THE FIRST TO \((k + 1)\)-TH SMALLEST WIENER
(HYPER-WIENER) INDICES OF CONNECTED GRAPHS

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Abstract. Let \(n\) and \(k\) be two nonnegative integers with \(n > 2k\), this paper presents the
first to \((k + 1)\)-th smallest Wiener indices, and the first to \((k + 1)\)-th smallest hyper-Wiener
indices among all connected graphs of order \(n\), respectively.

1. INTRODUCTION

Throughout this paper, we only concern with connected, undirected simple graphs.
Let \(\mathcal{G}(n)\) denote the set of all connected graphs of order \(n\). Let \(uv\) be an edge with
end vertices \(u\) and \(v\).

The distance \(d_G(u, v)\) between the vertices \(u\) and \(v\) of the graph \(G\) is equal to the
length of the shortest path that connects \(u\) and \(v\). Let \(\gamma(G, k)\) denote the number
of vertex pairs of $G$, whose distance is equal to $k$. There are two important graph-
based structure-descriptors, called Wiener index and hyper-Wiener index, based on
distances in a graph. The *Wiener index* $W(G)$ [1] is denoted by

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) = \sum_{k \geq 1} k\gamma(G,k),$$

and the *hyper-Wiener index* $WW(G)$ [2] is defined as

$$WW(G) = \frac{1}{2}W(G) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)^2 = \frac{1}{2} \sum_{k \geq 1} k(k + 1)\gamma(G,k).$$

The Wiener index was introduced long time ago [1]. Its chemical applications and
mathematical properties were stated in [3-5]. However, the hyper-Wiener index is
defined much later [2]. It rapidly gained popularity and numerous results on it were
raised [6-10].

Up to now, lots of relations between $W(G)$ and $WW(G)$ have been discovered
[11-13]. Suppose $n$ and $k$ are two nonnegative integers with $n > 2k$, this paper will
present the first to $(k + 1)$-th smallest Wiener indices, and the first to $(k + 1)$-th
smallest hyper-Wiener indices among all connected graphs of order $n$, respectively.
It’s surprising that the graphs which reach the $i$-th smallest Wiener indices even share
the $i$-th smallest hyper-Wiener indices of $G(n)$ for every $i \in \{1, 2, \ldots, k + 1\}$.

2. THE FIRST TO $(k + 1)$-TH SMALLEST WIENER INDICES OF $G(n)$

A graph $H$ is called a *subgraph* of $G$, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H$ is a connected subgraph of $G$ with $V(H) = V(G)$ and $E(H) \neq E(G)$,
then we called $H$ a **connected spanning subgraph** of $G$. Suppose that $V'$ is a nonempty
subset of $V$. The subgraph of $G$ whose vertex set is $V'$ and whose edge set is the set
of those edges of $G$ that have both ends in $V'$ is called the subgraph of $G$ induced by
$V'$ and is denoted by $G[V']$, and we say that $G[V']$ is an **induced subgraph** of $G$.

**Lemma 1.** If $G'$ is a connected spanning subgraph of $G$, then $W(G') > W(G)$. 

Proof. Suppose $P_{u,v}$ is a shortest path in $G'$ that connects $u$ and $v$, since $G'$ is a connected spanning subgraph of $G$, then $P_{u,v}$ is a path in $G$ that connects $u$ and $v$, thus $d_{G'}(u,v) \geq d_G(u,v)$. This implies that

$$W(G') = \sum_{\{u,v\} \subseteq V(G')} d_{G'}(u,v) \geq \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) = W(G).$$

Moreover, since $G'$ is a connected spanning subgraph of $G$, then $E(G') \subseteq E(G)$ but $E(G') \neq E(G)$. Assume $e = v_1v_2 \in E(G)$, but $e \notin E(G')$, then $d_{G'}(v_1, v_2) \geq 2 > 1 = d_G(v_1, v_2)$. Thus, $W(G') > W(G)$.

