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CONTINUITY FOR THE LITTLEWOOD-PALEY OPERATOR AND ITS COMMUTATOR ON HERZ TYPE HARDY SPACES

Liu Lanzhe and Wang Hua

College of Mathematics
Changsha University of Science and Technology
Changsha 410077, P. R. of China
(e-mail: lanzheliu@263.net)

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Abstract. In this paper, the continuity for Littlewood-Paley operators and its commutator on Herz type Hardy spaces are obtained.

1. PRELIMINARIES AND THEOREMS

Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+\varepsilon)}$,

(3) $|\psi(x+y) - \psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

Let b be a locally integrable function. The commutator of Littlewood-Paley operator is defined by

$$g^*_{\mu,b}(f)(x) = \left[\iint_{R^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_{b,t}(x,y)|^2 \frac{dydt}{t^{1+n}}\right]^{1/2}, \qquad \mu > 1$$

where

$$F_{b,t}(x,y) = \int_{R^n} \psi_t(y-z)f(z)(b(x)-b(z))dy,$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. We also define

$$g_\mu^*(f)(x) = \left[\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |f * \psi_t(y)|^2 \frac{dydt}{t^{1+n}}\right]^{1/2}, \qquad \mu > 1$$

which is the Littlewood-Paley operator ([15].) As well known, the operators is of great interest in harmonic analysis ([7, 8, 15]).

Theory of Herz type Hardy spaces have been recently developed ([4, 9, 10, 11]). Lu and Yang studied the boundedness of commutators on Herz type Hardy spaces ([12]). The main purpose of this paper is to consider the boundedness of Littlewood-Paley operator and its commutator on Herz type Hardy spaces. We will work on R^n, $n > 2$. Let us first introduce some definitions ([5, 10, 11]).

Let $B_k = \{x \in R^n : |x| \le 2^k\}, A_k = B_k \setminus B_{k-1}, k \in Z$ and $\chi_k = \chi_{A_k}$ for $k \in Z$, where χ_E is the characteristic function of the set E.

Definition 1. *Let $0 < p$, $q < \infty$ and $\alpha \in R$.*

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p} = \{f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}(R^n)} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}(R^n)} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||_{L^q}^p\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{K_q^{\alpha,p}(R^n)} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}(R^n)} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p}.$$

Definition 2. *Let $0 < p$, $q < \infty$ and $\alpha \in R$. For $k \in Z$ and measurable function f on R^n, $m_k(\lambda, f) = |\{x \in A_k : |f(x)| > \lambda\}|$. For $k \in N$, $\tilde{m}_k(\lambda, f) = m_k(\lambda, f)$ and $\tilde{m}_0(\lambda, f) = |\{x \in B_0 : |f(x)| > \lambda\}|$.*

(1) The homogeneous weak Herz space is defined by

$$W\dot{K}_q^{\alpha,p}(R^n) = \{f : ||f||_{W\dot{K}_q^{\alpha,p}(R^n)} < \infty\},$$

where

$$||f||_{W\dot{K}_q^{\alpha,p}} = \sup_{\lambda>0} \lambda \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} m_k(\lambda, f)^{p/q} \right]^{1/p};$$

(2) The nonhomogeneous weak Herz space is defined by

$$WK_q^{\alpha,p}(R^n) = \{f : ||f||_{WK_q^{\alpha,p}(R^n)} < \infty\},$$

where

$$||f||_{WK_q^{\alpha,p}(R^n)} = \sup_{\lambda>0} \lambda \left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, f)^{p/q} \right]^{1/p}.$$

Definition 3. *Let $\alpha \in R$, $1 < q \leq \infty$ and $b \in L_{loc}(R^n)$. A function $a(x)$ on R^n is called a central (α, q, b)-atom(or a central (α, q, b)-atom of restrict type), if*

1) $Suppa \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),

2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,

3) $\int_{R^n} a(x)dx = \int_{R^n} a(x)b(x)dx = 0$.

A temperate distribution f belongs to $H\dot{K}_{q,b}^{\alpha,p}(R^n)$ (or $HK_{q,b}^{\alpha,p}(R^n)$), if $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, where a_j is a central (α, q, b)-atom (or a central (α, q, b)-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$(or $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$). Moreover, $||f||_{H\dot{K}_{q,b}^{\alpha,p}}$ or $\left(||f||_{HK_{q,b}^{\alpha,p}}\right) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}$.

Our main results are following theorems.

