(\mu(C\backslash B)\big) < \epsilon$. The family of all Loeb measurable sets is denoted by $L(\mathcal{A})$, and let $L(\mu)$ denotes the natural extension of $\widetilde{\mu} = \mathrm{st}\,\mu$ on $L(\mathcal{A})$.*

Proposition 1.2. *$\langle X, L(\mathcal{A}), L(\mu)\rangle$ is a complete measure space with the σ– additive measure $L(\mu)$.*

Let B be a Banach space and $\langle X, \mathcal{A}, \mu, {}^*B\rangle$ an internal space with finitely additive $^*B$-valued measure μ. Let us suppose that the total variation $\vartheta(\mu, \cdot)$ defined by

$$\vartheta(\mu, A) := {}^*\mathrm{sup}_{\mathcal{P}} \sum \big\{ {}^*\|\mu(D)\| \mid D \in \mathcal{P} \big\},$$

where $\mathcal{P}$ ranges over set of all *-finite, $\mathcal{A}$-measurable partitions of X, is a finite internal positive measure on X. Recall that an element $x \in {}^*B$ (particularly $x \in {}^*\mathbb{R}$) is finite ($x \in \mathrm{fin}\,({}^*B)$) if $^*\|x\| \leq m$ for some $m \in \mathbb{R}$. The measure $|\mu| := \mathrm{var}(\mu, \cdot)$ is positive, hence let $\langle X, L(\mathcal{A}), L(|\mu|)\rangle$ be the completion of $\langle X, \mathcal{A}, |\mu|\rangle$. Our aim is to define

Loeb completion $\langle X, L(\mathcal{A}), L(\mu), ?\rangle$ of the measure space $\langle X, \mathcal{A}, \mu, {}^*B\rangle$. The natural candidate for ? is of course $\widehat{B}$, the nonstandard hull of Banach space B. Recall that

$$\widehat{B} := \text{fin}({}^*B)/_{\approx}\ ,$$

where $x \approx y$ means that $x - y$ is of infinitesimal norm. To simplify notation, $\|x\|$ will denote the norm of x for both $x \in B$ and $x \in \widehat{B}$. Analogously to the case $B = \mathbb{R}$ the quotient map $\text{fin}\,({}^*B) \longrightarrow \widehat{B}$ will be called the standard part map and denoted by st. We need the following well-known proposition.

Proposition 1.3. *Let $\langle X, L(\mathcal{A}), L(\mu)\rangle$ be the Loeb completion of the space $\langle X, \mathcal{A}, \mu\rangle$ with finitely-additive, positive, finite measure μ . Then, if $A \in L(\mathcal{A})$, then there exists $B \in \mathcal{A}$ such that $L(\mu)(A \vartriangle B) = 0$.*

Applying the last proposition to the measure space $\langle X, \mathcal{A}, |\mu|\rangle$, where $|\mu| := (\mu, \cdot)$, we get that $A \in L(\mathcal{A})$, iff there exists $B \in \mathcal{A}$ such that $L(|\mu|)(A \vartriangle B) = 0$. Also, if $B_1, B_2 \in \mathcal{A}$ are two sets which, in this sence, approximate A, then

$$\text{st}\,|\mu|\,(B_1 \vartriangle B_2) \leq L\,(|\mu|)\,(A \vartriangle B_1) + L\,(|\mu|)\,(A \vartriangle B_2) = 0,$$

which means that ${}^*\|\mu(B_1) - \mu(B_2)\| \in m(0)$, because ${}^*\|\mu(B_1) - \mu(B_2)\| \leq |\mu|\,(B_1 \vartriangle B_2)$.

This allows us to give the following definition.

Definition 1.4. *Let $\langle X, \mathcal{A}, \mu, {}^*B\rangle$ be an internal measure space with a finitely-additive ${}^*B$-valued measure μ so that total variation $|\mu| := \text{var}\,(\mu, \cdot)$ is finite. Let $\big\langle X, L(\mathcal{A}), L\big(|\mu|\big)\big\rangle$ be the Loeb space associated with the internal space $\langle X, \mathcal{A}, |\mu|\rangle$. For $A \in L(\mathcal{A})$ and $B \in \mathcal{A}$, with the property $L(|\mu|)(A \vartriangle B) = 0$, let*

$$L(\mu)(A) := \text{st}\big(\mu(B)\big), \quad \textit{where} \quad \text{st}\colon \text{fin}\,({}^*B) \longrightarrow \widehat{B}$$

is the mapping defined above.

$L(\mu)$ is obviously an additive $\widehat{B}$-valued measure.

