D-EQUIENERGETIC SELF-COMPLEMENTARY GRAPHS

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Abstract. The \(D\)-eigenvalues \(\{\mu_1, \mu_2, \ldots, \mu_n\}\) of a graph \(G\) are the eigenvalues of its distance matrix \(D\) and form the \(D\)-spectrum of \(G\) denoted by \(\text{spec}_D(G)\). The \(D\)-energy \(E_D(G)\) of the graph \(G\) is the sum of the absolute values of its \(D\)-eigenvalues. We describe here the distance spectrum of some self-complementary graphs in the terms of their adjacency spectrum. These results are used to show that there exists \(D\)-equienergetic self-complementary graphs of order \(n = 48t\) and \(24(2t + 1)\) for \(t \geq 4\).

1. INTRODUCTION

Let \(G\) be a simple graph on \(n\) vertices and let \(\lambda_1, \ldots, \lambda_n\) be the eigenvalues of its adjacency matrix \(A\). The energy of a graph is defined as

\[ E = E(G) = \sum_{i=1}^{n} |\lambda_i| . \]
For details on this currently much studied graph–spectral invariant see [4, 5, 6]. After the introduction of the analogous concept of Laplacian energy [7], it was recognized [1] that other energy-like invariants can be defined as well, among them the distance energy.

Let $G$ be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The distance matrix $D = D(G)$ of $G$ is defined so that its $(i, j)$-entry $d_{ij}$ is equal to $d_G(v_i, v_j)$, the distance between the vertices $v_i$ and $v_j$ of $G$. The eigenvalues of $D(G)$ are said to be the $D$-eigenvalues of $G$ and form the $D$-spectrum of $G$, denoted by $\text{spec}_D(G)$. Since the distance matrix is symmetric, all its eigenvalues $\mu_i$, $i = 1, 2, \ldots, n$, are real and can be labelled so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

The $D$-energy, $E_D(G)$, of $G$ is then defined as

$$E_D(G) = \sum_{i=1}^{n} |\mu_i|.$$  \hspace{1cm} (1)

The concept of $D$-energy, Eq. (1), is recently introduced [11]. This definition was motivated by the much older and nowadays extensively studied graph energy. This invariant was studied by Consonni and Todeschini [1] for possible use in QSPR modelling. Their study showed, among others, that the distance energy is a useful molecular descriptor, since the values of $E_D(G)$ or $E_D(G)/n$ appear among the best univariate models for the motor octane number of the octane isomers and for the water solubility of polychlorobiphenyls. For some recent works on $D$-spectrum and $D$-energy of graphs see [8, 9, 10, 11, 13].

Two graphs with equal $D$-energy are said to be $D$-equienergetic. $D$-cospectral graphs are evidently $D$-equienergetic. Therefore, in what follows we focus our attention to $D$-equienergetic non-$D$-cospectral graphs. In this paper we search for self-complementary graphs of this kind. A similar work on pairs of ordinary equienergetic self-complementary graphs is [12].

All graphs considered in this paper are simple and we follow [2] for spectral graph theoretic terminology. We shall need:

**Lemma 1.** [2] Let $G$ be an $r$-regular connected graph, with
\[ \text{spec}(G) = \{r, \lambda_2, \ldots, \lambda_n\}. \text{ Then} \]
\[ \text{spec}(L^2(G)) = \begin{pmatrix} 4r - 6 & \lambda_2 + 3r - 6 & \cdots & \lambda_n + 3r - 6 & 2r - 6 & -2 \\ 1 & 1 & \cdots & 1 & \frac{n(r-2)}{2} & \frac{n(r-2)}{2} \end{pmatrix}. \]

Let \( G \) be a graph. Then the following construction \[3\] results in a self-complementary graph \( \mathcal{H} \). Recall that a graph \( \mathcal{H} \) is said to be self-complementary if \( \mathcal{H} \cong \overline{\mathcal{H}} \), where \( \overline{\mathcal{H}} \) is the complement of \( \mathcal{H} \).