Theorem 1. *Let $0 < p < \infty$, $1 < q < \infty$, $n(1 - 1/q) \leq \alpha < n(1 - 1/q) + \varepsilon$ and $\mu > max(n/2, n/q)$. Then g_μ^* is bounded from $H\dot{K}_q^{\alpha,p}(R^n)$ to $\dot{K}_q^{\alpha,p}(R^n)$.*

Theorem 2. *Let $0 < p \leq 1 < q < \infty$, $\alpha = n(1-1/q)+\varepsilon$ and $\mu > max(n/2, n/q)$. Then g_μ^* is bounded from $H\dot{K}_q^{\alpha,p}(R^n)$ to $W\dot{K}_q^{\alpha,p}(R^n)$.*

Theorem 3. *Let $b \in BMO(R^n)$, $0 < p < \infty$, $1 < q < \infty$, $n(1 - 1/q) \leq \alpha < n(1 - 1/q) + \varepsilon$ and $\mu > \max(n/2, n/q)$. Then $g_{\mu,b}^*$ is bounded from $H\dot{K}_{q,b}^{\alpha,p}(R^n)$ to $\dot{K}_q^{\alpha,p}(R^n)$.*

Theorem 4. *Let $b \in BMO(R^n)$ and $0 < p \le 1 \le q < \infty$, $\alpha = n(1-1/q)+\varepsilon$ and $\mu > \max(n/2, n/q)$. Then, for any $\lambda > 0$ and $f \in HK_{q,b}^{\alpha,p}(R^n)$,*

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, g_{\mu,b}^*(f))^{p/q}\right]^{1/p} \le C\lambda^{-1}||f||_{HK_{q,b}^{\alpha,p}(R^n)} \left(1+\log^+(\lambda^{-1}||f||_{HK_{q,b}^{\alpha,p}(R^n)})\right).$$

2. PROOF OF THEOREMS

Proof of Theorem 1. Let $f \in H\dot{K}_q^{\alpha,p}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f([9]).

$$\begin{aligned}
||g_\mu^*(f)||_{\dot{K}_q^{\alpha,p}} &\le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3} |\lambda_j|\ ||g_\mu^*(a_j)\chi_k||_{L^q}\right)^p\right]^{1/p} \\
&\quad + C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|||g_\mu^*(a_j)\chi_k||_{L^q}\right)^p\right]^{1/p} \\
&= I_1 + I_2.
\end{aligned}$$

By the boundedness of g_μ^* on $L^q(R^n)$, when $\mu > \max(n/2, n/q)$([15]), we have

$$\begin{aligned}
I_2 &\le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|\ ||a_j||_{L^q}\right)^p\right]^{1/p} \\
&\le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j| 2^{-j\alpha}\right)^p\right]^{1/p} \\
&\le \begin{cases} C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=k-2}^{\infty} |\lambda_j|^p 2^{-j\alpha p}\right]^{1/p}, & 0<p\le 1 \\ C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|^p 2^{-j\alpha p/2}\right)(\sum_{j=k-2}^{\infty} 2^{-j\alpha p'/2})^{p/p'}\right]^{1/p}, & p>1 \end{cases} \\
&\le \begin{cases} C\left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}\right)\right]^{1/p}, & 0<p\le 1 \\ C\left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2}\right)\right]^{1/p}, & p>1 \end{cases} \\
&\le C\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \le C||f||_{H\dot{K}_q^{\alpha,p}(R^n)}.
\end{aligned}$$

Notice that for $j \le k-3$ and $x \in A_k$, by the vanishing moment of a_j, we gain for I_1

$$
\begin{aligned}
g_\mu^*(a_j)(x) &\le \left[\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left(\int_{B_j}|\psi_t(y-z)-\psi_t(y)||a_j(z)|dz\right)^2\frac{dydt}{t^{1+n}}\right]^{1/2}\\
&\le C\left[\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left(\int_{B_j}t^{-n}|a_j(z)|\frac{(|z|/t)^\varepsilon}{(1+|y|/t)^{n+1+\varepsilon}}dz\right)^2\frac{dydt}{t^{1+n}}\right]^{1/2}\\
&\le C|B_j|^{1+\varepsilon/n-\alpha/n-1/q}\left[\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{t^{1-n}}{(t+|y|)^{2(n+1+\varepsilon)}}dydt\right]^{1/2}\\
&\le C|B_j|^{1+\varepsilon/n-\alpha/n-1/q}\left[\int_0^\infty t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y|)^{2(n+1+\varepsilon)}}tdt\right]^{1/2}.
\end{aligned}
$$

