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PERIODIC SOLUTIONS OF FOURTH-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN

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Abstract. Sufficient conditions for the existence of at least one periodic solution of a nonlinear fourth order functional differential equations with $p-$Laplacian are established. Examples are presented to illustrate the main result.

1. INTRODUCTION

Fourth-order differential equations with or without $p-$Laplacian occur in beam theory [1,3]. The solvability of such equations with different boundary conditions has been studied in papers [4-11,13-30,32-34]. The methods used in above mentioned papers are fixed point theorems in cones in Banach spaces [1,9,13,16,17,21,19,20,28,30,33], Mawhin′s continuation theorem of coincidence degree [11,26,23,27], upper and lower solutions methods and monotone iterative technique [4-6,8,15,32,34], Leray-Schauder fixed point theorem [7,13,25,28,33], and so on.

Properties of solutions of fourth order ordinary or functional differential equations are also studied by many authors, for example, Amster and Mariani [2] studied the oscillatory properties of solutions of a fourth order differential equation. In paper [31], Tanigawa established oscillation and non-oscillation theorems for a class of fourth order differential equations with $p-$Laplacian.

However, results on existence of periodic solutions of fourth order functional differential equations with $p-$Laplacian have not been found in known literature.

To fill this gap, in this paper, we use Mawhin′s continuation theorem of coincidence degree (Theorem IV.13 of [12]) to establish sufficient conditions for the existence of at least one periodic solution of the following fourth order functional differential equations with $p-$Laplacian

$$[q(t)\phi(x''(t))]'' = f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))),\ t \in R, \tag{1}$$

where $T > 0$ is a constant, $\tau_k \in C^1(R)$ for all $k = 1, \cdots, m$, f is continuous and is $T-$periodic about t, $q \in C^0(R)$ is positive and $T-$periodic, $\phi(x) = |x|^{p-2}x$ with $p > 1$, which is called a $p-$Laplacian, and its inverse function is $\phi^{-1}(x) = |x|^{q-2}x$ with $1/q + 1/p = 1$.

The remainder is divided into three sections. In Section 2, we present the main results. In Section 3, we give some examples to illustrate the main theorems.

2. MAIN RESULTS

Let PC^0 be the set of all continuous $T-$periodic functions on R and $X = PC^0 \times PC^0$, the norm is defined by $||(x, y)|| = \max\{\max_{t\in[0,T]} |x(t)|, \max_{t\in[0,T]} |y(t)|\}$ for $(x, y) \in X$. Then X is a Banach space.

Let $D(L) = \{(x, y) \in X : x'' \in PC^0, (qy)'' \in PC^0\}$. Define the linear operator $L : D(L) \cap X \to X$ by

$$L\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x''(t) \\ (q(t)y(t))'' \end{pmatrix} \text{ for all } (x, y) \in D(L) \cap X.$$

Define the nonlinear operator $N : X \to X$ by

$$N\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \phi^{-1}(y(t)) \\ f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))) \end{pmatrix} \text{ for all } (x, y) \in X.$$

It is easy to show the following results. We omit their proofs since the proofs are simple and standard.

(i) $\mathrm{Ker}L = \{(a, b/q(t)) : a, b \in R\}$;

(ii) $\mathrm{Im}L = \{(u, v) \in X : \int_0^T u(s)ds = 0, \int_0^T v(t)dt = 0\}$;

(iii) L is a Fredholm operator of index zero;

(iv) there exist projectors $P : X \to X$ and $Q : X \to X$ such that $\mathrm{Ker}L = \mathrm{Im}P$ and $\mathrm{Ker}Q = \mathrm{Im}L$. There is an isomorphism $\wedge : \mathrm{Ker}L \to X/\mathrm{Im}L$.

(v) Let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is $L-$compact on $\overline{\Omega}$;

(vi) $(x, y) \in D(L)$ is a solution of the operator equation $L(x, y) = N(x, y)$ implies that x is a $T-$periodic solution of equation (1).

