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INHERITANCE PROPERTIES OF SOME CLASSES OF LOCALLY CONVEX SPACES[1]

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Abstract. In this paper we study the inheritance properties (3SP, subspace, quotient) of the classes of locally convex spaces for which, to our knowledge, these matters have not been considered.

1. INTRODUCTION

In [1] (resp. [6]), the classes of p-barrelled, p-semi-reflexive and p-reflexive (resp. k-barrelled, b-barrelled) spaces were studied thoroughly. P-semi reflexive and p-reflexive spaces are named polar-semi-reflexive and polar-reflexive in [5], and they play a significant role in the so-called Pontryagin duality (see [5], 23, 9 (5)). In [1] (resp. [6]), classes of p-spaces, p-distinguished spaces (resp. k-spaces, b-spaces) which are basically connected with precompact (resp. compact, bounded) disks of a locally convex spaces are also studied.

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In the definition of the aforementioned classes of spaces, precompact (resp. compact, bounded) disks of the space E and the topology E'_p (resp. E'_c, E'_β) of uniform convergence on them have the main role. Thus, for example, a locally convex space E is p-barrelled if each E'_p-precompact subset is $E-$equicontinuous. Locally convex space E is p-semi-reflexive (resp. p-reflexive) if $\left(E'_p\right)' = E$ (resp. $\left(E'_p\right)'_p = E$). In the first case it means that the topological dual of the space E'_p equals E, that is, that the topology E'_p of the uniform convergence on the precompact subsets is consistent with the dual pair $\langle E', E\rangle$, and in second case it means that, aside from algebraic equality $\left(E'_p\right)' = E$, we also have a topological equality: $\left(E'_p\right)'_p = E$.

A barrel (resp. disk) T of the space E is a $k-$barrel, $p-$barrel, b-barrel (resp. $k-$disk, $p-$disk, $b-$disk) if its intersection with compact, precompact, bounded disk is relative neighborhood of zero. Locally convex space E is $k-$barrelled, $p-$barrelled, $b-$barrelled (resp. $k-$space, $p-$space, $b-$space) if each $k-$barrel, $p-$barrel, $b-$barrel (resp. $k-$disk, $p-$disk, $b-$disk) in it is a neighborhood of zero. From [1] and [6] we know that E is $k-$space (resp. $p-$space, $b-$space) if and only if E is $k-$barrelled (resp. $p-$barrelled, $b-$barrelled) and E'_c (resp. E'_p, E'_β) is complete locally convex space.

From [1] and [6]) we know that a classes of $k-$barrelled, $p-$barrelled, $b-$barrelled, $k-$spaces, $p-$spaces, $b-$spaces are stable under to the arbitrary inductive topology, and therefore on the separated quotient, inductive limit and direct sum as well; then under to any product and completion. The stability under to the arbitrary subspace as well as to the three-space-problem was not discussed there.

2. RESULTS

Now, using the examples from [10], we shall prove that the class of $p-$barrelled (resp. $b-$barrelled) spaces is not three-space-stable; based on that as well as Komura's result from [4], Theorem 1.1, that each locally convex space is a closed subspace of some barrelled space, we conclude that these classes are not stable under to the closed subspace.

Theorem 1. *There exists a short exact sequence*

$$0 \to F \xrightarrow{j} E \xrightarrow{q} E/F \to 0,$$

so that the outer members F and E/F are $p-$barrelled spaces, and the middle member E is not.

Proof. From [10], 2.10, we know that there exists such short exact sequence

$$0 \to F \xrightarrow{j} E \xrightarrow{q} E/F \to 0,$$

such that F is Montel space, E/F is normed, therefore quasibarrelled, but not countably barrelled, and that bounded subsets in E are of finite dimension. The outer members F and E/F are apparently $p-$barrelled. Let us explain why the space E is not $p-$barrelled. Since the bounded subsets in E are of finite dimension, we have $E'_\sigma = E'_p = E'_{\beta.}$. Then we have that $E'_\sigma-$bounded is the same as $E'_\sigma-$precompact, and if such subsets are $E-$equicontinuous, then the space E is barrelled, wherefrom E/F is barrelled as well, that is countably barrelled, which is a contradiction. Hence, the space E is not $p-$barrelled, that is, the class of $p-$barrelled spaces is not three-space-stable. □

Corollary 2. *The class of $p-$barrelled paces is not stable under to the closed subspace.*

Proof. Let us take locally convex space E from the theorem 1. E is not $p-$barrelled space. According to Komura's result from [4], Theorem 1.1, we know that there exists a barrelled locally convex space, for instance X, such that E is its closed subspace. Since barrelled space is obviously $p-$barrelled the proof of the Corollary 2 follows. □

Remark 3. The previous example shows that the class of $b-$barrelled spaces is not three-space-stable and it is not stable under to the closed subspace either.

