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REMARK ON AN INTEGRAL INEQUALITY OF THE HARDY TYPE

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Abstract. We present a transparent proof of an integral inequality of Hardy type using Hölder's inequality. Some consequences of our result are pointed out.

1. INTRODUCTION

In a recent paper [1], the authors using mainly Jensen's inequality obtained the following result:

Theorem 1. *Let g be continuous and nondecreasing function on $[a,b]$, $0 \leq a < b < \infty$, with $g(x) > 0$ for $x > 0$. Let $q \geq p \geq 1$ and let $f(x)$ be nonnegative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a,b]$. Suppose δ is a real number such that $-\frac{p}{q} < \delta < 0$, then*

$$\left[\int_a^b g(x)^{\frac{q\delta}{p}} \left(\int_a^x f(t)dg(t)\right)^q dg(x)\right]^{\frac{1}{q}} \leq C(a,b,p,q,\delta) \left[\int_a^b g(x)^{(p-1)(1+\delta)} f^p(x)dg(x)\right]^{\frac{1}{p}}, \tag{1}$$

where

$$C(a,b,p,q,\delta) = (-\delta)^{\frac{q(1-p)}{p}} \left(\frac{p}{q\delta+p}\right)^{\frac{p}{q}} g(b)^{\frac{q\delta+p}{p}} \left(g(b)^{-\delta} - g(a)^{-\delta}\right)^{\frac{q(p-1)}{p}} > 0.$$

However, we observed that the constant $C(a,b,p,q,\delta)$ at the right hand side of (1) is wrong. The objective of this paper is to establish the above result using Hölder's inequality and also to obtain the correct constant.

2. MAIN RESULTS

Our result reads:

Theorem 2. *Let g be continuous and nondecreasing function on $[a,b]$, $0 \leq a < b < \infty$, with $g(x) > 0$ for $x > 0$. Let $q \geq p \geq 1$ and let $f(x)$ be nonnegative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a,b]$. Suppose δ is a real number such that $-\frac{p}{q} < \delta < 0$, then*

$$\left[\int_a^b g(x)^{\frac{q\delta}{p}} \left(\int_a^x f(t)dg(t)\right)^q dg(x)\right]^{\frac{1}{q}} \leq C(a,b,p,q,\delta) \left[\int_a^b g(x)^{(p-1)(1+\delta)} f^p(x)dg(x)\right]^{\frac{1}{p}}, \tag{2}$$

where

$$C(a,b,p,q,\delta) = (-\delta)^{\frac{1-p}{p}} \left(\frac{p}{q\delta+p}\right)^{\frac{1}{q}} \left(g(b)^{-\delta} - g(a)^{-\delta}\right)^{\frac{p-1}{p}} \left(g(b)^{\frac{q\delta+p}{p}} - g(a)^{\frac{q\delta+p}{p}}\right)^{\frac{1}{q}}.$$

Remark 1. *In Theorem 2, the special case $p = q$ we obtain from the condition $-\frac{p}{q} < \delta < 0$ that $-1 < \delta < 0$ and so inequality (2) becomes*

$$\int_a^b g(x)^{\delta} \left(\int_a^x f(t)dg(t)\right)^p dg(x) \leq C(a,b,p,\delta) \int_a^b g(x)^{(p-1)(1+\delta)} f^p(x)dg(x) \tag{3}$$

where

$$C(a,b,p,\delta) = (-\delta)^{\frac{1-p}{p}} \left(\frac{1}{\delta+1}\right)^{\frac{1}{p}} \left(g(b)^{-\delta} - g(a)^{-\delta}\right)^{\frac{p-1}{p}} \left(g(b)^{\delta+1} - g(a)^{\delta+1}\right)^{\frac{1}{p}}.$$

In particular, when $g(x) = x$ then (3) yields

$$\int_a^b x^\delta \left(\int_a^x f(t)dt\right)^p dx \le C(a,b,p,\delta) \int_a^b x^{(p-1)(1+\delta)} f^p(x)dx \tag{4}$$

where

$$C(a,b,p,\delta) = (-\delta)^{\frac{1-p}{p}} \left(\frac{1}{\delta+1}\right)^{\frac{1}{p}} \left(b^{-\delta} - a^{-\delta}\right)^{\frac{p-1}{p}} \left(b^{\delta+1} - a^{\delta+1}\right)^{\frac{1}{p}}.$$

We also observe that if we set $a = 0$ in (4), then we obtain

$$\int_0^b x^\delta \left(\int_0^x f(t)dt\right)^p dx \le C(b,p,\delta) \int_0^b x^{(p-1)(1+\delta)} f^p(x)dx$$

where

$$C(b,p,\delta) = (-\delta)^{\frac{1-p}{p}} \left(\frac{1}{\delta+1}\right)^{\frac{1}{p}} b^{\frac{-1}{p}[\delta(p-2)-1]}$$

and $-1 < \delta < 0$.