Let $F(t) = f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))$. We have, for $a, b \in R$, $(x, y) \in X$ and $(u, v) \in Y$, that

$$\begin{aligned}
P\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} x(0) \\ q(0)y(0)/q(t) \end{pmatrix}, \\
Q\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} &= \begin{pmatrix} \dfrac{1}{T}\displaystyle\int_0^T u(t)dt \\ \dfrac{1}{T}\displaystyle\int_0^T v(t)dt \end{pmatrix}, \\
K_p\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} &= \begin{pmatrix} \displaystyle\int_0^t (t-s)u(s)ds - \frac{1}{T}\int_0^T (T-s)u(s)ds \\ \dfrac{1}{q(t)}\left(\displaystyle\int_0^t (t-s)v(s)ds - \frac{1}{T}\int_0^T (T-s)v(s)ds\right) \end{pmatrix}, \\
K_p(I-Q)N\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= K_p(I-Q)\begin{pmatrix} \phi^{-1}(y(t)) \\ f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))) \end{pmatrix} \\
&= \begin{pmatrix} \displaystyle\int_0^t (t-s)\phi^{-1}(y(s))ds - \frac{1}{T}\int_0^T (T-s)\phi^{-1}(y(s))ds \\ \dfrac{1}{q(t)}\left(\displaystyle\int_0^t (t-s)F(s)ds - \frac{1}{T}\int_0^T (T-s)F(s)ds\right) \end{pmatrix}
\end{aligned}$$

$$-\begin{pmatrix} \frac{t^2}{2T}\int_0^T \phi^{-1}(y(s))ds - \frac{T}{2}\int_0^T \phi^{-1}(y(s))ds \\ \frac{1}{q(t)}\left(\frac{t^2}{2T}\int_0^T F(s)ds - \frac{T}{2}\int_0^T F(s)ds\right) \end{pmatrix},$$

$$\wedge\begin{pmatrix} a \\ b/q(t) \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}.$$

Suppose

(B_1). There exist numbers $\beta > 0$, $\theta > 1$, nonnegative functions $p_i, r \in X$ $(i = 0, \cdots, m)$, functions $g(t, x_0, \cdots, x_m)$, $h(t, x_0, \cdots, x_m)$ such that

$$f(t, x_0, \cdots, x_m) = g(t, x_0, \cdots, x_m) + h(t, x_0, \cdots, x_m),$$

$$g(t, x_0, x_1, \cdots, x_m)x_0 \leq -\beta|x_0|^{\theta+1},$$

and

$$|h(t, x_0, \cdots, x_m)| \leq \sum_{i=0}^{m} p_i(t)|x_i|^\theta + r(t),$$

for all $t \in R$, $(x_0, x_1, \cdots, x_m) \in R^{m+1}$.

(B_2). There exists a positive constant δ such that $q(t) > \delta$ for all $t \in [0, T]$, and nonnegative constants M_i^0 such that $|\tau_i(T) - \tau_i(0)| \leq M_i^0 T$ for $i = 1, \cdots, m$.

Lemma 2.1. *Let $\delta_i = \max_{t\in[0,T]} |\mu_i'(t)|$, where μ_i is the inverse function of $\tau_i (i = 1, \cdots, m)$. Suppose (B_1) and (B_2) hold. Let $\Omega_1 = \{(x, y) : L(x, y) = \lambda N(x, y),\ ((x, y), \lambda) \in [(D(L) \setminus KerL)] \times (0, 1)\}$. Then Ω_1 is bounded if*

$$||p_0|| + \sum_{i=1}^{m} ||p_i|| M_i^0 \delta_i^{\frac{\theta}{\theta+1}} < \beta. \tag{2}$$

Proof. For $(x, y) \in \Omega_1$, we have $L(x, y) = \lambda N(x, y)$, $\lambda \in (0, 1)$, i.e.

$$\begin{cases} x''(t) = \lambda\phi^{-1}(y(t)), \\ (q(t)y(t))'' = \lambda f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))). \end{cases}$$

It follows that

$$[q(t)\phi(x''(t))]'' = \phi(\lambda)\lambda f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))). \tag{3}$$

Thus

$$[q(t)\phi(x''(t))]''x(t) = \phi(\lambda)\lambda f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))x(t).$$

Integrating it from 0 to T, we get

$$\int_0^T q(t)\phi(x''(t))x''(t)dt = \phi(\lambda)\lambda \int_0^T f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))x(t)dt, \quad (4)$$