To prove its σ-additivity we need the following simple inequality:

For all $A \in L(\mathcal{A})$ holds $\|L(\mu)(A)\| \leq L(|\mu|)(A)$.

Proposition 1.5. *$L(\mu)$ is a σ-additive measure.*

Considering the fact that we will work with probability measures taking values in Banach space $\mathbb{R}^n$, this result produced by Živaljević (see [8]) will be applicable in the process of creating probability models. More precisely, probability vector-valued measure will have a finite total variation. Moreover, a somewhat stronger result given by Oswald and Sun is applicable as well (see [5]).

Oswald proved (see [4]) the vector-valued version of the following Keisler's result (see also [7]).

If $\langle A_1, \mathcal{F}_1, \mu_1\rangle$ and $\langle A_2, \mathcal{F}_2, \mu_2\rangle$ are two internal finitely additive measure spaces, we have the internal product space $\langle A_1 \times A_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2\rangle$, where $\mathcal{F}_1 \times \mathcal{F}_2$ is the internal product algebra generated by $\mathcal{F}_1$ and $\mathcal{F}_2$. This space has as a corresponding Loeb space $\langle A_1 \times A_2, L(\mathcal{F}_1 \times \mathcal{F}_2), L(\mu_1 \times \mu_2)\rangle$. Also we can construct from the Loeb spaces of $\langle A_1, \mathcal{F}_1, \mu_1\rangle$ and $\langle A_2, \mathcal{F}_2, \mu_2\rangle$ the standard product space

$$\langle A_1 \times A_2, L(\mathcal{F}_1) \times L(\mathcal{F}_2), L(\mu_1) \times L(\mu_2)\rangle \ .$$

Theorem 1.6. (Keisler-Fubini) *Let $f\colon A_1 \times A_2 \longrightarrow \overline{\mathbb{R}}$ be $L(\mu_1 \times \mu_2)$-measurable function. Then*

(1) For $L(\mu_2)$-almost all $a_2 \in A_2$, the function $f(\cdot, a_2)\colon A_1 \longrightarrow \overline{\mathbb{R}}$ is $L(\mu_1)$-measurable;

(2) If f is integrable, then

(a) for μ_2-almost a_2, the function $f(\cdot, a_2)$ is integrable over A_1,

(b) the function $g(a_2) = \int f(a_1, a_2)\, \mathrm{d}L(\mu_1)$ is integrable over A_2,

(c) $\int g(a_2)\, \mathrm{d}L(\mu_2) = \int f(a_1, a_2)\, \mathrm{d}L(\mu_1 \times \mu_2)$.

2. SYNTAX AND SEMANTICS

Let $\mathbb{A}$ be a countable admissible set, i.e. a well-behaved transitive model of Kripke-Platek set theory (for a definition of admissible sets see [1]). We will now discuss the situation when ω is not an element of $\mathbb{A}$ so that each formula is finite. We use $\vee\Phi$ $(\wedge\Phi)$ to denote disjunction (conjunction) of finite number of formulas.

We will assume that the rationals are defined is such a way that $\mathbb{Q} \subseteq \mathbb{A}$; by adding additional reals into $\mathbb{A}$ as urelements, we can obtain more probability quantifiers. That is, this construction suggest that n-tuples of rationals and reals are in the set $\mathbb{A}$. We use $\widetilde{L}_{\mathbb{A}P}$ to denote our logic.

Let L be a countable, Σ-definable set of finitary relation and constant symbols (no function symbols). Logical symbols that will by used are the same as in $L_{\mathbb{A}P}$ logic, except for the quantifiers which are defined in $\widetilde{L}_{\mathbb{A}P}$ logic by using n-tuples of reals, i.e. $(P\vec{x} \geq (r_1, \ldots, r_n))$, where $(r_1, \ldots, r_n) \in \mathbb{R}^n \cap \mathbb{A}$.

Also, the set of formulas are defined as in $L_{\mathbb{A}P}$ logic except for the part related to quantification, i.e. if φ is a formula of the $\widetilde{L}_{\mathbb{A}P}$ logic, then $(P\vec{x} \geq (r_1, \ldots, r_n))\varphi$ is also a formula of $\widetilde{L}_{\mathbb{A}P}$.

In the case of our logics, short forms cannot be used as in the logic $L_{\mathbb{A}P}$, i.e. we cannot write $(P\vec{x} < (r_1, \ldots, r_n))\varphi$ instead of $\neg(P\vec{x} \geq (r_1, \ldots, r_n))\varphi$, because the measure range is not linearly ordered.