**Construction of \( \mathcal{H} \):**
Replace each of the end vertices of \( P_4 \), the path on 4 vertices, by a copy of \( G \) and each of the internal vertices by a copy of \( \overline{G} \). Join the vertices of these graphs by all possible edges whenever the corresponding vertices of \( P_4 \) are adjacent.

### 2. DISTANCE SPECTRUM OF \( \mathcal{H} \)

**Theorem 1.** Let \( G \) be a connected \( k \)-regular graph on \( n \) vertices, with an adjacency matrix \( A \) and spectrum \( \{k, \lambda_2, \ldots, \lambda_n\} \). Then the distance spectrum of \( \mathcal{H} \) consists of \(- (\lambda_i + 2)\) and \( \lambda_i - 1 \), \( i = 2, 3, \ldots, n \), each with multiplicity 2, together with the numbers

\[ \frac{1}{2} \left[ 7n - 3 \pm \sqrt{(2k + 1)^2 + 45n^2 - 12nk - 6n} \right] \]

and

\[ \frac{1}{2} \left[ n + 3 \pm \sqrt{(2k + 1)^2 + 5n^2 + 4nk + 2n} \right]. \]

**Proof.** Let \( G \) be a connected \( k \)-regular graph on \( n \) vertices with an adjacency matrix \( A \) and spectrum \( \{k, \lambda_2, \ldots, \lambda_n\} \). Let \( \mathcal{H} \) be the self-complementary graph obtained from \( G \) by the above construction. Then the distance matrix \( D \) of \( \mathcal{H} \) has the form

\[ \begin{pmatrix}
 2(J - I) - A & J & 2J & 3J \\
 J & J - I + A & J & 2J \\
 2J & J & J - I + A & J \\
 3J & 2J & J & 2(J - I) - A
\end{pmatrix}. \]
As a regular graph, $G$ has the all-one vector $j$ as an eigenvector corresponding to eigenvalue $k$, while all other eigenvectors are orthogonal to $j$. Also corresponding to the eigenvalue $\lambda \neq k$ of $G$, $\overline{G}$ has the eigenvalue $-1 - \lambda$ such that both $\lambda$ and $-1 - \lambda$ have same multiplicities and eigenvectors.

Let $\lambda$ be an arbitrary eigenvalue of the adjacency matrix of $G$ with corresponding eigenvector $x$, such that $j^T x = 0$. Then $(\begin{array}{c} x \\ 0 \\ 0 \end{array})^T$ and $(\begin{array}{c} 0 \\ 0 \\ x \end{array})^T$ are the eigenvectors of $D$ corresponding to eigenvalue $-\lambda - 2$. Corresponding to an arbitrary eigenvalue $\lambda$ of $G$, $-\lambda - 2$ is an eigenvalue of $D$ with multiplicity 2. Similarly $(\begin{array}{c} 0 \\ x \\ 0 \end{array})^T$ and $(\begin{array}{c} 0 \\ 0 \\ x \end{array})^T$ are the eigenvectors of $D$ corresponding to the eigenvalue $\lambda - 1$.

In this way, forming eigenvectors of the form

$$(\begin{array}{c} x \\ 0 \\ 0 \end{array})^T, (\begin{array}{c} 0 \\ x \\ 0 \end{array})^T, (\begin{array}{c} 0 \\ 0 \\ x \end{array})^T$$

we can construct a total of $4(n - 1)$ mutually orthogonal eigenvectors of $D$. All these eigenvectors are orthogonal to the vectors

$$(\begin{array}{c} j \\ 0 \\ 0 \end{array})^T, (\begin{array}{c} 0 \\ j \\ 0 \end{array})^T, (\begin{array}{c} 0 \\ 0 \\ j \end{array})^T.$$  

The four remaining eigenvectors of $D$ are of the form $\Psi = (\alpha j, \beta j, \gamma j, \delta j)^T$ for some $(\alpha, \beta, \gamma, \delta) \neq (0, 0, 0, 0)$.

