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$L^{\mathrm{rat}}_{\mathbb{A}P}$ LOGIC AND COMPLETENESS THEOREM

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Abstract. To the list of axioms of $L_{\mathbb{A}P}$ logic is added axiom which will provide that measures on probability models have only rational values of $[0, 1]$ interval. Completeness Theorem is proved.

INTRODUCTION

In this paper we will introduce the logic $L^{\mathrm{rat}}_{\mathbb{A}P}$. This logic is similar to infinitary logic $L_{\mathbb{A}P}$ (see [2, 3]). Our logic will include a new types of axiom. A model of this logic is also a classical model with a probability measure in the universe, such that each relation is measurable and measure ranges are sets of rationals of $[0, 1]$ interval.

1. BASIC DEFINITION

Syntax. We assume that $\mathbb{A}$ is an admissible set (see [1]) such that $\mathbb{A} \subseteq \mathbb{HC}$ and $\omega \in \mathbb{A}$.

Let L be a countable, Σ-definable set of finitary relation and constant symbols (no function symbols).

We need the following logical symbols:

(1) The parentheses (,) .

(2) The variables $v_0, v_1, \ldots, v_n, \ldots, n \in \mathbb{N}$.

(3) The connectives $\neg$ and $\bigwedge$.

(4) The quantifiers $(P\vec{x} \geqslant r)$, where $r \in \mathbb{A} \cap [0,1]$

(5) The equality symbol $=$ (optional).

Definition 1.1. The formulas of $L_{\mathbb{A}P}^{\text{rat}}$ are defined as follows:

(1) An atomic formula of first-order logic is a formula of $L_{\mathbb{A}P}^{\text{rat}}$;

(2) If φ is a formula of $L_{\mathbb{A}P}^{\text{rat}}$, then $\neg\varphi$ is a formula of $L_{\mathbb{A}P}^{\text{rat}}$;

(3) If $\Phi \in \mathbb{A}$ is a set of formulas of $L_{\mathbb{A}P}^{\text{rat}}$ with only finitely many free variables, then $\wedge\Phi$ is a formula of $L_{\mathbb{A}P}^{\text{rat}}$;

(4) If φ is a formula of $L_{\mathbb{A}P}^{\text{rat}}$, then $(P\vec{x} \geqslant r)\varphi$ is a formula of $L_{\mathbb{A}P}^{\text{rat}}$;

We shall assume that $L_{\mathbb{A}P}^{\text{rat}} \subseteq \mathbb{A}$ where $\mathbb{A} = \mathbb{HC}$.

Definition 1.2. We shall use the following abbrevations:

(a) $(P\vec{x} < r)\varphi$ for $\neg(P\vec{x} \geqslant r)\varphi$;

(b) $(P\vec{x} \leqslant r)\varphi$ for $(P\vec{x} \geqslant 1 - r)\neg\varphi$;

(c) $(P\vec{x} > r)\varphi$ for $\neg(P\vec{x} \geqslant 1 - r)\neg\varphi$;

(d) The connectives $\vee$, $\to$ and $\leftrightarrow$ define as usual.

Models.

Definition 1.3. A probability model for L is a structure

$$\mathfrak{A} = \langle A,\ R_i^{\mathfrak{A}},\ c_j^{\mathfrak{A}},\ \mu\rangle_{i\in I,\ j\in J}$$

where $\langle A,\ R_i^{\mathfrak{A}},\ c_j^{\mathfrak{A}}\rangle_{i\in I,\ j\in J}$ is a classical model, μ is a countably additive probability measure on A. Such that each singleton is measurable, each n-placed relation $R_i^{\mathfrak{A}}$ is $\mu^{(n)}$-measurable and rang of measure μ is set of rationals of $[0,1]$ interval.