The relation $\leq$ on set $\mathbb{A} \cap [0,1]^n$, defined by

$$(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \Longleftrightarrow a_1 \leq b_1 \wedge \ldots \wedge a_n \leq b_n \,,$$

is a partial order on the given set.

Relation $<$ is defined by

$$\big(\exists i \in \{1, \ldots, n\}\big) a_i < b_i \wedge \neg\big(\exists j \in \{1, \ldots, n\}\big) a_j > b_j \,.$$

Similarly we can define $\geq$ and $>$.

Operations " + " and " $\cdot$ " will be defined on coordinates as follows:

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (\underline{a_1 + b_1}, \ldots, \underline{a_n + b_n}) \quad \text{where} \quad \underline{a_i + b_i} = \min(a_i + b_i, 1)$$
$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = (a_1 \cdot b_1, \ldots, a_n \cdot b_n) \,.$$

The satisfiability is defined in a usual way.

Definition 2.1. *A probability model is a structure* $\langle \mathfrak{A}, \mu_k \rangle_{k<\omega} = \langle \mathfrak{A}, \mu \rangle$, *where*

(1) $\mathfrak{A} = \langle A, R_i, c_j \rangle_{i \in I, j \in J}$ *is a model in the sense of first order logic;*

(2) Each μ_k*,* $k < \omega$*, is a (*σ*-additive) probability measure an* A^k*, each* μ_k *is vector-valued with values in* $[0,1]^n_{\mathbb{Q}}$ *and the sequence of measures* $\langle \mu_k \mid k < \omega \rangle$ *satisfies the Fubini property, that is:*

(i) For all $m, k \quad \mu_{m+k}$ *is an extension of the product measure* $\mu_m \times \mu_k$*;*

(ii) Each μ_k *is invariant under permutations. That is, whenever* π *is a permutation of* $\{1, 2, \ldots, k\}$*, and* $S \in \operatorname{dom}(\mu_k)$*,*
if $\pi S = \big\{ \big(a_{\pi(1)}, \ldots, a_{\pi(k)} \big) \mid (a_1, \ldots, a_k) \in S \big\}$*, then* $\pi S \in \operatorname{dom}(\mu_k)$ *and* $\mu_k(\pi S) = \mu_k(S)$*;*
If $S \in \operatorname{dom}(\mu_{m+k})$*, then*

(iii) For all $b \in A^k$*,* $\big\{ a \mid (a, b) \in S \big\} \in \operatorname{dom}(\mu_m)$;

(iv) For all $(r_1, \ldots, r_n) \in \mathbb{R}^n$*,* $\big\{ b \mid \mu_m\{ a \mid (a, b) \in S \} > (r_1, \ldots, r_n) \big\} \in \operatorname{dom}(\mu_k)$;

(v) $\mu_{m+k}(S) = \int \big(\int_S \mu_m(\mathrm{d}x) \big) \mu_k(\mathrm{d}y)$.

(3) Each atomic formula with k free variables is measurable with respect to μ_k*.*

3. A COMPLETE AXIOMATIZATION

Axioms and rules of inferences of logic $\widetilde{L}_{\mathbb{A}P}$ are:

(1) All axioms of logic $L_{\mathbb{A}}$ without quantifiers

(2) Monotonicity of the quantifier:

(i) $\big(P\vec{x} \geq (r_1, \ldots, r_n) \big) \varphi(\vec{x}) \longrightarrow \big(P\vec{x} \geq (s_1, \ldots, s_n) \big) \varphi(\vec{x})$, where $(s_1, \ldots, s_n) \leq (r_1, \ldots, r_n)$;

(ii) $\big(P\vec{x} > (r_1, \ldots, r_n) \big) \varphi(\vec{x}) \longrightarrow \big(P\vec{x} \geq (r_1, \ldots, r_n) \big) \varphi(\vec{x})$;

(iii) $\big(P\vec{x} \geq (r_1, \ldots, r_n) \big) \varphi(\vec{x}) \longleftrightarrow \big(P\vec{y} \geq (r_1, \ldots, r_n) \big) \varphi(\vec{y})$;

(3) Axioms and rules about probability:

(i) From $\psi \longrightarrow \varphi(\vec{x})$ infer $\psi \longrightarrow \big(P\vec{x} \geq (1,\dots,1)\big)\varphi(\vec{x})$;

(ii) $\big(P\vec{x} \geq (0,\dots,0)\big)\vec{x} \neq \vec{x}$, $\big(\mu_1(\varnothing) \geq (0,\dots,0)\big)$;

(iii) Additivity axioms.