together with (B_1) and $\int_0^T q(t)\phi(x''(t))x''(t)dt \geq 0$, we get that

$$\begin{aligned}
\beta \int_0^T |x(t)|^{\theta+1}dt &\leq -\int_0^T g(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))x(t)dt \\
&\leq \int_0^T h(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))x(t)dt \\
&\leq \int_0^T |h(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))||x(t)|dt \\
&\leq ||p_0|| \int_0^T |x(t)|^{\theta+1}dt + T^{\frac{\theta}{\theta+1}}||r|| \left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} \\
&\quad + \sum_{i=1}^m ||p_i|| \left(\int_0^T |x(\tau_i(t))|^{1+\theta}dt\right)^{\frac{\theta}{\theta+1}} \left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} \\
&\leq ||p_0|| \int_0^T |x(t)|^{\theta+1}dt + T^{\frac{\theta}{\theta+1}}||r|| \left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} \\
&\quad + \sum_{i=1}^m ||p_i|| \left|\int_{\tau_i(0)}^{\tau_i(T)} |x(s)|^{1+\theta} \frac{ds}{|\tau_i'(t)|}\right|^{\frac{\theta}{\theta+1}} \left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} \\
&\leq ||p_0|| \int_0^T |x(t)|^{\theta+1}dt + T^{\frac{\theta}{\theta+1}}||r|| \left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} \\
&\quad + \sum_{i=1}^m ||p_i||\delta_i^{\frac{\theta}{\theta+1}} M_i^0 \int_0^T |x(t)|^{\theta+1}dt.
\end{aligned}$$

Since

$$\beta > ||p_0|| + \sum_{i=1}^m \delta_i^{\frac{\theta}{\theta+1}} M_i^0 ||p_i||, \quad (5)$$

there is a constant $M_1 > 0$ such that $\int_0^T |x(t)|^{\theta+1}dt \leq M_1$. So there is $\xi \in [0, T]$ such that $|x(\xi)| \leq (M_1/T)^{\frac{1}{\theta+1}}$. Further more we have

$$\begin{aligned}
\int_0^T q(t)|x''(t)|^p dt &= \int_0^T q(t)\phi(x''(t))x''(t)dt \\
&= \phi(\lambda)\lambda \int_0^T f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))x(t)dt
\end{aligned}$$

$$
\begin{aligned}
&= \phi(\lambda)\lambda\int_0^T g(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t)))x(t)dt\\
&\quad +\phi(\lambda)\lambda\int_0^T h(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t)))x(t)dt\\
&\le \phi(\lambda)\lambda\int_0^T h(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t)))x(t)dt\\
&\le \int_0^T |h(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t)))||x(t)|dt\\
&\le ||p_0||\int_0^T |x(t)|^{\theta+1}dt + T^{\frac{\theta}{\theta+1}}||r||\left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}\\
&\quad +\sum_{i=1}^m ||p_i||\delta_i^{\frac{\theta}{\theta+1}}M_i^0\int_0^T |x(t)|^{\theta+1}dt\\
&\le ||p_0||M_1 + T^{\frac{\theta}{\theta+1}}||r||M_1^{\frac{1}{\theta+1}} + \sum_{i=1}^m ||p_i||\delta_i^{\frac{\theta}{\theta+1}}M_i^0 M_1.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
|x(t)| &= \left|x(\xi)+\int_\xi^t x'(s)ds\right| \le (M_1/T)^{\frac{1}{\theta+1}} + \int_0^T |x''(t)|dt\\
&\le (M_1/T)^{\frac{1}{\theta+1}} + \frac{T^{\frac{p-1}{p}}}{\mu^{\frac{1}{p}}}\left(\mu\int_0^T |x''(t)|^p dt\right)^{\frac{1}{p}}\\
&\le (M_1/T)^{\frac{1}{\theta+1}} + \frac{T^{\frac{p-1}{p}}}{\mu^{\frac{1}{p}}}\left(\int_0^T q(t)|x''(t)|^p dt\right)^{\frac{1}{p}}\\
&\le (M_1/T)^{\frac{1}{\theta+1}} + \frac{T^{\frac{p-1}{p}}}{\mu^{\frac{1}{p}}}\left(||p_0||M_1 + T^{\frac{\theta}{\theta+1}}||r||M_1^{\frac{1}{\theta+1}} + \sum_{i=1}^m ||p_i||\delta_i^{\frac{\theta}{\theta+1}}M_i^0 M_1\right)^{\frac{1}{p}}
\end{aligned}
$$

Hence there is a constant $M_2 > 0$ such that $||x|| \le M_2$.

It is easy to show that there are $\xi,\eta \in [0,T]$ such that $y'(\xi) = 0$ and $y(\eta) = 0$. Hence

$$
\begin{aligned}
|[q(t)y(t)]'| &= \left|\int_\xi^t [q(t)y(t)]''dt\right| \le \int_0^T |[q(t)y(t)]''|dt\\
&\le \int_0^T |f(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t)))|dt\\
&\le T\max_{t\in[0,T],|x_i|\le M_2,i=0,\cdots,m}|f(t,x_0,\cdots,x_m)|.
\end{aligned}
$$

So

$$
|q(t)y(t)| \le \int_0^T |(q(t)y(t))'|dt \le T^2\max_{t\in[0,T],|x_i|\le M_2,i=0,\cdots,m}|f(t,x_0,\cdots,x_m)|,\quad t\in[0,T].
$$