Definition 1.4. A graded probability model for L is a structure

$$\mathfrak{A} = \langle A,\ R_i^{\mathfrak{A}},\ c_j^{\mathfrak{A}},\ \mu_n\rangle_{i\in I,\ j\in J,\ n\in\mathbb{N}}$$

such that:

(1) $\langle A,\ R_i^{\mathfrak{A}},\ c_j^{\mathfrak{A}}\rangle_{i\in I,\ j\in J}$ is a classical model;

(2) Each μ_n is a countably additive probability measure on A^n;

(3) Every measure range of μ_n is subset of rationals of $[0,1]$ interval;

(4) For all $m,n\in\mathbb{N}$, μ_{m+n} is an extension of the product measure $\mu_m\times\mu_n$;

(5) Each μ_n is invariant under permutations, that is, whenever π is a permutation of $\{1,\ 2,\ \ldots,\ n\}$ and $B\in\mathrm{dom}(\mu_n)$, if

$$\pi B=\left\{\left(a_{\pi(1)},\ldots,a_{\pi(n)}\right)\mid(a_1,\ldots,a_n)\in B\right\},$$

then $\pi B\in\mathrm{dom}(\mu_n)$ and $\mu_n(\pi B)=\mu_n(B)$;

(6) $\langle\mu_n\mid n\in\mathbb{N}\rangle$ has the Fubini property: If B is μ_{m+n}-measurable, then

(a) for each $\vec{x}\in A^m$, the section $B\vec{x}=\{\vec{y}\in A^n\mid(\vec{x},\vec{y})\in B\}$ is μ_n-measurable;

(b) the function $f(\vec{x})=\mu_n(B\vec{x})$ is μ_m-measurable;

(c) $\int f(\vec{x})\mathrm{d}\mu_m=\mu_{m+n}(B)$.

(7) Each atomic formula with n free variables is measurable with respect to μ_n.

The satisfaction relation is defined recursively in the same way as it was for $L_{\mathbb{A}P}$.

Theorem 1.5. (Fubini theorem) *Let μ be a probability measure such that each singleton is measurable, and let $B\subseteq A^{m+n}$ be $\mu^{(m+n)}$-measurable. Then:*

(1) Every section $B\vec{x}=\{\vec{y}\in A^n\mid(\vec{x},\vec{y})\in B\}$ is $\mu^{(n)}$-measurable;

(2) The function $f(\vec{x})=\mu^{(n)}(B\vec{x})$ is $\mu^{(m)}$-measurable;

(3) $\mu^{(m+n)}(B)=\int f(\vec{x})\mathrm{d}\mu^{(m)}$.

Proof theory. We now give a list of axioms and rules of inference for $L_{\mathbb{A}P}^{\mathrm{rat}}$. In what follows, φ, ψ are arbitrary formulas of $L_{\mathbb{A}P}$, $\Phi\in\mathbb{A}$ is an arbitrary set of formulas of $L_{\mathbb{A}P}^{\mathrm{rat}}$ and $r,s\in\mathbb{A}\cap[0,1]$.

Definition 1.6. The axioms of the weak $L_{\mathbb{A}P}^{\mathrm{rat}}$ are the following:

(W_1) All axioms of $L_{\mathbb{A}}$ without quantifiers;

(W_2) Monotonicity: $(P\vec{x}\geqslant r)\varphi\to(P\vec{x}\geqslant s)\varphi$, where $r\geqslant s$;

(W_3) $(P\vec{x}\geqslant r)\varphi(\vec{x})\to(P\vec{y}\geqslant r)\varphi(\vec{y})$;

(W_4) $(P\vec{x}\geqslant 0)\varphi$;

(W_5) Finite additivity:

(a) $(P\vec{x} \leqslant r)\varphi \ \wedge \ (P\vec{x} \leqslant s)\psi \ \rightarrow \ (P\vec{x} \leqslant r+s)(\varphi \vee \psi)$;

(b) $(P\vec{x} \geqslant r)\varphi \ \wedge \ (P\vec{x} \geqslant s)\psi \ \wedge \ (P\vec{x} \leqslant 0)(\varphi \wedge \psi) \ \rightarrow \ (P\vec{x} \geqslant r+s)(\varphi \vee \psi)$;