$$\big(P\vec{x} \geq (1,\dots,1)\big)\big[\neg\big(\varphi(\vec{x}) \wedge \psi(\vec{x})\big)\big] \wedge \big(P\vec{x} \geq (r_1,\dots,r_n)\big)\varphi(\vec{x}) \wedge \big(P\vec{x} \geq (s_1,\dots,s_n)\big)\psi(\vec{x}) \longrightarrow \big(P\vec{x} \geq (r_1+s_1,\dots,r_n+s_n)\big)\big(\varphi(\vec{x}) \vee \psi(\vec{x})\big);$$

and

$$\big(P\vec{x} \leq (r_1,\dots,r_n)\big)\varphi(\vec{x}) \wedge \big(P\vec{x} \leq (s_1,\dots,s_n)\big)\psi(\vec{x}) \longrightarrow \big(P\vec{x} \leq (r_1+s_1,\dots,r_n+s_n)\big)\big(\varphi(\vec{x}) \vee \psi(\vec{x})\big);$$

(iv) Monotonicity of probability measure

$$\big(P\vec{x} \geq (1,\dots,1)\big)\big(\varphi(\vec{x}) \longrightarrow \psi(\vec{x})\big) \longrightarrow \Big(\big(P\vec{x} \geq (r_1,\dots,r_n)\big)\varphi(\vec{x}) \longrightarrow \big(P\vec{x} \geq (r_1,\dots,r_n)\big)\psi(\vec{x})\Big);$$

(v) Probability measure is continuous at $\vec{0}$. We use the following rule scheme: For any $k < \omega, k > 0$, from $\varphi \longrightarrow \neg\Big(P\vec{y} < \big(\frac{1}{k},\dots,\frac{1}{k}\big)\Big)\Big(P\vec{x} \in \Big[\big(r_1 - \frac{1}{m},\dots,r_n - \frac{1}{m}\big),(r_1,\dots,r_n)\Big)\Big)\psi$, $m = 1,2,\dots$ infer $\neg\varphi$, where $\Big[\big(r_1 - \frac{1}{m},\dots,r_n - \frac{1}{m}\big),(r_1,\dots,r_n)\Big)$ in the last formula represents a set of all n-tuples $(a_1,\dots,a_n)$ so that $(a_1,\dots,a_n) \geq \big(r_1 - \frac{1}{m},\dots,r_n - \frac{1}{m}\big)$ and $(a_1,\dots,a_n) < (r_1,\dots,r_n)$.

(4) Fubini property axiom:

(i) Permutation axiom:
If π is a permutation of $\{1,\dots,n\}$,

$$\big(Px_1,\dots,x_n \geq (r_1,\dots,r_n)\big)\varphi \longleftrightarrow \big(Px_{\pi_1},\dots,x_{\pi_n} \geq (r_1,\dots,r_n)\big)\varphi;$$

(ii) Product independence:

$$\big(P\vec{x} \geq (r_1, \ldots, r_n)\big)\big(P\vec{y} \geq (s_1, \ldots, s_n)\big)\varphi(\vec{x}, \vec{y}) \longrightarrow \big(P\vec{x}\vec{y} \geq (r_1 \cdot s_1, \ldots, r_n \cdot s_n)\big)\varphi(\vec{x}, \vec{y}),$$

and

$$\big(P\vec{x} \leq (r_1, \ldots, r_n)\big)\big(P\vec{y} \leq (s_1, \ldots, s_n)\big)\varphi(\vec{x}, \vec{y}) \longrightarrow \big(P\vec{x}\vec{y} \leq (r_1 \cdot s_1, \ldots, r_n \cdot s_n)\big)\varphi(\vec{x}, \vec{y}),$$

where all variables in $\vec{x}, \vec{y}$ are distinct.

(5) In addition, we will add a rule that secures the situation in which measure will take values only in the set $[0,1]^n_{\mathbb{Q}}$. Namely:

if $\varphi \longrightarrow \big(P\vec{x} \neq (q_1, \ldots, q_n)\big)\psi$, for every $(q_1, \ldots, q_n) \in [0,1]^n_{\mathbb{Q}}$, then we infer $\neg\varphi$;

Of course, the list of rules of inference includes also *modus ponens* and conjunction rule, as in the logic $L_{\mathbb{A}P}$.