Then

$$|y(t)| \leq \frac{T^2}{\mu} \max_{t\in[0,T],|x_i|\leq M_2,i=0,\cdots,m} |f(t,x_0,\cdots,x_m)|, \ \ t \in [0,T].$$

This implies that

$$||y|| \leq \frac{T^2}{\mu} \max_{t\in[0,T],|x_i|\leq M_2,i=0,\cdots,m} |f(t,x_0,\cdots,x_m)|, \ \ t \in [0,T].$$

It follows that, for $(x,y) \in \Omega_1$, one has that there is $H > 0$ such that $||(x,y)|| \leq H$. Hence Ω_1 is bounded. □

Suppose

(B_3). There exists a constant $M > 0$ such that

$$a\int_0^T f(t,a,\cdots,a)dt > 0 \text{ for all } |a| > M$$

or

$$a\int_0^T f(t,a,\cdots,a)dt < 0 \text{ for all } |a| > M.$$

Lemma 2.2. *Suppose (B_3) holds. Then $\Omega_2 = \{(x,y) \in KerL: \ N(x,y) \in ImL\}$ is bounded.*

Proof. For $(a, b/q(t)) \in \text{Ker}L$, we have $N(a,b) = (\phi^{-1}(b/q(t)), f(t,a,\cdots,a))$. $Nx \in \text{Im}L$ implies that

$$\int_0^T \phi^{-1}(b/q(t))dt = 0, \quad \int_0^T f(t,a,\cdots,a)dt = 0.$$

It follows from condition (B_3) that $|a| \leq M$ and $b = 0$. Thus Ω_2 is bounded. □

Lemma 2.3. *Suppose (B_3) holds. Then either $\Omega_3 = \{(x,y) \in KerL: \ \lambda \wedge (x,y) + (1-\lambda)QN(x,y) = 0, \ \lambda \in [0,1]\}$ or $\Omega_3 = \{(x,y) \in KerL: \ -\lambda \wedge (x,y) + (1-\lambda)QN(x,y) = 0, \ \lambda \in [0,1]\}$ is bounded, where $\wedge : KerL \to Y/ImL$ defined by $\wedge(a, b/q(t)) = (b,a)$.*

Proof. If the first inequality of (B_3) holds, consider

$$\Omega_3 = \{(x,y) \in \text{Ker}L: \ \lambda \wedge (x,y) + (1-\lambda)QN(x,y) = 0, \ \lambda \in [0,1]\}.$$

We will prove that Ω_3 is bounded. For $(a, b/q(t)) \in \Omega_3$, and $\lambda \in [0,1]$, we have

$$-(1-\lambda)\int_0^T \phi^{-1}(b/q(t))dt = \lambda bT, \ \ -(1-\lambda)\int_0^T f(t,a,\cdots,a)dt = \lambda aT.$$

If $\lambda = 1$, then $a = b = 0$. If $\lambda \neq 1$, and $|a| > M$, it follows from (B_3) that

$$0 \geq -(1-\lambda)a\int_0^T f(t,a,\cdots,a)dt = \lambda a^2 T > 0,$$

a contradiction. So $|a| \leq M$. Similarly, we get $|b| \leq M$. Hence Ω_3 is bounded.

If the second inequality of (B_3) holds, consider

$$\Omega_3 = \{(x,y) \in \mathrm{Ker}L : \ -\lambda \wedge (x,y) + (1-\lambda)QN(x,y) = 0, \ \lambda \in [0,1]\},$$

Similarly, we can get a contradiction. So Ω_3 is bounded. $\square$

Theorem L. *Suppose that (B_1), (B_2) and (B_3) hold. Then equation (1) has at least one $T-$periodic solution if (2) holds.*

Proof. We know that L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. Since (x,y) is a solution of $L(x,y) = N(x,y)$ implies that x is a solution of equation (1). It suffices to get a solution (x,y) of $L(x,y) = N(x,y)$. To do this, we construct an open bounded set Ω such that (i), (ii) and (iii) of Theorem IV.13 of [12] hold.

Set Ω be a open bounded subset of X centered at zero such that $\Omega \supset \cup_{i=1}^3 \overline{\Omega_i}$. By the definition of Ω, we have $\Omega \supset \overline{\Omega_1}$ and $\Omega \supset \overline{\Omega_2}$, thus, from Lemma 2.1 and Lemma 2.2, that $L(x,y) \neq \lambda N(x,y)$ for $(x,y) \in D(L) \setminus \mathrm{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$; $N(x,y) \notin \mathrm{Im}L$ for $(x,y) \in \mathrm{Ker}L \cap \partial\Omega$.

