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ON THE BOUNDEDNESS AND THE STABILITY RESULTS FOR THE SOLUTIONS OF CERTAIN THIRD ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we show the asymptotic stabilty of the trivial solution $x = 0$ for $p \equiv 0$ and the boundedness as well as the ultimate boundedness result for $p \neq 0$ with the use of a single complete Lyapunov function. The results obtained here improves on the results already obtained for this class of third order nonlinear differential equations.

1. INTRODUCTION

In this paper, we study the third order differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = P(t), \qquad (1.1)$$

where a and b are positive constants. The functions h and P are continuous in the respective argument displayed explicitly.

The corresponding linear equation to (1.1) assumes the form

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0 \qquad (1.2)$$

To the above, it is well known that all solutions tend to the trivial solution, as $t \to \infty$, provided that the Routh-Hurwitz conditions $a > 0$, $ab - c > 0$ are satisfied.

Intresting results have been obtained by several authors on the boundedness and stability properties of solutions for various equations of 2nd, 3rd, 4th and even 5th order. Some of these results have been summarised in [10].

In earlier studies Andres[1-2], Chukwu[4], Ezeilo[5-9], and Tejumola[11] have studied (1.1) using Lyapunov functions to investigate the boundedness and ultimate boundeness of solution on one side and stability and asymptotic stability on the other side. In their work, they all employed incomplete Lyapunov functions for their studies, except for Chukwu[4] who made an attempt and indeed used a complete (Yoshizawa) function with the use of signum function. By using a complete Lyapunov function not necessarily with signum function, ultimate boundedness of solution could easily be discussed while with incomplete functions, there may be need to have the space path or trajectories examination before conclusion on ultimate boundedness of solution could be made.

In particular the equation (1.1) is better handled as a system of three-coupled first order equations by letting;

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -az - by - h(x) + P(t) \end{aligned} \tag{1.3}$$

In this our study, we shall use a single complete Lyapunov function without the use of any signum function to show that the equation considered has stable and bounded solutions on a real line.

Definition 1.1. *A Lyapunov function V defined as $V : I \times \Re^n \to \Re$ is said to be COMPLETE if for $X \in \Re^n$*
(i) $V(t, X) \geq 0$
(ii) $V(t, X) = 0$, if and only if $X = 0$
and
(iii) $\dot{V}\mid_{1.3}(t, X) \leq -c\,|X|$ where c is any positive constant and $|X|$ given by $|X| = (\sum (x_i^2))^{\frac{1}{2}}$ such that

$|X| \to \infty$ *as* $X \to \infty$

Definition 1.2. *A Lyapunov function* V *defined as* $V : I \times \Re^n \to \Re$ *is said to be INCOMPLETE if for* $X \in \Re^n$ *(i) and (ii) of the above definition is satisfied and in addition*

(iii) $\dot{V}(t, X) \mid_{1.3} \leq -c\,|X|_*$ *where c is any positive constant and* $|X|_*$ *given by* $|X|_* = (x^2 + y^2)^{\frac{1}{2}}$ *or* $|X|_* = (y^2 + z^2)^{\frac{1}{2}}$ *or* $|X|_* = (x^2 + z^2)^{\frac{1}{2}}$ *or* $|X|_* = (x^2 + y^2)^{\frac{1}{2}}$ *or* $|X|_* = (x^2)^{\frac{1}{2}}$ *or* $|X|_* = (y^2)^{\frac{1}{2}}$ *or* $|X|_* = (z^2)^{\frac{1}{2}}$ *such that*

$|X|_* \to \infty$ *as* $X \to \infty$

2. FORMULATION OF RESULTS

We considered (1.1) in two major ways and we have prove the following . In the case when $P \equiv 0$ we shall prove:

Theorem 2.1. *Let* h *be continuous with the following conditions*

(i) $H_0 = \frac{h(x)-h(0)}{x} \in I_0$ *,*$x \neq 0$ *with* $I_0 = [\delta, ab]$

(ii) $ab \geq H_0$*,* $\forall x \in \Re$.