Remark 3.1. The adapted Hoover's Iterated integration axioms can be derived from the axioms Product independence:

Let $\mathcal{P}$ is partition of intervals $[0,1]^n$ consisted from $S_1, S_2, \ldots, S_k$ where

$$S_i = [s^i_1, s^i_2) \times [s^i_3, s^i_4) \times \cdots \times [s^i_{2n-1}, s^i_{2n}),$$

where are all specified intervals are subset of $[0,1]$;

$$\bigwedge_{1 \leq l \leq k} \big(P\vec{x} \geq (r^l_1, \ldots, r^l_n)\big)\,(P\vec{y} \in S_l)\varphi(\vec{x}, \vec{y}) \longrightarrow \Big(P\vec{x}\vec{y} \geq \sum (r^l_1, \ldots, r^l_n) \cdot (s^l_1, s^l_3, \ldots, s^l_{2n-1})\Big)\,\varphi(\vec{x}, \vec{y}),$$

where $(r^l_1, \ldots, r^l_n) \in [0,1]^n, 1 \leq l \leq k$;

and let $\mathcal{P}$ is partition of intervals $[0,1]^n$ consisted from $S_1, S_2, \ldots, S_k$ where

$$S_i = [s^i_1, s^i_2] \times [s^i_3, s^i_4] \times \cdots \times [s^i_{2n-1}, s^i_{2n}],$$

where are all specified intervals are subset of $[0,1]$;

$$\bigwedge_{1\le l\le k} \big(P\vec{x} \le (r_1^l,\dots,r_n^l)\big)\,(P\vec{y}\in S_l)\varphi(\vec{x},\vec{y}) \longrightarrow \Big(P\vec{x}\vec{y} \le \sum (r_1^l,\dots,r_n^l)\cdot(s_2^l,s_4^l,\dots,s_{2n}^l)\Big)\,\varphi(\vec{x},\vec{y}),$$

where $(r_1^l,\dots,r_n^l)\in[0,1]^n, 1\le l\le k$.

The proof is rather technical, and axioms Product independence are used k times.

Definition 3.2. *A weak model for $\widetilde{L}_{\mathbb{A}P}$ is a structure*

$$\mathfrak{A} = \big\langle M, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}, \mu_k\big\rangle_{i\in I, j\in J, k\in\mathbb{N}},$$

such that $\big\langle M, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}\big\rangle_{i\in I,j\in J}$, is a classical model, each μ_k is a finitely additive probability measure on A^k with each singleton measurable and takes values in the set $[0,1]^n\cap\mathbb{Q}^n$ and with the set $\big\{\vec{c}\in A^k \mid \mathfrak{A}\models\varphi[\vec{a},\vec{c}]\big\}$ μ_k-measurable for each $\varphi(\vec{x},\vec{y})\in \widetilde{L}_{\mathbb{A}P}$ and each $\vec{a}\in A$.

In this case, we will define $\varphi^{\neg}$ like in the logic $L_{\mathbb{A}P}$, except for the part for quantification. Namely, since a negation in the logic $\widetilde{L}_{\mathbb{A}P}$ cannot "go" trough the formula containing quantifiers; under $\big((P\vec{x}\ge(r_1,\dots,r_n))\,\varphi\big)^{\neg}$ we will assume simply the negation of formula $\big(P\vec{x}\ge(r_1,\dots,r_n)\big)\varphi$.

Let C be a countable set of new constant symbols and let $K = L\cup C$. Then we form the logic $\widetilde{K}_{\mathbb{A}P}$ corresponding to K and we introduce a notion of a consistency property.

Definition 3.3. *A consistency property for $\widetilde{L}_{\mathbb{A}P}$ is a set S of countable sets s of sentences of $\widetilde{K}_{\mathbb{A}P}$ which satisfies the following conditions for each $s\in S$:*

(C_1) (Triviality rule) $\varnothing\in S$;

(C_2) (Consistency rule) Either $\varphi\notin s$ or $\neg\varphi\notin s$;

(C_3) ($\neg$-rule) If $\neg\varphi\in s$, then $s\cup\{\varphi^{\neg}\}\in S$;

(C_4) ($\wedge$-rule) If $\wedge\Phi\in s$, then for all $\varphi\in\Phi$,$s\cup\{\varphi\}\in S$;

(C_5) ($\vee$-rule) If $\vee\Phi \in s$*, then for some* $\varphi \in \Phi$*,* $s \cup \{\varphi\} \in S$*;*

(C_6) (P-rule) If $\big(P\vec{x} > (0,\ldots,0)\big)\varphi(\vec{x}) \in s$*, then for some* $\vec{c} \in C$*,* $s \cup \big\{\varphi(\vec{c})\big\} \in S$*;*

(C_7) If $\varphi(\vec{x}) \in \widetilde{K}_{\mathbb{A}P}$ *is an axiom, then*

(i) $s \cup \Big\{\big(P\vec{x} \geq (1,\ldots,1)\big)\varphi(\vec{x})\Big\} \in S$*;*

