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A NOTE ON TWO DIMENSIONAL BRATU PROBLEM

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Abstract. In this paper, we examined the Bratu problem

$$-\Delta U = \lambda exp[U(x,y)], \quad x, y \in D,$$

$$U(x,y) = 0, \quad x, y \in \partial D,$$

in 2-dimensions, where Δ is the Laplace operator. The non linear equation is solved using various methods including finite difference method, weighted residual method and analytical method. Both the near exact solution and weighted residual solution, provide the upper and the lower branch solutions while the finite difference method only give the lower branch solution.

1. INTRODUCTION

The model stimulates a thermal reaction process in a rigid material, where the process depends on a balance between chemically generated heat addition and heat transfer by conduction, Aris [3] and Bebernes and Eberly [4], discuss this problem in the context of combustion problems. Various numerical methods have been employed to solve this problem. These include finite difference method, finite element approximation method Kikuchi [6] and continuation-conjugate gradient methods Glowinski [5].

Aregbesola and Odejide [2] obtained the solution to the problem, in one dimension, using the parameter perturbation, the weighted residual method, the finite difference method and the analytical method. The method of weighted residual was used to solve the boundary value problem by Aregbesola [1]. The idea is to approximate the solution with a polynomial involving a set of parameters.

A polynomial of the form

$$V(x,y) = \phi_0(x,y) + \sum_{i=1}^{N} A_i \phi_i(x,y), \tag{1.1}$$

where $\phi_0(x,y)$ satisfies the given boundary conditions and each $\phi_i(x,y)$ satisfies the homogeneous form of the boundary conditions. The function $V(x,y)$ is then used as an approximation to the exact solution in the equation

$$L[V(x,y)] = Q(x,y), \tag{1.2}$$

to give

$$R(x,y) = L[V(x,y)] - Q(x,y). \tag{1.3}$$

The function $R(x,y)$ is the residual. The idea is to make $R(x,y)$ as small as possible $R(x,y)$ is set to zero at some points in the interval. The system of these equations is then solved to determine the parameters A_i. $V(x,y)$ is then considered as the approximate solution.

In this work we have considered a more compact trial function than that considered in Aregbesola [1]. The results obtained here are compared with those obtained using the finite difference method and the weighted residual method. The new approach is less cumbersome than the other two methods.

2. METHODS OF SOLUTION

We consider the solution of the Bratu problem in two-dimensions

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \lambda exp[U] = 0, \quad x,y \in D = [0,1] \times [0,1] \tag{2.1}$$

subject to the boundary conditions

$$U(x,y) = 0, \quad x,y \in \partial D. \tag{2.2}$$

(i) Near Exact Solution (NES)

A function of the form

$$U(x,y) = 2\ln[\cosh(\frac{\theta}{4})\cosh[(x-\frac{1}{2})(y-\frac{1}{2})\theta]/(\cosh[(x-\frac{1}{2})\frac{\theta}{2}]\cosh[(y-\frac{1}{2})\frac{\theta}{2}])] \tag{2.3}$$

where θ is a constant to be determined, which satisfies the boundary conditions (2.2) is carefully chosen and assumed to be the solution of the differential equation (2.1). Substituting (2.3) in (2.1), simplifying and collocating at the point $x = \frac{1}{2}$ and $y = \frac{1}{2}$ because it is the midpoint of the interval. Another point could be chosen, but low-order approximations are likely to be better if the collocation points are distributed somewhat evenly throughout the region. Then we have

$$\theta^2 = \lambda\cosh(\frac{\theta}{4})^2. \tag{2.4}$$

Obtaining $\frac{d\lambda}{d\theta}$ from equation (2.4) and equating to zero, the critical value λ_c satisfies

$$\theta = \frac{\lambda_c}{4}\sinh(\frac{\theta}{4})\cosh(\frac{\theta}{4}). \tag{2.5}$$

By eliminating λ from equations (2.4) and (2.5) we have the value of θ_c for the critical λ satisfying

$$\frac{\theta_c}{4} = \coth(\frac{\theta_c}{4}) \tag{2.6}$$

and $\theta_c = \pm 4.798714561$. From equation (2.5) $\lambda_c = 7.027661438$.

Results showing the variation of λ, θ and U_{max} are shown in Table 1. The parameter θ_1 gives the lower branch solution while θ_2 gives the upper branch solution.

