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BERTRAND CURVES IN GALILEAN SPACE AND THEIR CHARACTERIZATIONS

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Abstract. In this paper, we introduce the notion of Bertrand and Mannheim curve in Galilean space G_3 and give characterizations of such curves.

1. INTRODUCTION

The notion of Bertrand curves was discovered by J. Bertrand in 1850 and it plays an important role in classical differential geometry. A Bertrand curve is a curve in Euclidean 3-space whose principal normal is the principal normal of another curve [3]. We can see in most textbooks, a characteristic property of Bertrand curve which asserts the existence of a linear relation between curvature and torsion. The characteristic property-linear relation -is deduced as an application of the Frenet-Serret formulas. Well known, every space curve is uniquely determined by its curvature and torsion up to Euclidean motions. A space curve with prescribed curvature and torsion functions are obtained by integrations of a Riccati equations. However, in general, it is impossible to carry out the integration explicitly except for some simple cases e.g., helices. Although the general theorem due to Lie is of interest and valuable, it doesn't provide us sample information of Bertrand curves. J. A. Serret proved in 1850 that curves with prescribed curvature or curves with prescribed torsion can be found by quadratures. Based on this result, L. Bianchi proved that Bertrand curves with prescribed linear relation of curvature and torsion can be found by quadratures.

Recently, null Bertrand curves and nonnull Bertrand curves in 3-dimensional Lorentzian space are studied in [1,2]. In this regards, we introduce the notion of Bertrand curves in 3-dimensional Galilean space and investigate Bertrand curves in detail. Also we obtained a characterization on Mannheim curves for curves in 3-dimensional Galilean space.

2. PRELIMINARIES

Differential geometry of the Galilean space G_3 has been largely developed in O. Röschel's paper [5].

The Galilean space is a three dimensional complex projective space P_3 in which the absolute figure $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and two complex conjugate points $I_1, I_2 \in f$ (the absolute points).

We shall take, as a real model of the space G_3 , a real projective space P_3 with the absolute $\{w, f\}$ consisting of a real plane $w \subset G_3$ and a real line $f \subset w$ on which an elliptic involution ε has been defined.

Let it be in homogeneous coordinates

$$w \; ... \; x_0 = 0, \quad f \; ... \; x_0 = x_1 = 0$$
$$\varepsilon : (0 : 0 : x_2 : x_3) \rightarrow (0 : 0 : x_3 : -x_2).$$

In the nonhomogeneous coordinates the similarity group H_8 has the form

$$x' = a_{11} + a_{12}x,$$

$$y' = a_{21} + a_{22}x + a_{23}\cos\varphi y + a_{23}\sin\varphi z, \tag{2.1}$$
$$z' = a_{31} + a_{32}x - a_{23}\sin\varphi y + a_{23}\cos\varphi z,$$

where a_{ij} and φ are real numbers.

For $a_{12} = a_{23} = 1$ we have the subgroup B_6 -the group of Galilean motions:

$$\begin{aligned} x' &= a + x \\ B_6 \; ... \; y' &= b + cx + y\cos\varphi + z\sin\varphi \\ z' &= d + ex - y\sin\varphi + z\cos\varphi. \end{aligned}$$

In G_3 there are four classes of lines:

a) (proper) nonisotropic lines - they don't meet the absolute line f.

b) (proper) isotropic lines - lines that don't belong to the plane w but meet the absolute line f.

c) unproper nonisotropic lines - all lines of w but f.

d) the absolute line f.

Planes $x = const.$ are Euclidean and so is the plane w. Other planes are isotropic.

In what follows the coefficients a_{12} and a_{23} will play the special role.

In particular, for $a_{12} = a_{23} = 1$ (2.1) defines the group $B_6 \subset H_8$ of isometries of the Galilean space G_3 [4].

3. CURVES IN THE GALILEAN SPACE

Let $\alpha : I \longrightarrow G_3$, $I \subset \mathbb{R}$ be a curve given by

$$\alpha(t) = (x(t), y(t), z(t)),$$

where $x(t), y(t), z(t) \in C^3$ (the set of three times continuously differentiable functions) and t run through a real interval [4].