(iii) $h(0) = 0$

then every solution $(x(t), y(t), z(t))$ *of the system (1.3) satisfies* $x^2(t)+y^2(t)+z^2(t) \to 0$ *as* $t \to \infty$ *(Asymptotic stability)*

In the case when $P(t) \neq 0$:

Theorem 2.2. *Suppose the following conditions are satisfied:*

(i) Conditions(i)-(iii) of Theorem 2.1 hold; and

(ii) $|P(t)| \leq M$ *(constant) for all* $t \geq 0$ *then there exists a constant* $\mu(0 < \mu < \infty)$ *depending only on a, b,and* δ *such that every solution of (1.1) satisfies*

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) \leq e^{-\frac{1}{2}\mu t} \left\{ A_1 + A_2 \int_{t_0}^{t} |P(\tau)|\, e^{\frac{1}{2}\mu\tau} d\tau \right\}^2$$

for all $t \geq t_0$*, where the constant* $A_1 > 0$*, depends on a, b,* δ *as well as on* $t_0, x(t_0), \dot{x}(t_0)$ *and* $\ddot{x}(t_0)$*; and the constant* $A_2 > 0$ *depends on a, b, and* δ*. We now consider the case when* $P(t)$ *in (1.1) is replaced with* $P(t, x, \dot{x}, \ddot{x})$.

Theorem 2.3. *Following the assumptions of Theorem 2.2 and condition (ii) replaced with* $|P(t,x,\dot{x},\ddot{x})| = (|x|+|y|+|z|)\,r(t)$*, where* $r(t)$ *is a non negative and continuous function of* t*, and satisfies* $\int_0^t r(s)ds \le M < \infty M > 0$.

Then there exists a constant K_0 *which depends on* M, K_1, K_2 *and* t_0 *such that every solution* $x(t)$ *of the equation (1.1) satisfies*

$$|x(t)| \le K_0, \quad |\dot{x}| \le K_0, \quad |\ddot{x}| \le K_0$$

for all sufficiently large t.

Remark. When $h(x) = cx$, equation (1.1) reduces to the linear differential equation with constant coefficients

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = P(t)$$

and conditions (i) $H_0 = \frac{h(x)-h(0)}{x} = c$ and (ii) read as $ab > c$, i.e. $ab-c > 0$ which is the Routh Hurwitz criterion for stability of solution of third order differential equations.

Notations. Throughout this paper $K, K_0, K_1, \ldots K_{12}$ will denote finite positive constants whose magnitudes depend only on the functions h and P as well as constants a, b and δ but are independent of solutions of (1.1). $K_i's$ are not neccessarily the same for each time they occur, but each $K_i, i = 1, 2...$ retains its identity throughout.

3. PRELIMINARY RESULTS

The main tool besides the equation (1.1) itself in the proof of the Theorems (2.1)-(2.3) is the function $V = V(x,y,z)$ defined by

$$\begin{aligned} 2V(x,y,z) = &\left(\tfrac{\delta b\ell}{\ell-1}\right)x^2 + \delta\left\{\tfrac{b(b+1)(\ell-1)+a^2[(\ell-1)+a\ell}{ab(\ell-1}\right\}y^2 + \delta\left\{\tfrac{(b+1)(\ell-1)+a\ell}{ab(\ell-1}\right\}z^2 \\ &+ \tfrac{2a\delta\ell}{\ell-1}xy + \tfrac{2\delta\ell}{\ell-1}xz + 2\delta\left\{\tfrac{(\ell-1)+a\ell}{b(\ell-1)}\right\}yz \end{aligned} \tag{3.1}$$

where $\delta > 0$, $\ell > 1$ and $m^2 > 1$ for all x, y, z.