(ii) $s \cup \big\{\varphi(\vec{c})\big\} \in S$*, where* $\vec{c} \in C$*;*

(C_8) (Continuity rule) For each $k < \omega$*,* $(r_1,\ldots,r_n) \in [0,1]^n$ *and formula* $\varphi(\vec{x},\vec{y})$ *of* $\widetilde{K}_{\mathbb{A}P}$ *with only finitely many free variables there is some* $m < \omega$ *such that*

$$s\cup\Bigg\{\big(P\vec{y} < \big(\tfrac{1}{k},\ldots,\tfrac{1}{k}\big)\big)\Bigg(P\vec{x} \in \Big[\big(r_1 - \tfrac{1}{m},\ldots,r_n - \tfrac{1}{m}\big),(r_1,\ldots,r_n)\Big)\Bigg)\varphi(\vec{x},\vec{y})\Bigg\} \in S;$$

(C_9) For any formula $\varphi(\vec{x})$ *of logic* $\widetilde{K}_{\mathbb{A}P}$ *with only finitely many free variables, for some* $(q_1,\ldots,q_n) \in [0,1]^n_{\mathbb{Q}}$*, there is*

$$s \cup \big\{\big(P\vec{x} = (q_1,\ldots,q_n)\big)\varphi(\vec{x})\big\} \in S\,.$$

Theorem 3.4. *If* S *is a consistency property, then any* $s_0 \in S$ *has a probability model.*

The completeness theorem follows from this theorem because the set of all countable, consistent sets of formulas $\widetilde{K}_{\mathbb{A}P}$ is a consistency property.

Proof. This consist of two lemmas. First, we will find for s_0 a weak model, and then by applying Loeb's construction on internal weak model we will make a probability model for s_0. □

Lemma 3.5. *If* S *is consistency property, then any* $s_0 \in S$*, has a weak model.*

Proof. We define a complete sequence $\langle s_k \mid k < \omega\rangle$ of elements of S as follows:

Let $\langle \varphi_k \mid k < \omega\rangle$ be an enumeration of the sentences of $\widetilde{K}_{\mathbb{A}P}$. s_0 is given. Given s_k choose s_{k+1} to satisfy the following conditions:

(1) $s_k \subseteq s_{k+1}$;

(2) If $s_k \cup \{\varphi_k\} \in S$, then $\varphi_k \in s_{k+1}$;

(3) If $s_k \cup \{\varphi_k\} \in S$, $\varphi_k = \vee\Phi$, then for some $\theta \in \Phi$, $\theta \in s_{k+1}$;

(4) If $s_k \cup \{\varphi_k\} \in S$ and $\varphi_k = \big(P\vec{x} > (0,\dots,0)\big)\psi(\vec{x})$, then for some $\vec{c} \in C$, $\psi(\vec{c}) \in s_{k+1}$;

(5) (Continuity rule) Let $\big\langle\big(\psi_k(\vec{x},\vec{y}),\vec{r}_k\big) \mid k<\omega\big\rangle$ be an enumeration of the pairs of formulas of $\widetilde{K}_{\mathbb{A}P}$ which have only finitely many free variables, and n-tuples of reals, listed so that each pair occurs infinitely often. Then for some m,

$$\big(P\vec{y} < (\tfrac{1}{k},\dots,\tfrac{1}{k})\big)\Big(P\vec{x} \in \big[\vec{r}_k - (\tfrac{1}{m},\dots,\tfrac{1}{m}),\vec{r}_k\big)\Big)\psi_k(\vec{x},\vec{y}) \in s_{k+1}\,;$$

(6) Let $\big\langle\big(\psi_k(\vec{x}),\vec{q}_k\big) \mid k<\omega\big\rangle$ be an enumeration od the pairs of formulas of $\widetilde{K}_{\mathbb{A}P}$ which have only finitely many free variables, and n-tuples of rationales, then

$$\big(P\vec{x} = (q_1,\dots,q_n)_k\big)\psi_k(\vec{x}) \in s_{k+1}\,.$$

Now let $s_\omega = \bigcup_n s_n$. Let T be the set of constants of $\widetilde{K}_{\mathbb{A}P}$. For $c,d \in T$, let $c \sim d$ iff $c = d \in s_\omega$. Then, $\sim$ is an equivalence relation. Let $[c]$ denote the equivalence class of the constant c. Let $\mathfrak{A}$ have the universe set $A = \big\{[c] \mid c \in T\big\}$. If R is an n-placed relation symbol and $c_1,\dots,c_n \in C$, then $\mathfrak{A} \vDash R\big([c_1],\dots,[c_n]\big)$ iff $R(c_1,\dots,c_n) \in s_\omega$.

