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HERMITE INTERPOLATION AND A METHOD FOR EVALUATING CAUCHY PRINCIPAL VALUE INTEGRALS OF OSCILLATORY KIND

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Abstract. An alternative method to the method proposed in [10] for the numerical evaluation of integrals of the form $\int_{-1}^{1} e^{i\phi t} f(t)dt$, where $f(t)$ has a simple pole in $(-1, 1)$ and $\phi \in R$ may be large, has been developed. The method is based on a special case of Hermite interpolation polynomial and it is comparatively simpler and entails fewer function evaluations and thus faster, but the two methods are comparable in accuracy. The validity of the method is demonstrated in the provision of two numerical experiments and their results.

1. INTRODUCTION

The quadrature of oscillatory integrals has a wide range of applications in engineering, quantum physics, image analysis, and fluid dynamics. If the oscillation is high , and this is often the case in most applications, the classical methods of integration are unsuitable and a non-classical method is then required. The earliest numerical method for the treatment of this problem is probably due to Filon [3] who approximates the non-oscillatory factor of the integrand by second-degree polynomials over an even number of subintervals and analytically integrates out the oscillatory factor. A full account of this method is given in Davis and Rabinowitz [2, Eq.(1.10.2)]. A notable later work is due to Luke [8], who approximates f by polynomials of degree ≤ 10. Since then a considerable literature has evolved on the subject and one may, for example, see [4, 7, 9, 10].

In [10] the following integral

$$I_{\tau,\omega} = \int_{-1}^{1} \frac{e^{i\omega x}}{x-\tau} f(x)dx, \quad \omega > 0, \quad i^2 = -1, \quad -1 < \tau < 1, \quad f'(\tau) \neq 0 \tag{1}$$

which is oscillatory and of Cauchy type and therefore has two practical difficulties, was considered for numerical treatment over $[-1, 1]$ using a method based on a modified Lagrangian interpolation formula and on properties of some orthogonal polynomials.

In the current paper we propose an alternative approach based on a special Hermite interpolation polynomial which in general consists of the following [5,6]: Given any $(n + 1)$ distinct points $x_0, \ x_1, \ldots, x_n$ in $[a, b]$ and corresponding integers $u_r \geq 1, \ r = 0, \ 1, \ldots, n$, and given a function $f(x) \in C^{H-1}$[a,b], and where $H = \max\{u_0, u_1 \ldots, u_n\}$, and u_r is the multiplicity of the point x_r , a polynomial can be found such that

$$P^{(s)}(x_r) = f^{(s)}(x_r) = f_r^{(s)}, \quad s = 0, 1, \ldots, u_r - 1, \quad r = 0, 1, \ldots, n \tag{2}$$

where $f_r^{(s)}$ is the s-th derivative of f at x_r.

It is well known that the interpolating polynomial $P(x)$ can be given by

$$P(x) = \sum_{r=0}^{n} \sum_{j=0}^{u_r-1} \beta_{r,j}(x) f^{(j)}(x_r)$$

where each $\beta_{r,j}(x)$ is a polynomial of degree N $= \sum_{r=0}^{n} u_r - 1$, such that $\beta_{r,j}^{(j)}(x_r) = 1$ and $\beta_{r,j}^{(j)}(x_k) = 0, \ k \neq r; \quad k = 0, 1, \ldots, n.$

From here and throughout the rest of the paper we shall asssume that $f(x)$ is a monotone and $f(x) \in C^{N+1}[-1, 1]$. The outcome of this approach is a truncated asymptotic series in inverse powers of the frequency ω and the integral is approximated as a linear combination of the function value and derivatives at $x = \tau$, with coefficients that may depend on frequency ω.

2. THE NEW APPROACH

Consider the special Hermite polynomial $P_n[f,\tau](x)$ which satisfies the interpolating conditions

$$P_n^{(s)}[f,\tau](\tau) = f^{(s)}(\tau), \qquad s = 0, 1, 2, \ldots, n \tag{3}$$

where $f^{(s)}(\tau) = f_\tau^{(s)}$ and $f(\tau) = f_\tau$ (*this is a special case of the Hermite interpolation polynomial, which is a Taylor series*).

Then, $\forall x \in [-1, 1]$ and $-1 < \tau < 1$ we have

$$P_n[f,\tau](x) = f_\tau + (x-\tau)f'_\tau + \frac{(x-\tau)^2}{2!}f''_\tau + \cdots + \frac{(x-\tau)^n}{n!}f_\tau^{(n)} \tag{4}$$

and the interpolating error is given by

$$E[f,\tau](x) = f(x) - P_n[f,\tau](x) = \frac{(x-\tau)^{n+1}}{(n+1)!}f^{(n+1)}(\xi); \;\; \xi \in (x,\tau) \tag{5}$$

The proof follows from standard techniques in [6], and which is exact if f is a polynomial of degree ≤ n.