Now, suppose that $\nu$ is an eigenvalue of $D$ with an eigenvector $\Psi$. Then from $D\Psi = \nu\Psi$, we get

$$[2(n - 1) - k] \alpha + n\beta + 2n\gamma + 3n\delta = \nu\alpha \quad (2)$$
$$n\alpha + (n - 1 + k) \beta + n\gamma + 2n\delta = \nu\beta \quad (3)$$
$$2n\alpha + n\beta + (n - 1 + k) \gamma + n\delta = \nu\gamma \quad (4)$$
$$3n\alpha + 2n\beta + n\gamma + [2(n - 1) - k] \delta = \nu\delta. \quad (5)$$

Claim: $\alpha \neq 0$. If $\alpha = 0$, then by solving equations (3)–(5) we get $\beta = g_1 \gamma$ and $\delta = g_2 \gamma$ for some constants $g_1$ and $g_2$. Then using $\beta + 2\gamma + 3\delta = 0$, we obtain

$$[11n^2 + n(4k + 2) + 12k^2 + 12k + 3] \gamma = 0$$
which implies that $\gamma = \beta = \delta = 0$, which is impossible.

Thus $\alpha \neq 0$ and without loss of generality we may set $\alpha = 1$.

Then by solving equations (3)–(5) for $\beta, \gamma$, and $\delta$, and substituting these values into equation (2), we arrive at a biquadratic equation in $\nu$:

$$\left[\nu^2 - (7n - 3)\nu + n(n + 3k - 9) - (k^2 + k - 2)\right]$$

$$\times \left[\nu^2 + (n + 3)\nu - n(n + k - 1) - (k^2 + k - 2)\right] = 0$$

whose solutions

$$\frac{1}{2} \left[7n - 3 \pm \sqrt{(2k + 1)^2 + 45n^2 - 12nk - 6n}\right]$$

and

$$-\frac{1}{2} \left[n + 3 \pm \sqrt{(2k + 1)^2 + 5n^2 + 4nk + 2n}\right]$$

as easily seen, represent the four remaining eigenvalues of $D$. Hence the theorem. □

**Corollary 1.** Let $G$ be a connected $k$-regular graph on $n$ vertices with an adjacency matrix $A$ and spectrum $\{k, \lambda_2, \ldots, \lambda_n\}$. Let $H$ be the self-complementary graph obtained from $G$ by the above described construction. Then

$$E_D(H) = 7n - 3 + \sqrt{(2k + 1)^2 + 5n^2 + 4nk + 2n} + \sum_{i=2}^{n} |\lambda_i + 2| + \sum_{i=2}^{n} |\lambda_i - 1| .$$

3. A PAIR OF $D$-EQUIENERGETIC SELF-COMPLEMENTARY GRAPHS

In this section we demonstrate the existence of a pair of $D$-equienergetic self-complementary graphs on $n$ vertices for $n = 48t$ and $n = 24(2t + 1)$ for all $t \geq 4$. For this we first prove:

**Theorem 2.** For every $n \geq 8$, there exists a pair of $4$-regular non-cospectral graphs on $n$ vertices.
Proof. We shall consider the following two cases.

Case 1: \( n = 2t \), \( t \geq 4 \). In this case form two \( t \)-cycles \( u_1 u_2 \ldots u_t \) and \( v_1 v_2 \ldots v_t \) and join \( u_i \) to \( v_i \) for each \( i \). Let \( A \) be the resulting graph. Let \( B_1 \) be the graph obtained from \( A \) by making \( u_i \) adjacent with \( v_{i+1} \) for each \( i \) and \( B_2 \) be obtained by making \( u_i \) adjacent with \( v_{i+2} \) for each \( i \) where suffix addition is modulo \( t \). Then both \( B_1 \) and \( B_2 \) are 4-regular and the number of triangles in \( B_1 \) is \( 2t \) and that in \( B_2 \) is zero. Thus \( B_1 \) and \( B_2 \) are non-cospectral.

In Figure 1 we illustrate the above construction for \( t = 4 \).

\begin{center}
\includegraphics[width=\textwidth]{figure1.png}
\end{center}

**Figure 1.** The graphs \( B_1 \) and \( B_2 \) in the case \( t = 4 \).

Case 2: \( n = 2t + 1 \), \( t \geq 4 \).