Define μ_k on the subset of A^k definable by formulas of $\widetilde{L}_{\mathbb{A}P}$ with parameters from A, by

$$\mu_k\big\{\vec{a} \in A^k \mid \mathfrak{A} \vDash \varphi[\vec{a},\vec{c}\,]\big\} = (q_1,\dots,q_n) \quad \text{iff} \quad \big(P\vec{x} = (q_1,\dots,q_n)\big)\varphi(\vec{x},\vec{c}\,) \in s_\omega\,.$$

The axioms about probability guarantee that the μ_k are finitely additive probability measures with ranges in $[0,1]^n_{\mathbb{Q}}$. It is routine to check that $\mathfrak{A} \vDash \varphi\big([c_1],\dots,[c_n]\big)$ whenever $\varphi(c_1,\dots,c_n) \in s_\omega$. Therefore $\mathfrak{A}$ is a weak model of s_ω, and hence a model of s_0. □

Lemma 3.6. *Let $\langle\mathfrak{A},\mu\rangle$ be a weak model for $L_{\mathbb{A}P}$. Then the model $\big\langle{}^*\mathfrak{A},L(\mu)\big\rangle$, obtained by applying the Loeb process to $\langle{}^*\mathfrak{A},{}^*\mu\rangle$, is a probability model, and for each $\varphi(\vec{x}) \in \widetilde{L}_{\mathbb{A}P}$, $\vec{a} \in A$*

$$\langle\mathfrak{A},\mu\rangle \vDash \varphi[\vec{a}] \quad \textit{iff} \quad \big\langle{}^*\mathfrak{A},L(\mu_n)\big\rangle \vDash \varphi[\vec{a}]\,.$$

Proof. Let $V(S)$ be a superstructure over S and $\mathbb{R} \cup A \subseteq S$. We suppose that a formula $\varphi(\vec{x}, \vec{a})$ with parameters from A, a weak model $\mathfrak{A}$, and the relation $\models$ are represented by sets in $V(S)$. Then ${}^*\varphi(\vec{x}, \vec{a})$ and ${}^*\mathfrak{A}$ are sets in the nonstandard universe $V({}^*S)$, and ${}^*\models$ is an internal relation. If the context is clear we write simply $\models$.

At the beginning of this paper we proved that in this case it is possible to infer Loeb's process.

$\left\langle {}^*\mathfrak{A}, L(\mu) \right\rangle$ satisfies the Fubini property because $\langle \mathfrak{A}, \mu \rangle$ satisfies the Fubini property axioms (form which it follows that $\langle {}^*\mathfrak{A}, {}^*\mu \rangle$ satisfies them in the nonstandard sense), and because of the very mode of constructing measure on the model.

The main step in our proof is to show that for each $\varphi(\vec{x}) \in \widetilde{L}_{\mathbb{A}P}$ and $\vec{a} \in \mathbb{A}$

$$\langle \mathfrak{A}, \mu \rangle \models \varphi[\vec{a}] \quad \text{iff} \quad \left\langle {}^*\mathfrak{A}, L(\mu) \right\rangle \models \varphi[\vec{a}] \; . \tag{1}$$

To prove (1) we prove by induction on formulas that for $\varphi(\vec{x}, \vec{y}) \in \widetilde{L}_{\mathbb{A}P}$, $\vec{a} \in A$

$$L(\mu_k) \Big(\left\{ \vec{c} \in {}^*A \mid \left\langle {}^*\mathfrak{A}, L(\mu) \right\rangle \models \varphi[\vec{c}, \vec{a}] \right\} \vartriangle \left\{ \vec{c} \in {}^*A \mid {}^* \langle \mathfrak{A}, \mu \rangle \models {}^*\varphi[\vec{c}, \vec{a}\,] \right\} \Big) = \vec{0}.$$

Since in our logic $\widetilde{L}_{\mathbb{A}P}$ we have only finite formulas, the only nontrivial case for the last equality is $\varphi(\vec{y}) = \big(P\vec{x} \geq (r_1, \ldots, r_n)\big)\psi(\vec{x}, \vec{y})$.

From now on we shall suppress parameters from A.