Lemma 3.1. *Subject to the assumptions of Theorem 2.1 there exist positive constants* $K_i = K_i(a, b, \ell, m, \delta), i = 1, 2$ *such that*

$$K_1(x^2+y^2+z^2) \le V(x,y,z) \le K_2(x^2+y^2+z^2). \tag{3.2}$$

Proof. Clearly $V(0,0,0) \equiv 0$

By rearranging (3.1) we have

$$\begin{aligned} 2V(x,y,z) = \; & \left(\tfrac{\delta b\ell}{\ell-1}\right)\left(\tfrac{b}{m}x + my + \tfrac{m}{a}z\right)^2 + \tfrac{\delta(m^2-1)\ell}{m^2(\ell-1)}x^2 \\ & + \tfrac{\delta\ell}{b(\ell-1)}\left\{z + \tfrac{[(\ell-1)+a\ell(1-m^2)]}{b\ell(\ell-1)}y\right\}^2 \\ & + \delta\left\{\tfrac{4b^2\ell(b+1)(\ell-1)^3+4a^2b\ell(\ell-1)^2[(\ell-1)+a\ell]+a(\ell-1)[2a\ell(m^2-1)-(\ell-1)]}{4ab^2\ell(\ell-1)^2}\right. \\ & \qquad \left. + \tfrac{-a^3\ell[4bm^2(\ell-1)-(1-m^2)^2]}{4ab^2\ell(\ell-1)^2}\right\}y^2 + \delta\left\{\tfrac{(b+1)(\ell-1)-am^2\ell}{ab(\ell-1)}\right\}z^2 \end{aligned} \tag{3.3}$$

$$\begin{aligned} 2V(x,y,z) \geq \; & \tfrac{\delta(m^2-1)\ell}{m^2(\ell-1)}x^2 + \delta\left\{\tfrac{(b+1)(\ell-1)-am^2\ell}{ab(\ell-1)}\right\}z^2 \\ & + \delta\left\{\tfrac{4b^2\ell(b+1)(\ell-1)^3+4a^2b\ell(\ell-1)^2[(\ell-1)+a\ell]+a(\ell-1)[2a\ell(m^2-1)-(\ell-1)]}{4ab^2\ell(\ell-1)^2}\right. \\ & \qquad \left. + \tfrac{-a^3\ell[4bm^2(\ell-1)-(1-m^2)^2]}{4ab^2\ell(\ell-1)^2}\right\}y^2 \end{aligned} \tag{3.4}$$

$$\geq K_1(x^2+y^2+z^2) \tag{3.5}$$

where

$$K_1 = \delta \cdot \min\left\{\left(\tfrac{(m^2-1)\ell}{m^2(\ell-1)}\right), \left(\tfrac{(b+1)(\ell-1)-am^2\ell}{ab(\ell-1)}\right), \Delta\right\}$$

with

$$\Delta = \left\{\left(\tfrac{4b^2\ell(b+1)(\ell-1)^3+4a^2b\ell(\ell-1)^2[(\ell-1)+a\ell]+a(\ell-1)[2a\ell(m^2-1)-(\ell-1)]-a^3\ell[4bm^2(\ell-1)-(1-m^2)^2]}{4ab^2\ell(\ell-1)^2}\right)\right\}.$$

Therefore,

$$2V(x,y,z) \geq K_1(x^2+y^2+z^2).$$

Also from (3.1), by using the Schwartz inequality $|xy| \leq \frac{1}{2}\left|x^2+y^2\right|$, we have

$$\begin{aligned} 2V \leq \; & \left(\tfrac{b\delta\ell}{\ell-1}\right)x^2 + \delta\left\{\tfrac{b(b+1)(\ell-1)+a^2[(\ell-1)+a\ell}{ab(\ell-1)}\right\}y^2 + \left\{\tfrac{(b+1)(\ell-1)+a\ell}{ab(\ell-1)}\right\}z^2 \\ & + \left(\tfrac{a\delta\ell}{\ell-1}\right)(x^2+y^2) + \left(\tfrac{\delta\ell}{\ell-1}\right)(x^2+z^2) + \left(\tfrac{\delta\ell}{b(\ell-1)}\right)\{(\ell-1)+a\ell\}(y^2+z^2). \\ \leq \; & \left(\tfrac{\delta}{\ell-1}\right)(b\ell+a\ell+\ell)x^2 + \left(\tfrac{\delta}{\ell-1}\right)\left\{\tfrac{b(b+1)(\ell-1)+a^2[(\ell-1)+a\ell}{ab} + \tfrac{(\ell-1)+a\ell}{b} + a\ell\right\}y^2 \\ & + \left(\tfrac{\delta}{\ell-1}\right)\left\{\tfrac{(b+1)(\ell-1)+a\ell}{ab} + \ell + \tfrac{(\ell-1)+a\ell}{b}\right\}z^2. \end{aligned} \tag{3.6}$$