Suppose we set $L[x,\tau] = \dfrac{1}{x-\tau}P_n[f,\tau](x)$
then,

$$L[x,\tau] = \frac{f_\tau}{x-\tau} + \sum_{k=1}^{n}\frac{(x-\tau)^{k-1}}{k!}f_\tau^{(k)} \tag{6}$$

and it follows that

$$\int_{-1}^{1} L[x,\tau]e^{i\omega x}dx = f_\tau\int_{-1}^{1}\frac{e^{i\omega x}}{x-\tau}dx + \sum_{k=1}^{n}\frac{f_\tau^{(k)}}{k!}\int_{-1}^{1}(x-\tau)^{k-1}e^{i\omega x}dx \tag{7}$$

Theorem. *If* $\delta_{k-1} = \displaystyle\int_{-1}^{1}(x-\tau)^{k-1}e^{i\omega x}dx$ *and* $\delta_m = 0,\;\; \forall m < 0,\; m \in \mathbf{Z}$*, then* δ_{k-1} *satisfies the linear non-homogeneous recurrence relation*

$$i\omega\delta_{k-1} + (k-1)\delta_{k-2} = [(1-\tau)^{k-1}e^{i\omega} + (-1)^k(1+\tau)^{k-1}e^{-i\omega}]; \;\; k = 1, \ldots, n \tag{8}$$

Proof. The proof follows by integrating δ_{k-1} by parts.

Thus, in view of (7) and the preceding theorem, our desired quadrature rule becomes

$$
\begin{aligned}
I_{\tau,\omega} \approx \tilde{I}_{\tau,\omega} &= \psi_{\tau,\omega} f_\tau + \sum_{k=1}^{n} \frac{\delta_{k-1}}{k!} f_\tau^{(k)} && (9)\\
\delta_0 &= \frac{2}{\omega}\sin\omega + 0i && (10)
\end{aligned}
$$

and where

$$\psi_{\tau,\omega} = \int_{-1}^{1} \frac{e^{i\omega x}}{x-\tau} dx$$

we have shown in [10] the following analytical results.

$$\mathrm{Re}[\psi_{\tau,\omega}] = \cos\tau\omega\,\mathrm{Ci}\,(u_1) - \sin\tau\omega\,\mathrm{Si}\,(u_1) + \sin\tau\omega\,\mathrm{Si}\,(u_2) - \cos\tau\omega\,\mathrm{Ci}\,(u_2) \qquad (11)$$

$$\mathrm{Im}[\psi_{\tau,\omega}] = \cos\tau\omega\,\mathrm{Si}\,(u_1) + \sin\tau\omega\,\mathrm{Ci}\,(u_1) - \cos\tau\omega\mathrm{Si}\,(u_2) - \sin\tau\omega\,\mathrm{Ci}\,(u_2) \qquad (12)$$

$$u_1 = \omega(1-\tau), \quad u_2 = -\omega(1+\tau) \qquad (13)$$

Ci and Si are cosine and sine integrals defined in [1] by

$$
\begin{aligned}
\mathrm{Ci}\,(z) &= \gamma + \ln z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{2n(2n)!}, \qquad \gamma = \text{ Euler's constant.} && (14)\\
\mathrm{Si}\,(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!} && (15)\\
\mathrm{Ci}\,(-z) &= \mathrm{Ci}\,(z) - i\pi, \quad 0 < \arg \mathrm{z} < \pi && (16)\\
\mathrm{Si}\,(-z) &= -\mathrm{Si}\,(z) && (17)
\end{aligned}
$$

Asymptotic Expansions

$$
\begin{aligned}
\mathrm{Si}(z) &= \frac{\pi}{2} - f(z)\cos z - g(z)\sin z && (18)\\
\mathrm{Ci}(z) &= f(z)\sin z - g(z)\cos z && (19)\\
f(z) &\sim \frac{1}{z}(1 - \frac{2!}{z^2} + \frac{4!}{z^4} - \frac{6!}{z^6} + \cdots), \quad |\arg z| < \pi && (20)\\
g(z) &\sim \frac{1}{z^2}(1 - \frac{3!}{z^2} + \frac{5!}{z^4} - \frac{7!}{z^6} + \cdots), \quad |\arg z| < \pi && (21)
\end{aligned}
$$

To evaluate each $f^{(s)}(\alpha),\ s \geq 1$, one may have to assume that f can be continued analytically into the complex plane and then use the Cauchy's integral formula to obtain

$$f^{(s)}(\alpha) = \frac{s!}{2\pi} \int_0^{2\pi} \frac{\mathrm{f}(\alpha + \cos\theta + i\sin\theta)}{(\cos s\theta + i\sin s\theta)} d\theta, \quad s \geq 1 \tag{22}$$

Any quadrature rule (e.g. Newton–Cotes) can then be applied to (22) but since we started with real variables only the real part of the result may be considered.