Let $M(g)$ be the Hardy-Littlewood maximal operator of g. Notice that, for a locally integrable function g,

$$
t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu}g(y)dy \le CM(g)(x)
$$

and

$$
\int_0^\infty \frac{tdt}{(t+|x|)^{2(n+1+\varepsilon)}} = C|x|^{-2(n+\varepsilon)}.
$$

We can conclude

$$
\begin{aligned}
g_\mu^*(a_j)(x) &\le C|B_j|^{1+\varepsilon/n-\alpha/n-1/q}\left(\int_0^\infty \frac{tdt}{(t+|x|)^{2(n+1+\varepsilon)}}\right)^{1/2}\\
&\le C2^{jn(1+\varepsilon/n-\alpha/n-1/q)-k(n+\varepsilon)}.
\end{aligned}
$$

Starting with the fact that $n(1-1/q)\le\alpha<n(1-1/q)+\varepsilon$, we get

$$
\begin{aligned}
I_1 &\le C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{jn(1+\varepsilon/n-\alpha/n-1/q)-k(n+\varepsilon)+kn/q}\right)^p\right]^{1/p}\\
&\le \begin{cases} C\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=j+3}^{\infty}2^{(j-k)(n(1-1/q)+\varepsilon-\alpha)p}\right)\right]^{1/p}, & 0<p\le 1\\ C\left[\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p 2^{(j-k)(n(1-1/q)+\varepsilon-\alpha)p/2}\right)\right.\\ \qquad\left.\cdot\left(\sum_{j=-\infty}^{k-3}2^{(j-k)(n(1-1/q)+\varepsilon-\alpha)p'/2}\right)^{p/p'}\right]^{1/p}, & p>1\end{cases}\\
&\le C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\le C||f||_{H\dot K_q^{\alpha,p}(R^n)}.
\end{aligned}
$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. Let $f(x)=\sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f

$$\begin{aligned}
\|g_\mu^*(f)\|_{W\dot K_q^{\alpha,p}(R^n)} &\le C\sup_{\lambda>0}\lambda\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}m_k\left(\lambda/2,\sum_{j=-\infty}^{k-3}|\lambda_j|g_\mu^*(a_j)\right)^{p/q}\right]^{1/p}\\
&\quad + C\sup_{\lambda>0}\lambda\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}m_k\left(\lambda/2,g_\mu^*\left(\sum_{j=k-2}^{\infty}|\lambda_j|a_j|\right)\right)^{p/q}\right]^{1/p}\\
&= J_1+J_2.
\end{aligned}$$

Similar to I_2, we have for J_2

$$\begin{aligned}
J_2 &\le C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left\|\sum_{j=k-2}^{\infty}\lambda_j a_j\right\|_{L^q}^p\right]^{1/p}\le C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p}\right)\right]^{1/p}\\
&\le C\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p}\right)\right]^{1/p}\le C\|f\|_{H\dot K_q^{\alpha,p}(R^n)};
\end{aligned}$$

Notice that $\alpha=n(1-1/q)+\varepsilon$. Similar to the proof of Theorem1, we have for J_1

$$g_\mu^*(a_j)(x)\le C2^{j(n(1-1/q)+\varepsilon-\alpha)-k(n+\varepsilon)}=C2^{-k(n+\varepsilon)}.$$

Thus

$$J_1\le C\sup_{\lambda>0}\lambda\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}m_k\left(\lambda/2,C2^{-k(n+\varepsilon)}\sum_{j=-\infty}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p}.$$

If for a fixed $\lambda>0$, $\left|\left\{x\in A_k: C2^{-k(n+\varepsilon)}\sum_{j=-\infty}^{\infty}|\lambda_j|>\lambda/2\right\}\right|\ne 0$ then

$$2^{k(n+\varepsilon)}\le C\lambda^{-1}\sum_{j=-\infty}^{\infty}|\lambda_j|.$$

If K_λ is the maximal integer k which satisfies this estimate, then

$$\begin{aligned}
J_1 &\le C\sup_{\lambda>0}\lambda\left(\sum_{k=-\infty}^{K_\lambda}2^{k(\alpha+n/q)}\right)^{1/p}\le C\sup_{\lambda>0}\lambda 2^{K_\lambda(n+\varepsilon)}\\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|\le C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\le C\|f\|_{HK_q^{\alpha,p}(R^n)}.
\end{aligned}$$