Hence,

$$2V \leq K_2(x^2+y^2+z^2). \tag{3.7}$$

where

$$\begin{aligned} K_2 = \left(\tfrac{\delta}{\ell-1}\right)\max\Big\{ & \ell(a+b+1), \tfrac{b(b+1)(\ell-1)+a^2[(\ell-1)+a\ell]}{ab} + \tfrac{(\ell-1)+a\ell}{b} + a\ell, \\ & \tfrac{(b+1)(\ell-1)+a\ell}{ab} + \ell + \tfrac{(\ell-1)+a\ell}{b}\Big\} > 0. \end{aligned}$$

From (3.5) and (3.7), we have

$$K_1(x^2+y^2+z^2) \leq V(x,y,z) \leq K_2(x^2+y^2+z^2). \tag{3.8}$$

This proves Lemma 3.1.

Lemma 3.2. Suppose that the conditions of Theorem 2.2 hold and in addition, let $\epsilon_i > 0 (i = 1, 2)$ and σ be constant such that,

$$H_0 = \frac{h(x)-h(0)}{x} \leq H_1 = \min\left\{\frac{4(\sigma-1)^2}{\sigma^2\epsilon_2^2}, \frac{4(\sigma-1)^2}{\sigma^2\epsilon_1^2}\right\},$$

then there are positive constants $K_j = K_j(a, b, \ell, m, \delta)(j = 3, 4)$ such that for any solution (x, y, z) of system (1.3),

$$\dot{V}\mid_{(1.3)} \equiv \tfrac{d}{dt}V\mid_{(1.3)}(x,y,z) \leq -K_3(x^2+y^2+z^2) + K_4(|x|+|y|+|z|)\,|P(t)|\,. \tag{3.9}$$

Proof. From (1.1) and (1.3) we have,

$$\begin{aligned}
\dot{V}\mid_{(1.3)} &= \tfrac{\partial V}{\partial x}\dot{x} + \tfrac{\partial V}{\partial y}\dot{y} + \tfrac{\partial V}{\partial z}\dot{z} \\
&= \tfrac{\partial V}{\partial x}y + \tfrac{\partial V}{\partial y}z + \tfrac{\partial V}{\partial z}(-az - by - h(x) + P(t)) \\
&= -h(x)x - y^2 - z^2 - \left\{\tfrac{(b+1)(\ell+1)+a\ell}{ab(\ell-1)}z + \tfrac{(\ell-1)+a\ell}{b(\ell-1)}y + \tfrac{\ell}{\ell-1}x\right\}h(x) \\
&\quad + \left\{\tfrac{(b+1)(\ell+1)+a\ell}{ab(\ell-1)}z + \tfrac{(\ell-1)+a\ell}{b(\ell-1)}y + \tfrac{\ell}{\ell-1}x\right\}P(t).
\end{aligned} \tag{3.10}$$