(W_6) The Archimedean property:

$$(P\vec{x} > r)\varphi \leftrightarrow \bigvee_{n\in\mathbb{N}} \Big(P\vec{x} \geqslant r + \frac{1}{n}\Big)\varphi$$

(W_7)

$$\Big(\psi \rightarrow \bigwedge_{q\in[0,1]\cap Q} (P\vec{x} \neq q)\varphi\Big) \rightarrow \ \neg\, \psi.$$

Definition 1.7. The axioms for graded $L^{\mathrm{rat}}_{\mathbb{A}P}$ consist of the axioms for weak $L^{\mathrm{rat}}_{\mathbb{A}P}$ plus following set of axioms:

(H_1) Countable additivity:

$$\bigwedge_{\Psi\subseteq\Phi} (P\vec{x} \geqslant r) \bigwedge \Psi \ \rightarrow \ (P\vec{x} \geqslant r) \bigwedge \Phi,$$

where Ψ ranges over the finite subset of Φ.

(H_2) Symmetry:

$$(Px_1 \ \ldots \ x_n \geqslant r)\varphi \ \leftrightarrow \ (Px_{\pi(1)} \ \ldots \ x_{\pi(n)} \geqslant r)\varphi,$$

where π is a permutation of $\{1,\ 2,\ \ldots,\ n\}$.

(H_3) Product independence:

$$(P\vec{x} \geqslant r)(P\vec{y} \geqslant s)\varphi \ \rightarrow \ (P\vec{x}\vec{y} \geqslant r \cdot s)\varphi,$$

where all variables in $\vec{x}\,, \vec{y}$ are distinct.

Definition 1.8. The axioms for the full $L^{\mathrm{rat}}_{\mathbb{A}P}$ consist of the axioms for graded $L^{\mathrm{rat}}_{\mathbb{A}P}$ plus the following Keisler's axiom:

(K) Product measurability:

$$(P\vec{x} \geqslant 1)(P\vec{y} > 0)(P\vec{z} \geqslant r)(\varphi(\vec{x}, \vec{z}) \ \leftrightarrow \ \varphi(\vec{y}, \vec{z}))$$

for each $r < 1$, where all variables in $\vec{x}\,, \vec{y}\,, \vec{z}$ are distinct.

The rules of inference for all of the above logics are: Modus ponens, Conjuction and Generalization, as in $L_{\mathbb{A}P}$ logic (see [3]).

2. COMPLETENESS THEOREM

Consistency properties and weak models.

Definition 2.1. A weak model for $L^{\text{rat}}_{\mathbb{A}P}$ is a structure

$$\mathfrak{A} = \langle A,\ R^{\mathfrak{A}}_i,\ c^{\mathfrak{A}}_j,\ \mu_n\rangle_{i\in I,\ j\in J,\ n\in\mathbb{N}}$$

such that $\langle A,\ R^{\mathfrak{A}}_i,\ c^{\mathfrak{A}}_j\rangle_{i\in I, j\in J}$ is a classical model, each μ_n is a finitely additive probability measure on A^n with each singleton measurable,every μ_n rang is subset of $[0,1]_{\mathbb{Q}}$, and with the set $\{\vec{c}\in A^n \mid \mathfrak{A}\vDash \varphi\,[\vec{a},\vec{c}]\}$ μ_n-measurable for each $\varphi(\vec{x},\vec{y}) \in L^{\text{rat}}_{\mathbb{A}P}$ and each $\vec{a}\in A$.

$\varphi^{\neg}$ is defined same as in $L_{\mathbb{A}P}$ logic. Without loss of generality, we can suppose in this chapter that $\mathbb{A}$ is a countable admissible set.

Let C be a countable set of new constant symbols, and let $K = L\cup C$. Then we form the logic $K^{\text{rat}}_{\mathbb{A}P}$ corresponding to K.