This completes the proof of Theorem 2. $\square$

Proof of Theorem 3. Let $f \in H\dot{K}^{\alpha,p}_{q,b}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x))$ be the atomic decomposition for f

$$\begin{aligned} ||g^*_{\mu,b}(f)||_{\dot{K}^{\alpha,p}_q(R^n)} &\leq C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3} |\lambda_j|\ ||g^*_{\mu,b}(a_j)\chi_k||_{L^q}\right)^p\right]^{1/p} \\ &\quad + C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|||g^*_{\mu,b}(a_j)\chi_k||_{L^q}\right)^p\right]^{1/p} \\ &= K_1 + K_2. \end{aligned}$$

By the boundedness of $g^*_{\mu,b}$ on $L^q(R^n)$, we have for K_2

$$\begin{aligned} K_2 &\leq C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|||a_j||_{L^q}\right)^p\right]^{1/p} \\ &\leq C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|2^{-j\alpha}\right)^p\right]^{1/p} \\ &\leq \begin{cases} C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p}\right]^{1/p}, & 0<p\leq 1 \\ C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p/2}\right)\left(\sum_{j=k-2}^{\infty} 2^{-j\alpha p'/2}\right)^{p/p'}\right]^{1/p}, & p>1 \end{cases} \\ &\leq \begin{cases} C\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}\right)\right]^{1/p}, & 0<p\leq 1 \\ C\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2})\right]^{1/p}, & p>1 \end{cases} \\ &\leq C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p} \leq C||f||_{H\dot{K}^{\alpha,p}_{q,b}(R^n)}. \end{aligned}$$

Let $b_j = |B_j|^{-1}\int_{B_j} b(x)dx$ with the properties of $BMO(R^n)$ by the vanishing moment of a_j where $x \in A_k$ and $j \leq k-3$. Similar to the proof of Theorem 1, we have for K_1

$$\begin{aligned} &g^*_{\mu,b}(a_j)(x) \\ &\leq \left[\iint_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left(\int_{B_j}|\psi_t(y-z)-\psi_t(y)||a_j(z)||b(x)-b(z)|dz\right)^2\frac{dydt}{t^{1+n}}\right]^{1/2} \\ &\leq C2^{-k(n+\varepsilon)}\left[|b(x)-b_j|2^{j(\varepsilon+n(1-1/q)-\alpha)} + 2^{j(\varepsilon-\alpha)}\left(\int_{B_j}|b(y)-b_j|^{q'}dy\right)^{1/q'}\right] \\ &\leq C2^{-k(n+\varepsilon)}\left[2^{j(\varepsilon+n(1-1/q)-\alpha)}|b(x)-b_k| + (k-j)2^{j(\varepsilon+n(1-1/q)-\alpha)}||b||_{BMO}\right]. \end{aligned}$$

Therefore

$$
\begin{aligned}
K_1 &\le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3} |\lambda_j| 2^{-k(n+\varepsilon)+j(\varepsilon+n(1-1/q)-\alpha)}\left(\int_{B_k} |b(x)-b_k|^q dx\right)^{1/q}\right)^p\right]^{1/p} \\
&\quad + C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3} |\lambda_j|(k-j) 2^{-k(n+\varepsilon)+j(\varepsilon+n(1-1/q)-\alpha)} 2^{kn/q}||b||_{BMO}\right)^p\right]^{1/p} \\
&\le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3} |\lambda_j|(k-j) 2^{(j-k)(\varepsilon+n(1-1/q))}\right)^p\right]^{1/p} ||b||_{BMO} \\
&\le \begin{cases} C||b||_{BMO}\left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=j+3}^{\infty} (k-j) 2^{(j-k)(\varepsilon+n(1-1/\varepsilon)-\alpha)p}\right)\right]^{1/p}, 0<p\le 1 \\ C||b||_{BMO}\left[\sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} |\lambda_j|^p (k-j)^p 2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)/2}\right)\right]^{1/p}, p>1 \end{cases} \\
&\le C\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \le C||f||_{H\dot{K}_q^{\alpha,p}(R^n)}.
\end{aligned}
$$