Set $\left(\frac{(b+1)(\ell+1)+a\ell}{ab(\ell-1)}\right) = \epsilon_1, \left(\frac{(\ell-1)+a\ell}{b(\ell-1)}\right)\epsilon_2, \left(\frac{\ell}{\ell-1}\right) = \epsilon_3$, then by the condition on $h(x)$

$$\begin{aligned}
\dot{V}\mid_{(1.3)} = &-\{H_0x^2 + y^2 + z^2 + H_0\epsilon_1 xz + H_0\epsilon_2 xy + H_0\epsilon_3 x^2\} \\
&- h(0)\,(\epsilon_1 z + \epsilon_2 y + (1+\epsilon_3)x) + (\epsilon_1 z + \epsilon_2 y + \epsilon_3 x)\,P(t),
\end{aligned} \tag{3.11a}$$

which can be written as

$$\begin{aligned}
\dot{V}\mid_{(1.3)} = &-\{(H_0x^2 + H_0\epsilon_2 xy + y^2) + (H_0\epsilon_3 x^2 + H_0\epsilon_1 xz + z^2)\} \\
&- h(0)\,(\epsilon_1 z + \epsilon_2 y + (1+\epsilon_3)x) + (\epsilon_1 z + \epsilon_2 y + \epsilon_3 x)\,P(t).
\end{aligned} \tag{3.11b}$$

Re-writing (3.11b) we have,

$$\begin{aligned}
\dot{V}\mid_{(1.3)} = &-\Big\{(\tfrac{H_0}{\tau}x^2 + \tfrac{1}{\tau}y^2 + \tfrac{1}{\tau}z^2) + (H_0\epsilon_3 x^2 + H_0\epsilon_1 xz + \tfrac{(\tau-1)}{\tau}z^2) \\
&\quad +(\tfrac{\tau-1}{\tau}H_0x^2 + H_0\epsilon_2 xy + \tfrac{\tau-1}{\tau}y^2)\Big\} \\
&-h(0)(\epsilon_1 z + \epsilon_2 y + (1+\epsilon_3)x) + (\epsilon_1 z + \epsilon_2 y + \epsilon_3 x)P(t).
\end{aligned} \tag{3.12}$$

Let

$$U = U_1 + U_2 + U_3 + U_4 - U_5 \tag{3.13}$$

with

$$U_1 = \tfrac{1}{\tau}\left(H_0 x^2 + y^2 + z^2\right), \tag{3.14}$$

$$U_2 = \left(H_0 \epsilon_3 x^2 + H_0 \epsilon_1 xz + \tfrac{1}{\tau} z^2\right), \tag{3.15}$$

$$U_3 = \left(\tfrac{\tau-1}{\tau} H_0 x^2 + H_0 \epsilon_2 xy + \tfrac{\tau-1}{\tau} y^2\right), \tag{3.16}$$

$$U_4 = h(0)\left(\epsilon_1 z + \epsilon_2 y + \epsilon_3 x\right). \tag{3.17}$$

$$U_5 = \left(\epsilon_1 z + \epsilon_2 y + \epsilon_3 x\right) P(t) \tag{3.18}$$

From (3.14),
$U_1 \leq K_3(x^2 + y^2 + z^2)$
where $K_3 = \frac{1}{\tau} \cdot \max\left(H_0, 1\right)$. We also have from (3.18) that,
$U_5 \leq K_4(|x| + |y| + |z|)P(t)$
with
$K_4 = \max\left(\epsilon_1, \epsilon_2, \epsilon_3 \ell\right)$.
But $U \geq U_1 + U_4 - |U_5|$,
by the hypothesis that $h(0) = 0$, then U_4 vanishes, hence

$$U \geq U_1 - |U_5|.$$

Note: U is positive definite, since U_2 and U_3 are quadratic forms in x and z, and x and y respectively. Since it is known that any quadratic form $AX^2 + BX + C$ is positive definite if $4AC - B^2 \geq 0$. Therefore,

$$\frac{dV}{dt} = \dot{V} = -U \leq -K_3(x^2 + y^2 + z^2) + K_4(|x| + |y| + |z|)\,|P(t)|. \tag{3.19}$$

Since

$$(|x| + |y| + |z|) \leq \sqrt{3}(x^2 + y^2 + z^2)^{\frac{1}{2}},$$

(3.19) becomes

$$\frac{dV}{dt} \leq -K_3(x^2 + y^2 + z^2) + K_5(x^2 + y^2 + z^2)^{\frac{1}{2}}\,|P(t)| \tag{3.20}$$

where $K_5 = \sqrt{3}K_4$.
This completes the proof of Lemma 3.2.