Remark 2. *For the case $g(x) = x$, then (2) takes the form*

$$\left[\int_a^b x^{\frac{q\delta}{p}} \left(\int_a^x f(t)dt\right)^q dx\right]^{\frac{1}{q}} \le C(a,b,p,q,\delta) \left[\int_a^b x^{(p-1)(1+\delta)} f^p(x)dx\right]^{\frac{1}{p}} \tag{5}$$

with constant

$$C(a,b,p,q,\delta) = (-\delta)^{\frac{1-p}{p}} \left(\frac{p}{q\delta+p}\right)^{\frac{1}{q}} \left(b^{-\delta} - a^{-\delta}\right)^{\frac{p-1}{p}} \left(b^{\frac{q\delta+p}{p}} - a^{\frac{q\delta+p}{p}}\right)^{\frac{1}{q}}.$$

Proof. By using the standard Hölder's inequality:

$$\int_a^x h(x,t)^{\frac{1}{pq}} d\lambda(t) \le \left(\int_a^x d\lambda(t)\right)^{1-\frac{1}{p}} \left(\int_a^x h(x,t)^{\frac{1}{q}} d\lambda(t)\right)^{\frac{1}{p}}, \tag{6}$$

with

$$h(x,t) = g(x)^{\delta q} g(t)^{pq(1+\delta)} f(t)^{pq}$$

and

$$d\lambda(t) = g(t)^{-(1+\delta)} dg(t)$$

respectively, then the left hand side of (6) becomes

$$\int_a^x g(x)^{\frac{\delta}{p}} g(t)^{(1+\delta)} f(t) g(t)^{-(1+\delta)} dg(t) = g(x)^{\frac{\delta}{p}} \int_a^x f(t) dg(t). \tag{7}$$

Similarly, the right hand side of (6) yields

$$\begin{aligned} &\left(\int_a^x g(t)^{-(1+\delta)} dg(t)\right)^{1-\frac{1}{p}} \left(\int_a^x g(x)^{\delta} g(t)^{p(1+\delta)} f^p(t) g(t)^{-(1+\delta)} dg(t)\right)^{\frac{1}{p}} \\ = \;& \left[-\delta^{-1}\right]^{1-\frac{1}{p}} \left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{1-\frac{1}{p}} \left(\int_a^x g(x)^{\delta} g(t)^{(p-1)(1+\delta)} f^p(t) dg(t)\right)^{\frac{1}{p}} \\ = \;& \left[-\delta^{-1}\right]^{\frac{1-p}{p}} \left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{\frac{p-1}{p}} g(x)^{\frac{\delta}{p}} \left(\int_a^x g(t)^{(p-1)(1+\delta)} f^p(t) dg(t)\right)^{\frac{1}{p}} \end{aligned} \tag{8}$$

By combining (7) and (8) and raising both sides to power q, inequality (6) yields

$$\begin{aligned} & g(x)^{\frac{q\delta}{p}} \left(\int_a^x f(t) dg(t)\right)^q \\ \leq \;& \left[-\delta^{-1}\right]^{\frac{q(1-p)}{p}} \left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{\frac{q(p-1)}{p}} g(x)^{\frac{q\delta}{p}} \left(\int_a^x g(t)^{(p-1)(1+\delta)} f^p(t) dg(t)\right)^{\frac{q}{p}}. \end{aligned} \tag{9}$$

Now, integrate both sides of (9) with respect to $g(x)$ from a to b and then raising both sides to power $\frac{p}{q}$ to obtain

$$\begin{aligned} & \left[\int_a^b g(x)^{\frac{q\delta}{p}} \left(\int_a^x f(t) dg(t)\right)^q dg(x)\right]^{\frac{p}{q}} \\ \leq \;& \left[-\delta^{-1}\right]^{1-p} \left[\int_a^b \left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{\frac{q(p-1)}{p}} g(x)^{\frac{q\delta}{p}} \right. \\ \times \;& \left. \left(\int_a^x g(t)^{(p-1)(1+\delta)} f^p(t) dg(t)\right)^{\frac{q}{p}} dg(x)\right]^{\frac{p}{q}}. \end{aligned} \tag{10}$$