The join of two vertex disjoint graphs $G_1$ and $G_2$, denoted by $G_1 \vee G_2$, is the graph having vertex set $V(G_1 \vee G_2) = V(G_1 \cup G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

Lemma 2. Let $G = G_1 \vee G_2$ with $|V(G)| = n$ and $|E(G)| = \binom{n}{2} - k$, where $k$ is a nonnegative integer, then $W(G) = \binom{n}{2} + k$.

Proof. Since $G = G_1 \vee G_2$, then $d_G(u,v) \leq 2$ holds for each vertex pair $\{u,v\} \subseteq V(G)$. Moreover, $d_G(u,v) = 1$ if and only if $uv \in E(G)$, and $d_G(u,v) = 2$ if and only if $uv \notin E(G)$. Thus, $\gamma(G,2) = k$. By the definition, we have

$$W(G) = \sum_{k \geq 1} k\gamma(G,k) = \gamma(G,1) + 2\gamma(G,2) = \binom{n}{2} - k + 2k = \binom{n}{2} + k.$$

This implies the result. $\square$

Suppose $k$ is a nonnegative integer, the notation $S(K_n - ke)$ denotes the set of all connected graphs raised from $K_n$ by deleting $k$ edges. By the definition, it follows that $S(K_n - 0e) = \{K_n\}$.

Lemma 3. If $H \in S(K_n - te)$ ($t \geq 1$), then there exists $G \in S(K_n - (t-1)e)$ such that $H$ is a connected spanning subgraph of $G$.

Proof. Suppose $H$ is obtained from $K_n$ by deleting the edges $e_1, \ldots, e_t$, then let $G$ be the graph raised from $K_n$ by deleting the edges $e_1, \ldots, e_{t-1}$. Since $H$ is connected, then $G$ is also connected. Thus, $G \in S(K_n - (t-1)e)$. It is easy to see that $H$ is a spanning connected subgraph of $G$. This completes the proof. $\square$
Theorem 1. Suppose $n$ and $k$ are nonnegative integers. If $n > 2k$, then the first to $(k + 1)$-th smallest Wiener indices of $G(n)$ is $\binom{n}{2}, \binom{n}{2} + 1, \ldots, \binom{n}{2} + k$. Moreover, $W(G) = \binom{n}{2} + i$ if and only if $G \in S(K_n - ie)$, where $0 \leq i \leq k$.

Proof. Clearly, $W(K_n) = \binom{n}{2}$. If $1 \leq i \leq k$ and $G \in S(K_n - ie)$, let $V_1 = \{v : v \in V(G)$ and $d(v) = n-1\}$, since $|E(G)| = \binom{n}{2} - i$ and $n > 2k \geq 2i$, then $V_1 \neq \emptyset$. Let $V_2 = V(G) \setminus V_1$, clearly $V_2 \neq \emptyset$. Set $G_1 = G[V_1]$ and $G_2 = G[V_2]$, then $G = G_1 \cup G_2$. By Lemma 2, we have $W(G) = \binom{n}{2} + i$.

Next we shall prove that if $H \in \mathcal{G}(n) \setminus \{K_n, S(K_n - e), \ldots, S(K_n - ke)\}$, then $W(H) > \binom{n}{2} + k$. Once this is proved, the conclusion holds. By Lemma 1, we only need to show that for each $H \in \mathcal{G}(n) \setminus \{K_n, S(K_n - e), \ldots, S(K_n - ke)\}$, there exists one graph $G \in S(K_n - ke)$ such that $H$ is a connected spanning subgraph of $G$. Now suppose $H \in S(K_n - te)$, where $t > k$, by Lemma 3 there exists some graph $H_1 \in S(K_n - (t - 1)e)$ such that $H$ is a connected spanning subgraph of $H_1$. Once again, by Lemma 3 there exists one graph $H_2 \in S(K_n - (t - 2)e)$ such that $H_1$ is a connected spanning subgraph of $H_2$. By the definition, $H$ is also a connected spanning subgraph of $H_2$. Repeat the above process, we can conclude that there must exist one graph $G \in S(K_n - ke)$ such that $H$ is a connected spanning subgraph of $G$. By Lemma 1, $W(H) > W(G) = \binom{n}{2} + k$. This completes the proof. \hfill \Box

By Theorem 1, the next Corollary follows at once.