4. PROOF OF THE MAIN RESULTS

We shall now give the proofs of the Theorems stated in Section 2 of this paper.

Proof of Theorem 2.1. From Lemma 3.1 and Lemma 3.2 it had been established (or shown) that the function $V(x, y, z)$ is a Lyapunov function of system (1.3). Hence, the trivial solution of system (1.3) is asymptotically stable. □

Proof of Theorem 2.2. Indeed from (3.20),

$$\frac{dV}{dt} \leq -K_3(x^2 + y^2 + z^2) + K_5(x^2 + y^2 + z^2)^{\frac{1}{2}} |P(t)|$$

from (3.5), we have

$$(x^2 + y^2 + z^2)^{\frac{1}{2}} \leq \left(\frac{2V}{K_1}\right)^{\frac{1}{2}}.$$

Thus (3.20) becomes

$$\frac{dV}{dt} \leq -K_6V + K_7V^{\frac{1}{2}} |P(t)| \tag{4.1}$$

We note that
$K_3(x^2 + y^2 + z^2) = K_3 \cdot \frac{V}{K_1}$
and

$$\frac{dV}{dt} \leq -K_6V + K_7V^{\frac{1}{2}} |P(t)| \tag{4.2}$$

where $K_6 = \frac{K_3}{K_2}$ and $K_7 = \frac{K_5}{K_2^{\frac{1}{2}}}$.
This implies that

$$\dot{V} \leq -K_6V + K_7V^{\frac{1}{2}} |P(t)|$$

and this can be written as

$$\dot{V} \leq -2K_8V + K_7V^{\frac{1}{2}} |P(t)| \tag{4.3}$$

where $K_8 = \frac{1}{2}K_6$.
Therefore

$$\dot{V} + K_8V \leq -K_8V + K_7V^{\frac{1}{2}} |P(t)| \tag{4.4}$$

$$\leq K_7 V^{\frac{1}{2}} \left\{ |P(t)| - K_9 V^{\frac{1}{2}} \right\}, \tag{4.5}$$

where $K_9 = \frac{K_8}{K_7}$.
Thus (4.5) becomes

$$\leq K_7 V^{\frac{1}{2}} V^* \tag{4.6}$$

where

$$V^* = |P(t)| - K_9 V^{\frac{1}{2}} \tag{4.7}$$

$$\leq V^{\frac{1}{2}} |P(t)|$$

$$\leq |P(t)|. \tag{4.8}$$

When $|P(t)| \leq K_9 V^{\frac{1}{2}}$,

$$V^* \leq 0 \tag{4.9}$$

and when $|P(t)| \geq K_9 V^{\frac{1}{2}}$,

$$V^* \leq |P(t)| \cdot \frac{1}{K_9}. \tag{4.10}$$

Substituting (4.9) into (4.5), we have,

$$\dot{V} + K_8 V \leq K_{10} V^{\frac{1}{2}} |P(t)|$$

where

$$K_{10} = \frac{K_7}{K_9}.$$

This implies that

$$V^{-\frac{1}{2}} \dot{V} + K_8 V^{\frac{1}{2}} \leq K_{10} |P(t)|. \tag{4.11}$$

Multiplying both sides of (4.11) by $e^{\frac{1}{2}K_8 t}$ we have,

$$e^{\frac{1}{2}K_8 t} \left\{ V^{-\frac{1}{2}} \dot{V} + K_8 V^{\frac{1}{2}} \right\} \leq e^{\frac{1}{2}K_8 t} K_{10} |P(t)| \tag{4.12}$$

i.e

$$2\frac{d}{dt} \left\{ V^{\frac{1}{2}} e^{\frac{1}{2}K_8 t} \right\} \leq e^{\frac{1}{2}K_8 t} K_{10} |P(t)|. \tag{4.13}$$