	θ		U_{max}	
λ	θ_1	θ_2	U_{max1}	U_{max2}
0.1	0.3172227274	19.19637291	0.006282809236	8.212027794
0.5	0.7185464179	14.98479987	0.03209723648	6.107219872
1.0	1.033569462	13.03823930	0.06603661666	5.135773050
2.0	1.517164599	10.93870277	0.1405392141	4.091467246
3.0	1.939726525	9.581699793	0.2264817040	3.421097726
4.0	2.357551054	8.507199571	0.3289524214	2.895531266
5.0	2.811554938	7.548098106	0.4580374660	2.433153338
6.0	3.373507764	6.576569259	0.6401466966	1.975266972
6.4	3.674358094	6.131465409	0.7464589086	1.770569562
6.8	4.108395792	5.56288431	0.9091426554	1.515096612
7.0	4.551853663	5.054342699	1.085158948	1.294585480
7.02	4.667812741	4.932041041	1.132617977	1.242742595
7.027661438	4.798690688	4.798714561	1.186832218	1.186842168

Table 1. The maximum values of U for various values of λ and θ

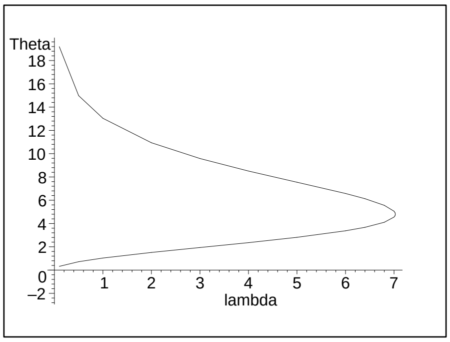

Figure 1a. The values of θ for various values of λ.

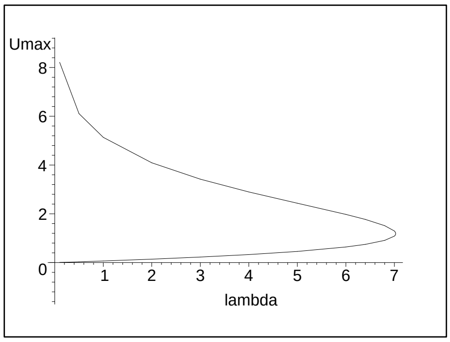

Figure 1b. U_{max} for various values of λ.

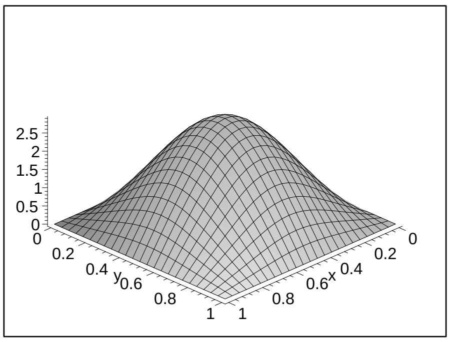

Figure 2. Upper Branch Solution for $\lambda = 4$ and $\theta_2 = 8.507199571$.

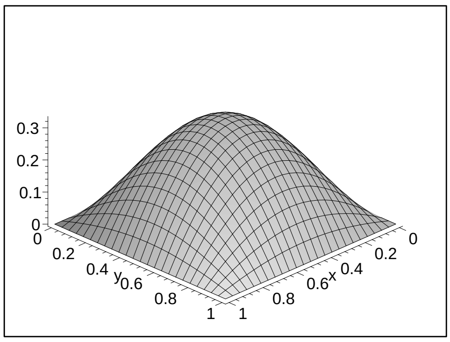

Figure 3. Lower Branch Solution for $\lambda = 4$ and $\theta_1 = 2.357551054$.

(ii) Finite Difference Method (FDM)

The finite difference formulation of the problem was considered. The region is subdivided into $N \times N$ equal subregions: $h \times h$, where $h = \frac{1}{N}$. The results obtained are shown in Table 2. In each of the cases considered, convergence is assumed if the maximum absolute difference of two consecutive values of U at all the nodes is less than 10^{-6}. With the FDM, it was only possible to obtain the lower branch solution.

$N \times N$	λ_c	U_{max}
5×5	6.739545	1.2295280
10×10	6.792610	1.3835357
20×20	6.80497	1.3911583
40×40	6.81565	1.3881487
100×100	7.12222	1.3565527

Table 2. λ_c with the values of U_{max} for various values of N.

(iii) Weighted Residual Method (WRM)

If we approximate U with modified double two point Taylor interpolation of the form Aregbesola [1], we have

$$\begin{aligned} U(x,y) \quad = \quad & C_{11}x^3(1-x)^2y^3(1-y)^2 + C_{21}x^3(1-x)y^3(1-y)^2 + C_{31}x(1-x)^3y^3(1-y)^2 \\ & +C_{41}x^2(1-x)^3y^3(1-y)^2 + C_{12}x^3(1-x)^2y^3(1-y) + C_{22}x^3(1-x)y^3(1-y) \\ & +C_{32}x(1-x)^3y^3(1-y) + C_{42}x^2(1-x)^3y^3(1-y) + C_{13}x^3(1-x)^2y(1-y)^3 \\ & +C_{23}x^3(1-x)y(1-y)^3 + C_{33}x(1-x)^3y(1-y)^3 + C_{43}x^2(1-x)^3y(1-y)^3 \\ & +C_{14}x^3(1-x)^2y^2(1-y)^3 + C_{24}x^3(1-x)y^2(1-y)^3 + C_{34}x(1-x)^3y^2(1-y)^3 \\ & +C_{44}x^2(1-x)^3y^2(1-y)^3 \end{aligned} \tag{2.7}$$