Let α be a curve in G_3, parameterized by arclength $t = s$, given in coordinate form

$$\alpha(s) = (s,\ y(s),\ z(s)). \tag{3.1}$$

Then the curvature $\kappa_\alpha(s)$ and the torsion $\tau_\alpha(s)$ are defined by

$$\kappa_\alpha(s) = \sqrt{y''^2(s) + z''^2(s)} \tag{3.2}$$
$$\tau_\alpha(s) = \frac{\det(\alpha'(s),\ \alpha''(s),\ \alpha'''(s))}{\kappa_\alpha^2}$$

and associated moving trihedron is given by

$$\begin{aligned} T_\alpha(s) &= \alpha'(s) = (1,\ y\prime(s),\ z'(s)) \\ N_\alpha(s) &= \frac{1}{\kappa_\alpha(s)}\alpha''(s) = \frac{1}{\kappa_\alpha(s)}(0,\ y''(s),\ z''(s)) \\ B_\alpha(s) &= \frac{1}{\kappa_\alpha(s)}(0, -z''(s),\ y''(s)). \end{aligned} \tag{3.3}$$

The vectors T_α, N_α, B_α are called the vectors of the tangent, principal normal and binormal line of α, respectively. For their derivatives the following Frenet formulas hold

$$\begin{aligned} T'_\alpha &= \kappa_\alpha N_\alpha \\ N'_\alpha &= \tau_\alpha B_\alpha \\ B'_\alpha &= -\tau_\alpha N_\alpha \end{aligned} \tag{3.4}$$

4. BERTRAND CURVES IN GALILEAN SPACE G_3

Definition 4.1. *Let α and $\overline{\alpha}$ be the curves with $\kappa_\alpha(s) \neq 0$, $\overline{\kappa}_\alpha(s) \neq 0$, $\tau_\alpha(s) \neq 0, \overline{\tau}_\alpha \neq 0$ for each $s \in I$ in G_3 and $\{T_\alpha,\ N_\alpha,\ B_\alpha\}$ and $\{\overline{T}_\alpha,\ \overline{N}_\alpha,\ \overline{B}_\alpha\}$ be the Frenet frames in G_3 along α and $\overline{\alpha}$, respectively. If $\{N_\alpha, \overline{N_\alpha}\}$ is linearly dependent, in other words if the normal lines of α and $\overline{\alpha}$ at $s \in I$ are parallel, then a pair of curves $(\alpha,\ \overline{\alpha})$ is said to be a Bertrand pair in G_3.*

The curve $\overline{\alpha}$ is called a Bertrand mate of α and vice versa. A Frenet framed curve is said to be a *Bertrand curve* if it admits a Bertrand mate.

By definition, for a Bertrand pair $(\alpha,\ \overline{\alpha})$, there exists a functional relation $\overline{s} = \overline{s}(s)$ such that

$$\overline{u}(\overline{s}(s)) = u(s)$$

Let $(\alpha, \overline{\alpha})$ be *Bertrand pair* in G_3. Then we can write

$$\overline{\alpha}(s) = \alpha(s) + u(s)N_\alpha(s). \tag{4.1}$$

Theorem 4.1. *Let $(\alpha, \overline{\alpha})$ be a Bertrand pair in G_3. Then the function u defined by relation (4.1) is a constant.*

Proof. Let $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{\overline{T}_\alpha, \overline{N}_\alpha, \overline{B}_\alpha\}$ be the Frenet frames in G_3 along α and $\overline{\alpha}$, respectively. Since $(\alpha, \overline{\alpha})$ is a *Bertrand pair,* from (4.1) we can write

$$\overline{\alpha} = \alpha + uN_\alpha. \tag{4.2}$$

By differentiation of the equation (4.2) with respect to s, we obtain

$$\overline{T}_\alpha \frac{d\overline{s}}{ds} = T_\alpha + u'N_\alpha + uN'_\alpha \tag{4.3}$$

where s and $\overline{s}$ are parameters on α and $\overline{\alpha}$, respectively and $\frac{d\overline{s}}{ds} \neq 0$. By using relation (3.4) we have

$$\overline{T}_\alpha \frac{d\overline{s}}{ds} = T_\alpha + u'N_\alpha + u\tau_\alpha B_\alpha. \tag{4.4}$$