Let $\varphi(\vec{y}) = \big(P\vec{x} \geq (r_1, \ldots, r_n)\big)\psi(\vec{x}, \vec{y})$. Then we have

$${}^*\Big(\big(P\vec{x} \geq (r_1, \ldots, r_n)\big)\psi(\vec{x}) \Big) \longleftrightarrow {}^*\mu_k \left\{ \vec{x} \mid {}^*\psi(\vec{x}) \right\} \geq (r_1, \ldots, r_n) \quad \text{in} \quad {}^*R^n$$

and

$$\big(P\vec{x} \geq (r_1, \ldots, r_n)\big)^*\psi(\vec{x}) \longleftrightarrow (\forall k \in \mathbb{N})\, {}^*\mu_k \big\{ \vec{x} \mid {}^*\psi(\vec{x}) \big\} \geq \big(r_1 - \tfrac{1}{k}, \ldots, r_n - \tfrac{1}{k}\big) \, .$$

On the basis of the rule of inference "measure continuity", we have (this is just the limited case)

$$\vec{0} = L(\mu_k) \Big\{ \vec{c} \in {}^*A \mid {}^*\langle \mathfrak{A}, \mu \rangle \models {}^*\Big(\big(P\vec{x} \geq (r_1, \ldots, r_n)\big)\psi(\vec{x}, \vec{c}) \Big) \\ \vartriangle \big(P\vec{x} \geq (r_1, \ldots, r_n)\big)^*\psi(\vec{x}, \vec{c}) \Big\} \, .$$

By the triangle argument

$$L\,(\mu_k)\left\{\vec{c}\mid {}^*\langle\mathfrak{A},\mu\rangle\vDash {}^*\varphi(\vec{c})\,\triangle\,\varphi(\vec{c})\right\}\le$$
$$\le L(\mu_k)\Big\{\vec{c}\mid {}^*\langle\mathfrak{A},\mu\rangle\vDash {}^*\Big(\big(P\vec{x}\ge(r_1,\dots,r_n)\big)\psi(\vec{x},\vec{c})\Big)\,\triangle\,\big(P\vec{x}\ge(r_1,\dots,r_n)\big)^*\psi(\vec{x},\vec{c})\Big\}+$$
$$+\,L(\mu_k)\Big(\Big\{\vec{c}\mid {}^*\langle\mathfrak{A},\mu\rangle\vDash\big(P\vec{x}\ge(r_1,\dots,r_n)\big)^*\psi(\vec{x},\vec{c})\Big\}\,\triangle$$
$$\triangle\,\Big\{\vec{c}\mid\big\langle{}^*\mathfrak{A},L(\mu)\big\rangle\vDash\big(P\vec{x}\ge(r_1,\dots,r_n)\big)\psi(\vec{x},\vec{c})\Big\}\Big)\,.$$

The first term is $\vec{0}$.

By applying the induction hypothesis:

$$L(\mu_{m+k})\Big(\big\{(\vec{d},\vec{c})\mid {}^*\langle\mathfrak{A},\mu\rangle\vDash {}^*\psi[\vec{d},\vec{c}]\big\}\,\triangle\,\big\{(\vec{d},\vec{c})\mid\langle{}^*\mathfrak{A},L(\mu)\rangle\vDash\psi[\vec{d},\vec{c}]\big\}\Big)=\vec{0}.$$

So, for all $\vec{c}$'s but a set of $L(\mu_k)$-measure $\vec{0}$ we have:

$$L(\mu_m)\Big(\big\{\vec{d}\mid {}^*\langle\mathfrak{A},\mu\rangle\vDash {}^*\psi[\vec{d},\vec{c}]\big\}\,\triangle\,\big\{\vec{d}\mid\langle{}^*\mathfrak{A},L(\mu)\rangle\vDash\psi[\vec{d},\vec{c}]\big\}\Big)=\vec{0}.$$

So, for all $\vec{c}$'s but a set of $L(\mu_k)$-measure $\vec{0}$ we have:

$$L(\mu_m)\big\{\vec{d}\mid {}^*\langle\mathfrak{A},\mu\rangle\vDash {}^*\psi[\vec{d},\vec{c}]\big\}\ge(r_1,\dots,r_n)$$

iff

$$L(\mu_m)\big\{\vec{d}\mid\big\langle{}^*\mathfrak{A},L(\mu)\big\rangle\vDash\psi[\vec{d},\vec{c}]\big\}\ge(r_1,\dots,r_n)\,.$$

Hence the second term in the inequality is $\vec{0}$.

If some definable set S in model $\langle\mathfrak{A},\mu\rangle$ had measure $\vec{q}_0$, then, as a consequence of (1), we have that set S will keep the same measure also in model $\langle{}^*\mathfrak{A},L(\mu)\rangle$, and *vice versa*, for any formula $\varphi(\vec{a})\in\widetilde{L}_{\mathbb{A}P}$. This concludes the proof of the lemma and theorem. □

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