By applying the modified Minkowski integral inequalities (3.1) and (3.2) in [6, pp. 21-22] to the right hand side of (10) we obtain

$$\left[\int_a^b g(x)^{\frac{q\delta}{p}} \left(\int_a^x f(t) dg(t)\right)^q dg(x)\right]^{\frac{p}{q}}$$

$$
\begin{aligned}
&\leq \left[-\delta^{-1}\right]^{1-p} \int_a^b g(t)^{(p-1)(1+\delta)} f^p(t) \left(\int_t^b \left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{\frac{q(p-1)}{p}} g(x)^{\frac{q\delta}{p}} dg(x) \right)^{\frac{p}{q}} dg(t) \\
&\leq \left[-\delta^{-1}\right]^{1-p} \left[g(b)^{-\delta} - g(a)^{-\delta}\right]^{p-1} \int_a^b g(t)^{(p-1)(1+\delta)} f^p(t) \left(\int_t^b g(x)^{\frac{q\delta}{p}} dg(x) \right)^{\frac{p}{q}} dg(t) \\
&= \left[-\delta^{-1}\right]^{1-p} \left(\frac{p}{q\delta + p} \right)^{\frac{p}{q}} \left[g(b)^{-\delta} - g(a)^{-\delta}\right]^{p-1} \\
&\quad \times \int_a^b g(t)^{(p-1)(1+\delta)} f^p(t) \left(g(b)^{\frac{q\delta+p}{p}} - g(t)^{\frac{q\delta+p}{p}} \right)^{\frac{p}{q}} dg(t) \\
&\leq \left[-\delta^{-1}\right]^{1-p} \left(\frac{p}{q\delta + p} \right)^{\frac{p}{q}} \left[g(b)^{-\delta} - g(a)^{-\delta}\right]^{p-1} \left(g(b)^{\frac{q\delta+p}{p}} - g(a)^{\frac{q\delta+p}{p}} \right)^{\frac{p}{q}} \\
&\quad \times \int_a^b g(t)^{(p-1)(1+\delta)} f^p(t) dg(t).
\end{aligned}
$$

From the above, it follows that

$$
\begin{aligned}
&\left[\int_a^b g(x)^{\frac{q\delta}{p}} \left(\int_a^x f(t) dg(t) \right)^q dg(x) \right]^{\frac{1}{q}} \\
&\leq \left[-\delta^{-1}\right]^{\frac{1-p}{p}} \left(\frac{p}{q\delta + p} \right)^{\frac{1}{q}} \left[g(b)^{-\delta} - g(a)^{-\delta}\right]^{\frac{p-1}{p}} \left(g(b)^{\frac{q\delta+p}{p}} - g(a)^{\frac{q\delta+p}{p}} \right)^{\frac{1}{q}} \\
&\quad \times \left[\int_a^b g(t)^{(p-1)(1+\delta)} f^p(t) dg(t) \right]^{\frac{1}{p}}. \qquad (11)
\end{aligned}
$$

From (11) the result follows and the proof is complete. □

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References

[1] A. G. Adeagbo-Sheikh and C. O. Imoru, An integral inequality of the Hardy's type, *Kragujevac J. Math.* **29** (2006), 57-61.

[2] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities,* Cambridge University Press, Cambridge, 1959.

[3] C. O. Imoru, On some integral inequalities related to Hardy's, *Canad. Math. Bull.* **20** (3) (1977), 307-312.

[4] A. Kufner, L. Maligranda and L.-E. Persson, *The Hardy Inequality. About its History and Some Related Results,* Vydavatelsky Servis Publishing House, Pilsen, 2007.

[5] A. Kufner and L-E. Persson, *Weighted Inequalities of Hardy Type,* World Scientific Publishing Co., New Jersey-London-Singapore-Hong Kong, 2003.

[6] B. Opic and A. Kufner, *Hardy-type Inequalities,* Harlow: Pitman Research Notes in Mathematics Series 219, Longman Scientific & Technical, Harlow, 1990.

[7] D. T. Shum, On integral inequalities related to Hardy's, *Canad. Math. Bull.* **14** (2) (1971), 225-230.