Since $\{T_\alpha, N_\alpha, B_\alpha\}$ is the Frenet frame on G_3 along α and $\overline{\alpha}$ is a Bertrand mate of α, we obtain

$$u' = 0.$$

This means that u is constant. Hence the proof is completed. □

Now, let us define $\overline{T}_\alpha$ by

$$\overline{T}_\alpha = \cos\theta T_\alpha + \sin\theta B_\alpha \tag{4.5}$$

such that θ is the angle between T_α and $\overline{T}_\alpha$.

If we differentiate of the equation (4.5) with respect to s, then we obtain

$$\overline{\kappa}_\alpha \overline{N}_\alpha \frac{d\overline{s}}{ds} = \frac{d(\cos\theta)}{ds} T_\alpha + (\kappa_\alpha \cos\theta - \tau_\alpha \sin\theta) N_\alpha + \frac{d(\sin\theta)}{ds} B_\alpha. \tag{4.6}$$

Since $\{T_\alpha, N_\alpha, B_\alpha\}$ is the Frenet frame on G_3 along α and $\overline{\alpha}$ is a Bertrand mate of α, we have

$$\theta = const. \tag{4.7}$$

Theorem 4.2. *Let α be a curve in G_3. Then α is a Bertrand curve if and only if α is a curve with constant torsion τ_α.*

Proof. Let $(\alpha, \overline{\alpha})$ be a *Bertrand pair.* By using (4.4) and Theorem 4.1., we have

$$\overline{T}_\alpha \frac{d\overline{s}}{ds} = T_\alpha + u\tau_\alpha B_\alpha. \tag{4.8}$$

If we consider (4.5) and (4.8), we get

$$u\tau_\alpha \cot\theta = 1.$$

Taking $\lambda = u\cot\theta$ using (4.7) as well as theorem 4.1, we get

$$\tau_\alpha = \frac{1}{\lambda}, \tag{4.9}$$

this means that τ_α is constant.

Conversely, let us assume that τ_α is constant i.e., λ is nonzero constant. Now let us define

$$\overline{\alpha} = \alpha + uN_\alpha \tag{4.10}$$

By using (4.8) and (4.9) we get

$$\frac{\overline{T}_\alpha}{d\overline{s}}\frac{d\overline{s}}{ds} = \frac{\kappa_\alpha - u\tau_\alpha^2}{\sqrt{1+u^2\tau_\alpha^2}}N_\alpha,$$

which means that, N_α and $\overline{N}_\alpha$ are linearly dependent. According to definition 4.1, it follows that $(\alpha, \overline{\alpha})$ is a Bertrand Pair, which completes the proof of the theorem. □

Theorem 4.3. (*Schell's Theorem*)

Let $(\alpha, \overline{\alpha})$ be a Bertrand pair in G_3. Then the product of torsions τ_α and $\overline{\tau}_\alpha$ at the corresponding points of the Bertrand curves is constant.

Proof. If we take $\overline{\alpha}$ instead of α, then we can write the equation (4.2) as follows:

$$\alpha = \overline{\alpha} - u\overline{N}_\alpha.$$

Hence, we have

$$T_\alpha = \overline{T}_\alpha \frac{d\overline{s}}{ds} - u\overline{\tau}_\alpha \overline{B}_\alpha \frac{d\overline{s}}{ds}. \tag{4.11}$$

Also, we can write

$$T_\alpha = \cos\theta \overline{T}_\alpha + \sin\theta \overline{B}_\alpha. \tag{4.12}$$