This completes the proof of Theorem 3. $\square$

Proof of Theorem 4. Let $f \in HK_{q,b}^{\alpha,p}(R^n)$ and $f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f

$$
\begin{aligned}
\left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, g_{\mu,b}^*(f))^{p/q}\right]^{1/p} &\le C\left[\sum_{k=0}^{3} 2^{k\alpha p} \tilde{m}_k(\lambda, g_{\mu,b}^*(f))^{p/q}\right]^{1/p} \\
&\quad + C\left[\sum_{k=4}^{\infty} 2^{k\alpha p} \tilde{m}_k\left(\lambda/2, \sum_{j=0}^{k-3} |\lambda_j| g_{\mu,b}^*(a_j)\right)^{p/q}\right]^{1/p} \\
&\quad + C\left[\sum_{k=4}^{\infty} 2^{k\alpha p} \tilde{m}_k\left(\lambda/2, g_{\mu,b}^*\left(\sum_{j=k-2}^{\infty} \lambda_j a_j\right)\right)^{p/q}\right]^{1/p} \\
&= L_1 + L_2 + L_3.
\end{aligned}
$$

By the boundedness of $g_{\mu,b}^*$ and $0 < p \le 1$, we have for L_1 and L_3

$$
\begin{aligned}
L_1 &\le C\lambda^{-1}\left[\sum_{k=0}^{3} 2^{k\alpha p} ||f||_{L^q}^p\right]^{1/p} \le C\lambda^{-1}\left(\sum_{j=0}^{\infty} |\lambda_j|^p ||a_j||_{L^q}^p\right)^{1/p} \\
&\le C\lambda^{-1}\left(\sum_{j=0}^{\infty} |\lambda_j|^p 2^{-j\alpha p}\right)^{1/p} \le C\lambda^{-1}\left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{1/p} \\
&\le C\lambda^{-1}||f||_{HK_{q,b}^{\alpha,p}(R^n)},
\end{aligned}
$$

$$
\begin{aligned}
L_3 &\leq C\lambda^{-1}\left[\sum_{k=4}^{\infty}2^{k\alpha p}\left\|\sum_{j=k-2}^{\infty}\lambda_j a_j\right\|_{L^q}^p\right]^{1/p} \leq C\lambda^{-1}\left[\sum_{k=4}^{\infty}2^{k\alpha p}\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p}\right]^{1/p}\\
&\leq C\lambda^{-1}\left[\sum_{j=0}^{\infty}|\lambda_j|^p\sum_{k=0}^{j+2}2^{(k-j)\alpha p}\right]^{1/p} \leq C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}\\
&\leq C\lambda^{-1}||f||_{HK_{q,b}^{\alpha,p}(R^n)}.
\end{aligned}
$$

Similar to the proof of Theorem 1, we have for L_2

$$
g_{\mu,b}^{*}(a_j)(x) \leq C2^{-k(n+\varepsilon)}(|b(x)-b_k| + k||b||_{BMO}),
$$

Therefore

$$
\begin{aligned}
L_2 &\leq C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/4, C2^{-k(n+\varepsilon)}|b(x)-b_k|\sum_{j=0}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p}\\
&\quad + C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/4, C2^{-k(n+\varepsilon)}k||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p}\\
&= L_2^{(1)} + L_2^{(2)}.
\end{aligned}
$$

By using John-Nirenberg inequality ([15]), we gain for $L_2^{(1)}$

$$
\begin{aligned}
L_2^{(1)} &\leq C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\left(\exp\left(-\frac{C2^{k(n+\varepsilon)}\lambda}{||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)2^{kn}\right)^{p/q}\right]^{1/p}\\
&\leq C\left[\sum_{k=0}^{\infty}2^{k(n+1)p}\exp\left(-\frac{C\lambda 2^{k(n+\varepsilon)}}{||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)\right]^{1/p}\\
&\leq C\left[\int_0^{\infty}x^{p-1}\exp\left(-\frac{c\lambda x}{||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)dx\right]^{1/p}\\
&= C\lambda^{-1}||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|\left(\int_0^{\infty}t^{p-1}e^{-t}dt\right)^{1/p}\\
&\leq C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p} \leq C\lambda^{-1}||f||_{HK_{q,b}^{\alpha,p}(R^n)}.
\end{aligned}
$$

If there exists $u > 1$, such that $2^x/x \leq u$ for $x \geq 3$, then $2^x \leq cu\log^+ u$. By using this fact, we have for $L_2^{(2)}$ that if

$$
\left|\left\{x \in A_k : C2^{-k(n+\varepsilon)}k||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j| > \lambda/4\right\}\right| \neq 0,
$$

then

$$1 < 2^{k(n+\varepsilon)/k(n+\varepsilon)} < C\lambda^{-1}||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|.$$