In this case form the \((t + 1)\)-cycle \( v_1 v_2 \ldots v_{t+1} \) and the \( t \)-cycle \( u_1 u_2 \ldots u_t \). Now make \( v_{t-1} \) adjacent with \( v_1 \) and \( v_i \) with \( u_i \), \( i = 1, \ldots, t \). Then join \( v_j \) to \( u_{j+2} \), \( j = 2, \ldots, t-2 \), \( v_t \) to \( u_2 \) and then \( v_{t+1} \) to \( u_1 \) and \( u_3 \). Let \( F_1 \) be the resulting graph. Then \( F_1 \) is 4-regular and contains two triangles \( v_1 v_2 v_3 \) and \( v_5 u_1 v_1 \) for \( t = 4 \) and only one triangle \( v_{t+1} u_1 v_1 \) for \( t \geq 5 \).

To get the other 4-regular graph, form the \((2t + 1)\)-cycle \( v_1 v_2 \ldots v_{t+1} \ldots v_{2t+1} \). Join \( v_i \) to \( v_{i+2} \), \( i = 1, 3, 5, \ldots, 2t + 1, 2, 4, 6, \ldots, 2t \). Let \( F_2 \) be the resulting graph. Then it is 4-regular and contains \( 2t + 1 \) triangles. Thus the graphs \( F_1 \) and \( F_2 \) are not cospectral.

In Figure 2 we illustrate the above construction for \( t = 4 \).
\[ \square \]
Theorem 3. Let $G$ be a connected 4-regular graph on $n$ vertices, with an adjacency matrix $A$ and spectrum $\{4, \lambda_2, \ldots, \lambda_n\}$. Let $H = L^2(G)$ and $\mathcal{H}$ be the $P_4$ self-complementary graph obtained from $H$, according to the above described construction. Then

$$E_D(\mathcal{H}) = 3[8(3n - 1) + \sqrt{20n^2 + 28n + 49}] .$$

Proof follows from Theorem 1, Lemma 1, and the fact that both $\lambda_i + 3r - 4$ and $\lambda_i + 3r - 7$ are positive when $r = 4$.

Theorem 4. For every $n = 48t$ and $n = 24(2t + 1)$, $t \geq 4$, there exists a pair of $D$-equienergetic self-complementary graph.

Proof. Case 1: $n = 48t$

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the two non-cospectral 4-regular graphs on $2t$ vertices as given by Theorem 2. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively denote their second iterated line graphs. Then both are on $12t$ vertices and are 6-regular. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the respective self-complementary graphs on $48t$ vertices. Then by Theorem 3, $\mathcal{B}_1$ and $\mathcal{B}_2$ are $D$-equienergetic.

The other case $n = 24(2t + 1)$ can be proven in a similar manner by considering the two non-cospectral 4-regular graphs on $2t + 1$ vertices whose structure is outlined in Theorem 2.
4. D-ENERGY OF SOME SELF-COMPLEMENTARY GRAPHS

The $D$-energy of some self-complementary graphs $\mathcal{H}$ is easily deduced from the adjacency spectra of the respective parent graphs $G$.

1. If $G \cong K_n$, the complete graph on $n$ vertices, then

$$E_D(\mathcal{H}) = \begin{cases} 
4 + 2\sqrt{10} & \text{for } n = 1 \\
6 + 3\sqrt{17} + \sqrt{41} & \text{for } n = 2 \\
22 + 2\sqrt{85} & \text{for } n = 3 \\
13n - 9 + \sqrt{13n^2 - 6n + 1} & \text{for } n \geq 4 .
\end{cases}$$

2. If $G \cong K_{p,p}$, the complete bipartite graph on $n = 2p$ vertices, then

$$E_D(\mathcal{H}) = 15n - 17 + \sqrt{8n^2 + 4n + 1} .$$

3. If $G \cong CP(n)$, the cocktail party graph on $n$ vertices, then

$$E_D(\mathcal{H}) = 13n - 9 + \sqrt{13n^2 - 18n + 9} .$$

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**References**