Definition 2.2. A consistency property for $L^{\text{rat}}_{\mathbb{A}P}$ is a set S of countable sets s of sentences of $K^{\text{rat}}_{\mathbb{A}P}$ which satisfies the following conditions for each $s\in S$:

(C_1) (*Triviality rule*) $\varnothing\in S$;

(C_2) (*Consistency rule*) Either $\varphi\notin s$ or $\neg\varphi\notin s$;

(C_3) ($\neg$-*rule*) If $\neg\varphi\in s$, then $s\cup\{\varphi^{\neg}\}\in S$;

(C_4) ($\bigwedge$-*rule*) If $\bigwedge\Phi\in s$, then for all $\varphi\in\Phi$, $s\cup\{\varphi\}\in S$;

(C_5) ($\bigvee$-*rule*) If $\bigvee\Phi\in s$, then for some $\varphi\in\Phi$, $s\cup\{\varphi\}\in S$;

(C_6) (*P-rule*) If $(P\vec{x}>0)\varphi(\vec{x})\in s$, then for some $\vec{c}\in C$, $s\cup\{\varphi(\vec{c})\}\in S$;

(C_7) If $\varphi(\vec{x})\in K^{\text{rat}}_{\mathbb{A}P}$ is an axiom,then

(a) $s\cup\{(P\vec{x}\geqslant 1)\varphi(\vec{x})\}\in S$,

(b) $s\cup\{\varphi(\vec{c})\}\in S$, where $\vec{c}\in C$.

Theorem 2.3. **(Model Existence Theorem)** *If S is a consistency property, then any $s_0 \in S$ has a weak model.*

Proof. Let $\varphi_0, \ \varphi_1, \ \varphi_2, \ \ldots$ be an enumeration of the sentences of $K_{\mathbb{A}P}^{\text{rat}}$. We shall construct a sequence $s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots$ of elements of S as follows. s_0 is given. Given s_n choose s_{n+1} to satisfy the following conditions:

(1) $s_n \subseteq s_{n+1}$.

(2) If $s_n \cup \{\varphi_n\} \in S$, then $\varphi_n \in s_{n+1}$.

(3) If $s_n \cup \{\varphi_n\} \in S, \ \varphi_n = \bigvee \Phi$, then for some $\theta \in \Phi, \ \theta \in s_{n+1}$.

(4) If $s_n \cup \{\varphi_n\} \in S, \ \varphi_n = (P\vec{x} > 0)\psi(\vec{x})$, then for some $\vec{c} \in C, \ \psi(\vec{c}) \in s_{n+1}$.

We now define a model $\mathfrak{A}$ of s_0. Let $s_\omega = \bigcup_{n<\omega} s_n$. Let T be a set of constants of $K_{\mathbb{A}P}^{\text{rat}}$. For c, $d \in T$, let $c \sim d$ iff $c = d \in s_\omega$. Then, $\sim$ is an equivalence relation. Let $[c]$ denote the equivalence class of the constant c. Let $\mathfrak{A}$ have the universe set $A = \{[c] \mid c \in T\}$. If R is an n-placed relation symbol and $c_1, \ \ldots, \ c_n \in C$, then

$$\mathfrak{A} \vDash R([c_1], \ \ldots, \ [c_n]) \quad \text{iff} \quad R(c_1, \ \ldots, \ c_n) \in s_\omega \, .$$

Define μ_n on the subsets of A^n definable by formulas of $L_{\mathbb{A}P}^{\text{rat}}$ with parameters from A, by

$$\mu_n \{\vec{a} \in A^n \mid \mathfrak{A} \vDash \varphi[\vec{a}, \vec{c}]\} = q \quad \text{iff} \quad (P\vec{x} = q)\varphi(\vec{x}, \vec{c}) \in s_\omega \, .$$

It is not difficult to show that everything is well-defined, μ_n's are finitely additive probability measures, which ranges are subsets of $[0,1]_{\mathbb{Q}}$, and it is routine to check that

$$\mathfrak{A} \vDash \varphi \, [[c_1], \ldots, \ [c_n]] \quad \text{iff} \quad \varphi(c_1, \ \ldots, \ c_n) \in s_\omega \, .$$

Therefore $\mathfrak{A}$ is a weak model of s_ω, and hence a model of s_0. □

Theorem 2.4. **(Weak Completeness Theorem)** *Every countable set T of sentences which is consistent in weak $L_{\mathbb{A}P}^{rat}$ has a weak model.*

Proof. Let S be the set of all countable sets s of sentences of $K_{\mathbb{A}P}^{\text{rat}}$ such that only finitely many $c \in C$ occur in s and not $\vdash_{K_{\mathbb{A}P}^{\text{rat}}} \neg \bigwedge s$.