Corollary 1. If $n > 18$, then the first to tenth smallest Wiener indices of $G(n)$ is $\binom{n}{2}, \binom{n}{2} + 1, \ldots, \binom{n}{2} + 9$. Moreover, $W(G) = \binom{n}{2} + i$ if and only if $G \in S(K_n - ie)$, where $0 \leq i \leq 9$.

3. THE FIRST TO $(k + 1)$-TH SMALLEST HYPER-WIENER INDICES OF $G(n)$

Suppose $k$ is a nonnegative integer. In this section, we shall list the first to $(k + 1)$-th smallest hyper-Wiener indices of $G(n)$ for any $n > 2k$.

Lemma 4. If $G'$ is a connected spanning subgraph of $G$, then $WW(G') > WW(G)$.
Proof. Suppose $P_{u,v}$ is a shortest path in $G'$ that connects $u$ and $v$, since $G'$ is a connected spanning subgraph of $G$, then $P_{u,v}$ is a path in $G$ that connects $u$ and $v$, thus $d_{G'}(u, v) \geq d_G(u, v)$ and $d_{G'}(u, v)^2 \geq d_G(u, v)^2$. By Lemma 1 it follows that

$$\sum_{\{u,v\} \subseteq V(G')} d_{G'}(u, v) > \sum_{\{u,v\} \subseteq V(G)} d_G(u, v),$$

and

$$\sum_{\{u,v\} \subseteq V(G')} d_{G'}(u, v)^2 \geq \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)^2.$$

Thus,

$$WW(G') = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G')} d_{G'}(u, v) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(G')} d_{G'}(u, v)^2 > \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d_G(u, v) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)^2 = WW(G).$$

This completes the proof of this lemma.

Theorem 2. Suppose $n$ and $k$ are nonnegative integers. If $n > 2k$, then the first to $(k+1)$-th smallest hyper-Wiener indices of $G(n)$ is \(\binom{n}{2}\), \(\binom{n}{2} + 2\), \ldots, \(\binom{n}{2} + 2k\). Moreover, $WW(G) = \binom{n}{2} + 2i$ if and only if $G \in S(K_n - ie)$, where $0 \leq i \leq k$.

Proof. Clearly, $WW(K_n) = \binom{n}{2}$. If $1 \leq i \leq k$ and $G \in S(K_n - ie)$, let $V_1 = \{v : v \in V(G) \text{ and } d(v) = n - 1\}$, and $V_2 = V(G) \setminus V_1$. By the proof of Theorem
1, $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$. Set $G_1 = G[V_1]$ and $G_2 = G[V_2]$, then $G = G_1 \vee G_2$. By Lemma 5, we have $WW(G) = \binom{n}{2} + 2i$.

Next we shall prove that if $H \in \mathcal{G}(n) \setminus \{K_n, S(K_n - e), \ldots, S(K_n - ke)\}$, then $WW(H) > \binom{n}{2} + 2k$. By the proof of Theorem 1, we can conclude that for each $H \in \mathcal{G}(n) \setminus \{K_n, S(K_n - e), \ldots, S(K_n - ke)\}$, there exists one graph $G \in S(K_n - ke)$ such that $H$ is a connected spanning subgraph of $G$. Thus, $WW(H) > WW(G) = \binom{n}{2} + 2k$ by Lemma 4. By combining the above discussion, the results follow.

By Theorem 2, the next Corollary follows at once.

**Corollary 2.** If $n > 18$, then the first to tenth smallest hyper-Wiener indices of $\mathcal{G}(n)$ is $\binom{n}{2}$, $\binom{n}{2} + 2$, $\ldots$, $\binom{n}{2} + 18$. Moreover, $W(G) = \binom{n}{2} + 2i$ if and only if $G \in S(K_n - ie)$, where $0 \leq i \leq 9$.

**Remark.** Suppose $n$ and $k$ are two nonnegative integers with $n > 2k$. Theorem 1 and Theorem 2 imply that the graphs which reach the $i$-th smallest Wiener indices even share the $i$-th smallest hyper-Wiener indices of $\mathcal{G}(n)$ for each $i \in \{1, 2, \ldots, k+1\}$.

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**References**