From (4.11) and (4.12), we get

$$\frac{\cos\theta}{\sin\theta} = -\frac{1}{u\overline{\tau}_\alpha}. \tag{4.13}$$

Using Theorem 4.2. and (4.13) we have

$$\tau_\alpha \overline{\tau}_\alpha = -\frac{\sin^2\theta}{u^2\cos^2\theta} = const.$$

This completes the proof. □

Let $(\alpha, \overline{\alpha})$ be a *Bertrand pair* in G_3. Let P, $\overline{P}$ be two corresponding points of $(\alpha, \overline{\alpha})$ and M and $\overline{M}$ be the curvature centers at these points. Then

$$\begin{aligned} \|\overline{P}M\| &= u - \rho_\alpha = u - \frac{1}{\kappa_\alpha} \\ \|P\overline{M}\| &= u + \overline{\rho}_\alpha = u + \frac{1}{\overline{\kappa}_\alpha} \end{aligned}$$

where ρ_α and $\overline{\rho}_\alpha$ are the curvature radiuses of α and $\overline{\alpha}$, respectively. Then we have

$$\frac{\|\overline{P}M\|}{\|PM\|} : \frac{\|\overline{P}\overline{M}\|}{\|P\overline{M}\|} = (u\kappa_\alpha - 1)(u\overline{\kappa}_\alpha + 1) \neq const.,$$

since κ_α and $\overline{\kappa}_\alpha$ are not constant. Therefore ,we have following Theorem which is valid in 3-dimensional Euclidean space R^3 and 3-dimensional Lorentzian space L^3.

Theorem 4.4. *Mannheim Theorem for Bertrand curves in Galilean space G_3 is not valid.*

5. MANNHEIM CURVES IN GALILEAN SPACE G_3

Definition 5.1. *Let γ be a curve in Galilean space G_3. If its principal normal is the binormal of another curve then γ is called Mannheim curve in G_3.*

Theorem 5.1. *Let γ be a curve in Galilean space G_3. Then γ is Mannheim curve if and only if its curvature κ_γ and torsion τ_γ satisfy the relation $\kappa_\gamma = c\tau_\gamma^2$ for some constant c.*

Proof. Let $\gamma = \gamma(s)$ be a *Mannheim curve.* Let us denote by $\{T_\gamma, N_\gamma, B_\gamma\}$ the Frenet frame field of γ.

Assume that $\overline{\gamma} = \overline{\gamma}(\overline{s})$ is a curve whose binormal direction coincides with the principal normal of γ. Namely let us denote by $\{\overline{T}_\gamma, \overline{N}_\gamma, \overline{B}_\gamma\}$ the Frenet frame field of $\overline{\gamma}$. Then $\overline{B}_\gamma(\overline{s}) = \pm\, N_\gamma(s)$.

The curve $\overline{\gamma}$ is parametrized by arclength s as

$$\overline{\gamma}(s) = \gamma(s) + c(s)N_\gamma(s) \tag{5.1}$$

for some function $c(s) \neq 0$. Differentiating (5.1) with respect to s, we find

$$\overline{\gamma}' = T_\gamma + c'N_\gamma + c\tau_\gamma B_\gamma. \tag{5.2}$$

Since the binormal direction of $\overline{\gamma}$ coincides with the principal normal of γ, we have $c' = 0$. Hence c is constant. The second derivative $\overline{\gamma}''$ with respect to s is

$$\overline{\gamma}'' = (\kappa_\gamma - c\tau_\gamma^2)N_\gamma + c\tau_\gamma' B_\gamma. \tag{5.3}$$

Since N_γ is in the binormal direction of $\overline{\gamma}$, we have

$$\kappa_\gamma - c\tau_\gamma^2 = 0.$$

Conversely, let γ be a curve in Galilean space G_3 with $\kappa_\gamma = c\tau_\gamma$, for some constant c. Then the curve

$$\overline{\gamma}(s) = \gamma(s) + cN_\gamma(s)$$

has binormal direction $N_\gamma(s)$. It follows that γ is a Mannheim curve which proves the theorem. □

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