Therefore

$$2^{k(n+\varepsilon)} \leq C\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\log^{+}\left(\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\right),$$

If K_λ is the maximal integer k which satisfies this estimate, then

$$\begin{aligned}
L_2^{(2)} &\leq C\left(\sum_{k=4}^{K_\lambda}2^{k\alpha p}2^{knp/q}\right)^{1/p} \leq C2^{K_\lambda(n+\varepsilon)}\\
&\leq C\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\log^{+}\left(\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\right)\\
&\leq C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}\log^{+}\left(\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}\right)\\
&\leq C\lambda^{-1}||f||_{HK_{q,b}^{\alpha,p}(R^n)}\log^{+}\left(\lambda^{-1}||f||_{HK_{q,b}^{\alpha,p}(R^n)}\right).
\end{aligned}$$

Now, summing up the above estimates, we have

$$\left[\sum_{k=0}^{\infty}2^{k\alpha p}\tilde{m}_k(\lambda, g_{\mu,b}^{*}(f))^{p/q}\right]^{1/p} \leq C\lambda^{-1}||f||_{H_{q,b}^{\alpha,p}(R^n)}\left(1+\log^{+}\left(\lambda^{-1}||f||_{HK_{q,b}^{\alpha,p}(R^n)}\right)\right).$$

This completes the proof of Theorem 4. □

If we relax the condition of b in Theorem 4, then we get the following result.

Theorem 5. *Let $b \in L^\infty(R^n)$, $0 < p \leq 1 \leq q < \infty$ and $\alpha = n(1-1/q)+1$,$\mu > max(n/2, n/q)$. Then, $g_{\mu,b}^{*}$ is bounded from $H\dot{K}_{q,b}^{\alpha,p}(R^n)$ to $W\dot{K}_{q}^{\alpha,p}(R^n)$.*

Proof. Let $f \in H\dot{K}_{q,b}^{\alpha,p}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f.

$$\begin{aligned}
\sum_{k=-\infty}^{\infty}2^{k\alpha p}\tilde{m}_k(\lambda, g_{\mu,b}^{*}(f))^{p/q} &= \sum_{k=-\infty}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/2, \sum_{j=-\infty}^{k-3}|\lambda_j|g_{\mu,b}^{*}(a_j)\right)^{p/q}\\
&\quad + C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/2, g_{\mu,b}^{*}\left(\sum_{j=k-2}^{\infty}\lambda_j a_j\right)\right)^{p/q}\\
&= M_1 + M_2.
\end{aligned}$$

By the boundedness of $g^*_{\mu,b}$ and $0 < p \le 1$, we have for M_2

$$\begin{aligned} M_2 &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\lambda^{-p} \sum_{j=k-2}^{\infty} |\lambda_j|^p ||a_j||^p_{L^q} \\ &\le C\lambda^{-p} \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p} \\ &\le C\lambda^{-p}||f||^p_{H\dot{K}^{\alpha,p}_{q,b}(R^n)}; \end{aligned}$$

Similar to the proof of Theorem 1, we have for M_1

$$g^*_{\mu,b}(a_j)(x) \le C2^{-k(n+\varepsilon)}||b||_{L^\infty},$$

and if $\left|\left\{x \in A_k : C2^{-k(n+\varepsilon)}||b||_{L^\infty} \sum_{j=-\infty}^{k-3} |\lambda_j| > \lambda/2\right\}\right| \ne 0$ then

$$2^{k(n+\varepsilon)} \le C\lambda^{-1}||b||_{L^\infty} \sum_{j=-\infty}^{\infty} |\lambda_j| \le C\lambda^{-1} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p},$$

If K_λ is the maximal integer k which satisfies this estimate, then

$$\begin{aligned} M_1 &\le C \sum_{k=-\infty}^{K_\lambda} 2^{k\alpha p} 2^{knp/q} \le C2^{K_\lambda(n+\varepsilon)p} \\ &\le C\lambda^{-p} \sum_{j=-\infty}^{\infty} |\lambda_j|^p \le \lambda^{-p}||f||^p_{H\dot{K}^{\alpha,p}_{q,b}(R^n)}. \end{aligned}$$

This finishes the proof of Theorem 5. □

Remark. Theorem 1, 2, 3 and 5 also hold for nonhomogeneous Herz-type space.

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