3. THE ERROR

The resulting quadrature error may be expressed as

$$E_{\tau,\omega}(k) = \sum_{k=n}^{\infty} \frac{\delta_k}{(k+1)!} f_\tau^{(k+1)} \tag{23}$$

and we have assumed that f is analytic in $[-1, 1]$.

Rewriting (8),

$$\delta_k + rk\delta_{k-1} = rb_k, \quad k = 1, 2, \ldots \tag{24}$$

$$\delta_0 = \frac{2}{\omega} \sin\omega + 0i$$

$$b_0 = 0 + 2\sin\omega i$$

where $r = \dfrac{1}{i\omega}$, and $b_k = [(1-\tau)^k e^{iw} + (-1)^{k+1}(1+\tau)^k e^{-iw}]$

The particular solution of (24) using standard techniques can be expressed by

$$\delta_k = (-1)^k (k!) r^k \delta_0 + \sum_{s=0}^{k-1} (-1)^s \frac{k!}{(k-s)!} r^{s+1} b_{k-s}, \quad k = 1, 2, \ldots \tag{25}$$

and by direct calculation obtain

$\delta_1 = -r\delta_0 + rb_1$

$\delta_2 = 2r^2\delta_0 - 2r^2 b_1 + rb_2$

$\delta_3 = -6r^3\delta_0 + 6r^3 b_1 - 3r^2 b_2 + rb_3$

$\delta_4 = 24r^4\delta_0 - 24r^4 b_1 + 12r^3 b_2 - 4r^2 b_3 + rb_4$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

Without proof but by observation, we note that if $\omega \to \infty$, then $r \to 0$ and $\delta_k \approx rb_k$. It can be shown also that $|b_k| \leq 2(1+\tau)^k$. Since f is analytic and bounded in $[-1,1]$ and $|\delta_k| < M$, for each k $\in N$, then by (23) $E_{\tau,\omega}(k) \to 0$ as $k \to \infty$, which is an indication of convergence.

4. NUMERICAL EXAMPLES

For our numerical experiment, two simple but typical problems are presented and their numerical results are shown to demonstrate the potential of the method developed. All computations were done with MATLAB (version 6.5 release 13) running on Windows 2000.

(a) Consider the integral [10]

$$\mathrm{Im}(I_{0,12}) = \int_{-1}^{1} \frac{e^x \sin 12x}{x} dx$$

which has the exact value

$$2\mathrm{Si}(12) - 2\sum_{m=1}^{\infty} \frac{1}{(2m)!} \sum_{j=0}^{2m-1} j! \binom{2m-1}{j} \frac{1}{12^{j+1}} \cos(12 + \frac{\pi}{2} j) = 2.929140054093$$

Note that for this integral $\tau = 0,\ \omega = 12$. Therefore for some fixed values of n we obtained the following approximate values to the integral using eqn.(9)

n	$\mathrm{Im}(\tilde{I}_{0,12})$	$\lvert E_{\tau,\omega}\rvert$
7	2.92914374390664	3.6e−6
9	2.92914009129263	3.7e−8
11	2.92914005434454	2.5e−10
15	2.92914005409237	6.3e−13

(b) Our final experiment is with the integral

$$\mathrm{Re}(I_{-\frac{1}{2},100}) = \int_{-1}^{1} \frac{\cosh x \cos 100x}{x + \frac{1}{2}} dx$$

For this problem also, $\tau = -\frac{1}{2},\ \omega = 100$. Since ω is large, we have used the asymptotic values of $\mathrm{Ci}(x)$ and $\mathrm{Si}(x)$ in our computation. The results follow.

n	$\mathrm{Re}(\tilde{I}_{-\frac{1}{2},100})$
7	-0.91524640126011
9	-0.91524639524348
11	-0.91524639520922
15	-0.91524639520912

Although the exact value of this integral is not immediately known, we believe the result in the last row is correct to 12 decimal digits.

An Extension:

Eqn (9) can be modified without much extra effort to deal with the numerical evaluation of the integrals of the form

$$\tilde{I}_{\tau,\omega} = \int_{1}^{1} \frac{e^{i\omega x} f(x)}{(x-\tau)^2} dx \tag{26}$$

in which the point $x = \tau$ is a pole of order 2. Consequently, the modification of eqn (9) then gives,

$$\begin{aligned} \tilde{I}_{\tau,\omega} &= (z + i\omega\psi_{\tau,\omega}) f_\tau + \psi_{\tau,\omega} f'_\tau + \frac{\sin\omega}{\omega} f''_\tau + \sum_{k=3}^{n} \frac{\delta_{k-2}}{k!} f_\tau^{(k)} && (27) \\ z &= -2(1-\tau^2)^{-1}(\cos\omega + i\tau\sin\omega) && (28) \end{aligned}$$

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