In fact, let $H((x,y),\lambda) = \pm\lambda \wedge (x,y) + (1-\lambda)QN(x,y)$. According the definition of Ω, we know $\Omega \supset \overline{\Omega_3}$, thus $H((x,y),\lambda) \neq 0$ for $(x,y) \in \partial\Omega \cap \mathrm{Ker}L$, thus, from Lemma 2.3, by homotopy property of degree,

$$\begin{aligned} & \deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) = \deg(H(\cdot,0), \Omega \cap \mathrm{Ker}L, 0) \\ = \ & \deg(H(\cdot,1), \Omega \cap \mathrm{Ker}L, 0) = \deg(\pm\wedge, \Omega \cap \mathrm{Ker}L, 0) \neq 0 \text{ since } 0 \in \Omega. \end{aligned}$$

Thus by Theorem IV.13 of [12], $L(x,y) = N(x,y)$ has at least one solution in $D(L) \cap \overline{\Omega}$, then x is a $T-$solution of equation (1). The proof is completed. $\square$

3. EXAMPLES

In this section, we present examples to illustrate the main result in section 2.

Example 3.1. Consider the problem

$$x''''(t) = -\frac{[x(t)]^{\frac{3}{5}}}{1+2[\sin x(t)]^8} + \sum_{i=1}^{m} p_i(t)[x(t-\tau_i)]^{\frac{3}{5}} + r(t), \tag{6}$$

where $p_i, r \in X$ and p_i are all non-positive or all non-negative with $\sum_{i=1}^{m}\int_0^T p_i(t)dt < 0$ or $\sum_{i=1}^{m}\int_0^T p_i(t)dt > 0$, $\tau_i > 0 (i = 1, \cdots, m)$ are constants. Corresponding to the assumptions of Theorem L, we set

$$g(t, x_0, x_1, \cdots, x_m) = -\frac{x_0^{\frac{3}{5}}}{1+2(\sin x_0)^8},$$

and

$$h(t, x_0, \cdots, x_m) = \sum_{i=1}^{m} p_i(t) x_i^{\frac{3}{5}} + r(t)$$

and $\beta = 1/3$, $p = 2$, $\theta = 3/5$. It is easy to see that (B_1) holds, and

$$c\int_0^T f(t, c, \cdots, c)dt = \int_0^T \left(-\frac{c^2}{1+2c} + \sum_{i=1}^{m} p_i(t) c^{\frac{8}{5}} + cr(t) \right) dt$$

implies that there is $M > 0$ such that $c\int_0^T f(t, c, \cdots, c)dt > 0$ for all $|c| > M$ if $\sum_{i=1}^{m}\int_0^T p_i(t)dt > 0$ or $c\int_0^T f(t, c, \cdots, c)dt < 0$ for all $|c| > M$ if $\sum_{i=1}^{m}\int_0^T p_i(t)dt < 0$. So (B_3) holds. It is easy to see that $\delta_i = T + \tau_i, M_i^0 = 1$, it follows that (B_2) holds. It follows from Theorem L that (6) has at least one $T-$periodic solution if

$$\sum_{i=1}^{m} (T+\tau_i)^{\frac{3}{8}} ||p_i|| < \frac{1}{3}.$$

Example 3.2. Consider the problem

$$[((\sin t)^2 + 2)\phi(x''(t))]'' = -\frac{[x(t)]^5}{1+2[\sin x(t)]^8} + \sum_{i=1}^{m} p_i(t)[x(t-\tau_i)]^5 + r(t), \tag{7}$$

where $\phi(x) = |x|^4 x$, $q(t) = (\sin t)^2 + 2$, which means that $\mu = 2$, $p_i, r \in X$ and p_i are all non-positive or all non-negative with $\sum_{i=1}^{m}\int_0^T p_i(t)dt < 0$ or $\sum_{i=1}^{m}\int_0^T p_i(t)dt > 0$. Corresponding to the assumptions of Theorem L, we set

$$g(t, x_0, x_1, \cdots, x_m) = -\frac{x_0^5}{1+2(\sin x_0)^8},$$

and

$$h(t, x_0, \cdots, x_m) = \sum_{i=1}^{m} p_i(t) x_i^5 + r(t)$$

and $\beta = 1/3$, $p = 6$, $\theta = 5$. It is easy to see that $\delta_i = T + \tau_i$, $M_i^0 = 1$, it follows that (B_2) holds. It follows from Theorem L that problem (7) has at least one solution if

$$\frac{1}{3} > \sum_{i=0}^{m} ||p_i||(T + \tau_i)^{\frac{5}{6}}.$$

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