It is not difficult to check that S is a consistency property.

Theorem 2.5. (Graded Completeness Theorem) *Every countable set T of sentences which is consistent in graded $L_{\mathbb{A}P}^{rat}$ has a graded model.*

Proof. Let $\mathbf{V}(S)$ be a superstructure over S and $\mathbb{R} \cup A \subseteq S$. We suppose that a formula $\varphi(\vec{x}, \vec{a})$ with parameters from A, a weak model $\mathfrak{A}$ of T, and the relation $\models$ are represented by sets in $\mathbf{V}(S)$. Then ${}^*\varphi(\vec{x}, \vec{a})$ and ${}^*\mathfrak{A}$ are sets in the nonstandard universe $\mathbf{V}({}^*S)$, and ${}^*\models$ is an internal relation. If the context is clear we write simply $\models$.

Let $\mathbb{A}$ be countable, and assume L has countably many constants not appearing in T. From the proof of the weak completeness theorem, T has a weak model $\mathfrak{A} = \langle A,\, R_i,\, c_j,\, \mu_n\rangle_{i\in I,\, j\in J,\, n\in\mathbb{N}}$ such that $\mathfrak{A}$ satisfies each theorem of graded $L_{\mathbb{A}P}^{\mathrm{rat}}$, $A = \{c_j \mid j \in J\}$, and the domain of each μ_n is the set od $L_{\mathbb{A}P}^{\mathrm{rat}}$-definable subsets of A^n. Form the internal structure ${}^*\mathfrak{A}$, let $\widehat{\mathfrak{A}} = \langle {}^*A,\, {}^*R_i,\, {}^*c_j,\, L(\mu_n)\rangle$, where $L(\mu_n)$ is the Loeb measure of μ_n. Every $L(\mu_n)$-measurable set can be approximated above and below by $*$-definable sets in n variables. Using this fact and axioms (H_2) and (H_3) in $\mathfrak{A}$, it can be shown that $\widehat{\mathfrak{A}}$ is a graded probability structure. An induction of formulas will show that $\widehat{\mathfrak{A}}$ is $L_{\mathbb{A}P}^{\mathrm{rat}}$-equivalent to $\mathfrak{A}$. □

Remark 2.6. It is needed to be stressed that measures on graded model $\langle {}^*\mathfrak{A}, L(\mu_n)\rangle$ also have for measure rang, set of rationals of $[0,1]$ interval. Fact that some definable set S has measure $q \in [0,1]$ can be written by

$$\langle {}^*\mathfrak{A}, L(\mu_n)\rangle \models (P\vec{y} = q)\varphi_0(\vec{y}, \vec{c}\,)$$

for some formula $\varphi_0(\vec{x}) \in L_{\mathbb{A}P}^{\mathrm{rat}}$.

Considering $L_{\mathbb{A}P}^{\mathrm{rat}}$-equivalent graded model $\widehat{\mathfrak{A}} = \langle {}^*\mathfrak{A}, L(\mu_n)\rangle$ and weak model $\mathfrak{A}$, we'll have that $\langle \mathfrak{A}, \mu_n\rangle \models (P\vec{y} = q)\varphi_0(\vec{y}, \vec{c}\,)$, where we can conclude that measure of set S is rational number of $[0,1]$ interval.

As consequence of Completness Theorem for full $L_{\mathbb{A}P}$ (see [3, 4]) and theorem 2.5 we have

Theorem 2.7. (Completeness Theorem for full $L_{\mathbb{A}P}^{\mathrm{rat}}$) *Every countable consistent set T of sentences of $L_{\mathbb{A}P}^{rat}$ has a probability model.*

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