Proof. The notions "$b-$barrelled" and "$p-$barrelled" for space E in Theorem 1 are equivalent. □

Remark 4. For the time being, we do not know whether the class of $k-$barrelled spaces is three-space-stable.

Similarly to $p-$barrelled (resp. $b-$barrelled) spaces, it follows that the class of $k-$barrelled spaces is not stable under to closed subspace. Namely, from [4], 1.1.5, we know that the quasibarrelled space is barrelled if and only if it is $k-$barrelled. Therefore, if E is quasibarrelled which is not barrelled, it is, according to Komura's result [4], Theorem 1.1, closed subspace of some barrelled, that is $k-$barrelled space. Therefore E is closed subspace of $k-$barrelled space which is not $k-$barrelled.

Let final P be some property in the class of all locally convex space. If from being barrelled it follows P, then obviously according to Komura's result [4], Theorem 1.1 P is not stable property under to closed subspace. Based on that, again we have that properties being $k-$barrelled, $p-$barrelled, $b-$barrelled are not stable under to the closed subspace.

If Q is the property also in the class of all locally convex spaces and if bornological space satisfies the property Q, then Q is not stable under to the closed subspace. Based on that we have:

Theorem 5. *A class of $p-$space (resp. $b-$space) is not stable under to the closed subspace.*

Proof. If E is Banach space of infinite dimension, then, according to [9], Theorem, a locally convex space E_σ is not the space of the type D_b, that is, it is not $b-$space. Since the dimension of the space E is less that the first inaccessible cardinal, then according to Komura's result from [4], Lemma 1.3 E_σ is closed subspace of some bornological space, for instance X. Since X is clearly $p-$space, that is $b-$space, these two properties are not stable under to the closed subspace. (For subspaces of b-spaces see also [3], examples 2, 3 and 4 (i), (ii).) □

Remark 6. A bornological space is $p-$space, because disk U absorbs all precompact subsets, if and only if it absorbs all bounded subsets.

Remark 7. We do not know whether the class of $k-$spaces (Kelly's spaces by some authors) is stable under to the closed subspace.

As for $p-$semi-reflexive and $p-$reflexive locally convex spaces, in view of their inheritance properties, classes of semi-reflexive and reflexive spaces are not three-space-stable. Based on that and the example 1.5 from [10], we have the following results:

Theorem 8. *There exists a $p-$semi-reflexive locally convex space E and its closed subspace F, such that the corresponding quotient E/F is not $p-$semi-reflexive.*

Proof. Let E be a semi-reflexive locally convex space, and let F be its closed subspace, so that E/F is not semi-reflexive (such one exists according to [5] or [11]). Since the property "semi-reflexive" depends on dual pair only, it means that space E_σ is semi-reflexive, and that the corresponding quotient $(E/F)_\sigma$ is not semi-reflexive. The notions "semi-reflexive" and "p-semi-reflexive" being equivalent for the spaces E_σ and $(E/F)_\sigma$, we have got the proof of the theorem. □

Theorem 9. *The property "$p-$semi-reflexive" is not three-space-stable.*

Proof. Let

$$0 \to F \xrightarrow{j} E \xrightarrow{q} E/F \to 0$$

be a short exact sequence of locally convex spaces, such that the outer members F and E/F are semi-reflexive, and the middle member E is not semi-reflexive (such one exists according to [10], 1.5). From this it follows that the following sequence

$$0 \to F_\sigma \xrightarrow{j} E_\sigma \xrightarrow{q} (E/F)_\sigma \to 0$$

is exact and the outer members F_σ and $(E/F)_\sigma$ are $p-$semi-reflexive, and the middle member is not. The theorem is proved. □

By using the method presented in [7], Theorem 1., that is [10], 4.2, we can prove:

Theorem 10. *If $F'_p = E'_p/F^o$ and if the outer members F and E/F of short exact sequence*

$$0 \to F \xrightarrow{j} E \xrightarrow{q} E/F \to 0$$

are $p-$semi-reflexive, then the middle member E is such as well.