The expression (2.7) satisfies the boundary conditions in (2.2). Cases with $N \times N = 2 \times 2$, 4×4 and 6×6 were considered. As N increases the solution of the equations, becomes more difficult to solve. Considering higher values of N shows negligible difference in the results. Results obtained, with $N = 4$ are satisfactory and are representative of those of higher values of N. The results are as shown in Table 3 and 4. For $N = 4$, and $\lambda_c = 6.939571823$ the coefficients C_{ij} are shown below.

$$\begin{pmatrix} C_{11} & C_{21} & C_{31} & C_{41} \\ C_{12} & C_{22} & C_{32} & C_{42} \\ C_{13} & C_{23} & C_{33} & C_{43} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{pmatrix} = \begin{pmatrix} 167.6975431 & 30.88327324 & 30.88327324 & 167.6975431 \\ 30.88327324 & 20.19885325 & 20.19885325 & 30.88327324 \\ 30.88327324 & 20.19885325 & 20.19885325 & 30.88327324 \\ 167.6975431 & 30.88327324 & 30.88327324 & 167.6975431 \end{pmatrix} \tag{2.8}$$

For $N = 4$, and $\lambda = 4.0$ the coefficients C_{ij} for the **lower branch** solution are shown below.

$$\begin{pmatrix} C_{11} & C_{21} & C_{31} & C_{41} \\ C_{12} & C_{22} & C_{32} & C_{42} \\ C_{13} & C_{23} & C_{33} & C_{43} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{pmatrix} = \begin{pmatrix} 27.56411891 & 9.9814042676 & 9.9814042676 & 27.56411891 \\ 9.9814042676 & 8.377293071 & 8.377293071 & 9.9814042676 \\ 9.9814042676 & 8.377293071 & 8.377293071 & 9.9814042676 \\ 27.56411891 & 9.9814042676 & 9.9814042676 & 27.56411891 \end{pmatrix} \tag{2.9}$$

For $N = 4$, and $\lambda = 4.0$ the coefficients C_{ij} for the **upper branch** solution are shown below.

$$\begin{pmatrix} C_{11} & C_{21} & C_{31} & C_{41} \\ C_{12} & C_{22} & C_{32} & C_{42} \\ C_{13} & C_{23} & C_{33} & C_{43} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{pmatrix} = \begin{pmatrix} 990.575779 & -54.0503105 & -54.0503105 & 990.575779 \\ -54.0503105 & 38.3209736 & 38.3209736 & -54.0503105 \\ -54.0503105 & 38.3209736 & 38.3209736 & -54.0503105 \\ 990.575779 & -54.0503105 & -54.0503105 & 990.575779 \end{pmatrix} \tag{2.10}$$

$N \times N$	λ_c	U_{max}
2×2	6.6218298	1.265886857
4×4	6.939511823	1.453247267
6×6	6.7796325	1.369513771

Table 3. λ_c with the values of U_{max} for various values of N.

λ	U_{max1}	U_{max2}
0.1	0.00739148232	9.64736670
0.5	0.03781451306	7.309600958
1.0	0.0779460505	6.22778797
2.0	0.1665928797	5.04587189
3.0	0.2698543078	4.262350004
4.0	0.3945269608	3.62365745
5.0	0.554387275	3.037781167
6.0	0.7871142546	2.433301716
6.4	0.9292292746	2.153841
6.8	1.169825663	1.781480172
6.939571828	1.453226755	1.453344057

Table 4. The maximum values for various values of λ with $N \times N = 4 \times 4$.

3. RESULTS AND CONLUSION

We have used three methods to solve this problem: the finite difference method, the weighted residual method, and the near exact solution. The finite difference method produce only the lower branch solution while the analytical solution and the weighted residual method produce both the lower and the upper branch solutions. Figures 2 and 3 show the upper and the lower branch solutions for $\lambda = 4$, $\theta_1 = 2.357551054$ and $\theta_2 = 8.507199571$ which is inclose agreement with that obtained by Aregbesola [1].

We have used Table 1 to generate Fig 1a and Fig 1b showing the upper and the lower branch solutions from the near exact solution approach. The critical values of λ_c obtained by the weighted residual method, for $N = 4$, is 6.939571823, that of the finite difference method is 7.12222 compared with that obtained through the near exact solution 7.027661438. The results obtained with the near exact solution is compact and more accurate. The computations can be performed with ease. To get an accurate result with the finite difference method, it may require $N \times N = 100 \times 100$ mesh points. This involves much more computations. As N increases, with weighted residual method, the computation for the solution of the equations becomes more tedious. The near exact analytical method provides a very good approach.

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