Integrating both sides of (4.13) from t_0 to t, gives

$$\left\{ V^{\frac{1}{2}} e^{\frac{1}{2}K_8 \gamma} \right\}_{t0}^{t} \leq \int_{t0}^{t} \frac{1}{2} e^{\frac{1}{2}K_8 \tau} K_{10} |P(\tau) d\tau| \tag{4.14}$$

which implies that

$$\left\{V^{\frac{1}{2}}(t)\right\} e^{\frac{1}{2}K_8 t} \leq V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_8 t_0} + \frac{1}{2}K_{10}\int_{t0}^{t} |P(\tau)|\, e^{\frac{1}{2}K_8\tau} d\tau,$$

or

$$V^{\frac{1}{2}}(t) \leq e^{-\frac{1}{2}K_8 t}\left\{V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_8 t_0} + \frac{1}{2}K_{10}\int_{t0}^{t} |P(\tau)|\, e^{\frac{1}{2}K_8\tau} d\tau\right\}.$$

Using (3.5) and (3.7) we have

$$\begin{aligned} K_1(x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)) \leq e^{-\frac{1}{2}K_8 t}\Big\{&K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))e^{\frac{1}{2}K_8 t_0} \\ &+\frac{1}{2}K_{10}\int_{t0}^{t} |P(\tau)|\, e^{\frac{1}{2}K_8\tau} d\tau\Big\}^2 \end{aligned} \tag{4.15}$$

for all $t \geq t_0$ Thus,

$$\begin{aligned} x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t) &\leq \frac{1}{K_1}\Big\{e^{-\frac{1}{2}K_8 t}\Big\{K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))e^{\frac{1}{2}K_8 t_0} \\ &\qquad +\frac{1}{2}K_{10}\int_{t0}^{t} |P(\tau)|\, e^{\frac{1}{2}K_8\tau} d\tau\Big\}^2\Big\} \\ &\leq \left\{e^{-\frac{1}{2}K_8 t}\left\{A_1 + A_2\int_{t0}^{t} |P(\tau)|\, e^{\frac{1}{2}K_8\tau} d\tau\right\}^2\right\}. \end{aligned} \tag{4.16}$$

By substituting $K_8 = \mu$ in the inequality (4.16), we have

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t) \leq e^{-\frac{1}{2}\mu t}\left\{A_1 + A_2\int_{t_0}^{t} |P(\tau)|\, e^{\frac{1}{2}\mu\tau} d\tau\right\}^2. \tag{4.17}$$

Hence, the completion of the proof. □

Proof of Theorem 2.3. From the function V defined above and the conditions of Theorem 2.3, the conclusion of Lemma 3.1 can be obtained, as

$$V \geq K_1\left(x^2+y^2+z^2\right) \tag{4.18}$$

and since $P \neq 0$ we can revise the conclusion of Lemma 3.2 i.e

$$\dot{V} \leq -K_3(x^2+y^2+z^2) + K_4(|x|+|y|+|z|)\,|P(t)|\,,$$

and we obtained

$$\dot{V} \leq K_4(|x|+|y|+|z|)^2 r(t) \tag{4.19}$$

By using the Schwartz inequalities on (4.19), we have

$$\dot{V} \leq K_{11}(x^2 + y^2 + z^2)r(t) \tag{4.20}$$

where $K_{11} = 3K_4$
From equations (4.18) and (4.20) we have,

$$\dot{V} \leq K_{11}Vr(t). \tag{4.21}$$

Integrating equation (4.21) from 0 to t, we obtain

$$V(t) - V(0) \leq K_{12} \int_0^t V(s)r(s)ds. \tag{4.22}$$

where $K_{12} = \frac{K_{11}}{K_1} = \frac{3K_4}{K_1}$ Using the condition (ii) of Theorem 2.3 we have

$$V(t) \leq V(0) + K_{11} \int_0^t V(s)r(s)ds \tag{4.23}$$

By Grownwall-Bellman inequality equation (4.23) yields

$$V(t) \leq V(0)exp\left(K_{12} \int_0^t r(s)ds\right). \tag{4.24}$$

this completes the proof of Theorem 2.3. □

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