Similarly or directly using the example 1.5 from [10], we have that the property "$p-$reflexive" is also not three-space-stable.

In [8], β^*-semi-reflexive and β^*-reflexive spaces have been defined, by taking strongly bounded subsets of given locally convex space instead of bounded. Namely, locally convex space E is β^*-semi-reflexive (resp. β^*-reflexive) if $\left(E'_{\beta^*}\right)' = E \left(\text{resp.} \left(E'_{\beta^*}\right)'_{\beta^*} = E\right)$ where E'_{β^*} is the topology of uniform convergence on a family of strongly bounded subsets of space E (see [8] for details). There it is proven that the class of semi-reflexive spaces is true subclass of β^*-semi-reflexive, as well as that a space is semi-reflexive if and only if it is $p-$semi-reflexive and β^*-semi-reflexive. As well as the other classes of different semi-reflexive spaces, this class is also stable under to the closed subspace.

For quotient and three-space-problem the answer is negative.

Theorem 11. *A class of β^*-semi-reflexive (resp. β^*-reflexive) spaces is not stable under to the separated quotient, and under to the three-space-problem.*

Proof. Indeed, from [5], that is [11], we know that there exists Frechet-Montel space which has quotient which is isomorph to the space l_1; wherefrom it follows that a class β^*-semi-reflexive (resp. β^*-reflexive) spaces is not stable under to the separated quotient. □

By using the example 1.5 from [10] we have that the property "being β^*-semi-reflexive (resp. β^*-reflexive) is not three-space stable either. Indeed, that example represents exact sequence in which the outer members are Montel and Freche-Montel space respectively, and the middle member is barrelled space which is not semi-reflexive. Since the notions "β^*-semi-reflexive" (resp. β^*-reflexive) and "semi-reflexive" (resp. reflexive) are equal for barrelled spaces, we have that the mentioned classes of spaces are not three-space stable.

Similarly to the theorem 10 we have:

Theorem 12. *If $F'_{\beta^*} = E'_{\beta^*}/F^o$, and if the outer members F and E/F of short exact sequence are β^*-semi-reflexive, then the middle member E is such as well.*

References

[1] J. Dazord, M. Jourlin, *Sur quelques classes d'espaces localement convexes*, Publ. du Dép. de Math. Lyon, **8-2** (1969), 39–69.

[2] Z. Kadelburg, S. Radenović, *Three-space-problem for some classes of linear topological spaces*, Comment. Math. Univ. Carolinae, **37** (1996), 507–514.

[3] J. Kakol, *On Subspaces of $b-$Spaces in the Sense of Noureddine*, Math. Nachr., **120** (1985), 43–51

[4] Y. Kōmura, *On linear topological spaces*, Kumamoto J. Sci., **5** (1962), 148–157.

[5] G. Köthe, *Topological Vector Spaces I*, 2^{and} ed., Springer, Berlin-Heidelberg-New York (1983).

[6] K. Noureddine, *Nouvelles classes d'espaces localment convexes*, Publ. Dep. Math. Lyon, **10, 3** (1973), 105–123.

[7] S. Radenović, Z. Kadelburg, *Three-space-problem for inductively (semi-)reflexive locally convex spaces*, Publ. de l'institut math. Nouvelle série. (to appear)

[8] S. Radenović, *Sur un type de la réflexivité d'espaces localment convexes*, Publ. de l'institut Math., Nouvelle série **24 (38)** (1978), 133–137.

[9] S. Radenović, Z. Kadelburg, *Weak topology in locally convex spaces with a fundamental sequence of bounded subsets*, Matematički Vesnik, **49** (1997), 225–227.

[10] W. Roelcke, S. Dierolf, *On the three-space-problem for topological vector spaces*, Collect. Mat., **32** (1981), 13–25.

[11] H. H. Schaefer, *Topological Vector Spaces*, 2^{and} ed., Springer, Berlin-Heidelberg-New York (1971).