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AN INTEGRAL INEQUALITY OF THE HARDY'S TYPE

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Abstract. In this paper, we establish an integral inequality of the type of Hardy's mainly by Jensen's Inequality. Our result is a generalization of the earlier results of the authors.

1. INTRODUCTION

Various methods have been employed for the establishment of necessary and sufficient conditions on p, q, v, w for the Hardy-type inequality

$$\left[\int_a^b |u(x)|^q w(x)dx\right]^{\frac{1}{q}} \leq C\left[\int_a^b |u'(x)|^p v(x)dx\right]^{\frac{1}{p}} \tag{1.1}$$

to hold, where C is a constant depending on p and q. A recent trend in inequalities has been to establish , mainly by Jensen's inequality (see for example [1] and [3]) and its generalization due to Steffensen, some general inequalities that include as special cases many that are of independent interest and which were originally proved by quite different methods.

In this paper, we follow this trend in using Jensen's inequality in conjunction with a form of Minkowski's integral inequality to establish a result that generalizes inequality (1.1).

Jensen's Inequality. Let φ be continuous and convex, $h(x,t)$ non-negative for $x \geq 0,\ t \geq 0,\ \lambda$ nondecreasing and $-\infty \leq \xi(x) \leq \eta(x) \leq \infty$. Suppose φ has a continuous inverse φ^{-1} which is necessarily concave, then

$$\varphi^{-1}\left[\frac{\int_{\xi(x)}^{\eta(x)} h(x,t)d\lambda(t)}{\int_{\xi(x)}^{\eta(x)d\lambda(t)}}\right] \geq \frac{\int_{\xi(x)}^{\eta(x)} \varphi^{-1}\left[h(x,t)\right]d\lambda(t)}{\int_{\xi(x)}^{\eta(x)} d\lambda(t)}. \tag{1.2}$$

The inequality (1.2) is the Jensen's inequality for convex functions.

Taking the convex function $\varphi(u) = u^p,\ p \geq 1$ and letting $\xi(x) = a,\ \eta(x) = x$ we obtain as a consequence of (1.2) the inequality

$$\left[\frac{\int_a^x h(x,t)d\lambda(t)}{\int_a^x d\lambda(t)}\right]^{\frac{1}{p}} \geq \frac{\int_a^x h(x,t)^{\frac{1}{p}}d\lambda(t)}{\int_a^x d\lambda(t)}.$$

For $1 \leq p \leq q$, we have

$$\left[\frac{\int_a^x h(x,t)^{\frac{1}{q}}d\lambda(t)}{\int_a^x d\lambda(t)}\right]^{\frac{1}{p}} \geq \frac{\int_a^x h(x,t)^{\frac{1}{pq}}d\lambda(t)}{\int_a^x d\lambda(t)},$$

which we write as

$$\int_a^x h(x,t)^{\frac{1}{pq}}d\lambda(t) \leq \left[\int_a^x d\lambda(t)\right]^{1-\frac{1}{p}}\left[\int_a^x h(x,t)^{\frac{1}{q}}d\lambda(t)\right]^{\frac{1}{p}}. \tag{1.3}$$

The inequality (1.3) is our main tool in the proof of the main result of this paper.

2. MAIN RESULTS

Our main result is the following:

Theorem 2.1. *Let g be continuous and nondecreasing on $[a,b]$, $0 \leq a \leq b < \infty$, with $g(x) > 0$ for $x > 0$. Let $q \geq p \geq 1$ and $f(x)$ be nonnegative and*

Lebesgue-Stieltjes integrable with respect to $g(x)$ *on* $[a,b]$. *Suppose* δ *is a real number such that* $\frac{-p}{q} < \delta < 0$, *then*

$$\left[\int_a^b g(x)^{\frac{\delta q}{p}}\left(\int_a^x f(t)dg(t)\right)^q dg(x)\right]^{\frac{1}{q}} \le C(a,b,p,q,\delta)\left[\int_a^b g(x)^{(p-1)(1+\delta)} f(x)^p dg(x)\right]^{\frac{1}{p}}, \tag{2.1}$$

where

$$C(a,b,p,q,\delta) = (-\delta)^{\frac{q(1-p)}{p}}\left(\frac{p}{p+\delta q}\right)^{\frac{p}{q}} g(b)^{p+\delta q}\left(g(b)^{-\delta} - g(a)^{-\delta}\right)^{\frac{q}{p}(p-1)} > 0.$$

We shall need the following result of Minkowski for the proof of the theorem.

Lemma 2.2. **[2]** *Let* $K(x,y)$ *be a nonngative measurable function on* $(a,b)\times(c,d)$, *and let* $r \ge 1$. *Then,*

$$\left[\int_a^b\left[\int_c^d K(x,y)dy\right]^r dx\right]^{\frac{1}{r}} \le \int_c^d\left[\int_a^b K^r(x,y)\right]^{\frac{1}{r}} dy. \tag{2.2}$$

If $K(x,y) = H(x)G(y)$ *where* $x \in (a,b)$ *and* $y \in (c,d)$, *then (2.2) reduces to*

$$\left(\int_a^b G(x)\left[\int_a^b H(y)dy\right]^r dx\right)^{\frac{1}{r}} \le \int_a^b H(y)\left[\int_y^b G(x)dx\right]^{\frac{1}{r}} dy\ . \tag{2.3}$$

Proof of Theorem 2.1. In the inequality (1.3) let

$$\begin{aligned} h(x,t) &= g(x)^{\delta q} g(t)^{pq(1+\delta)} f(t)^{pq} \\ d\lambda(t) &= g(t)^{-(1+\delta)} dg(t). \end{aligned}$$

Then the left-hand side of (1.3) becomes

$$\int_a^x g(x)^{\frac{\delta}{p}} g(t)^{(1+\delta)} f(t) g(t)^{-(1+\delta)} dg(t) = g(x)^{\frac{\delta}{p}}\int_a^x f(t)dg(t).$$

and the right-hand side reduces to

$$\begin{aligned} &\left[\int_a^x g(t)^{-(1+\delta)}dg(t)\right]^{\frac{p-1}{p}}\left[\int_a^x g(x)^\delta g(t)^{(p-1)(1+\delta)} f(t)^p dg(t)\right]^{\frac{1}{p}} \\ &\quad= \left\{\left[(-\delta)^{-1} g(t)^{-\delta}\right]_a^x\right\}^{\frac{p-1}{p}}\left[\int_a^x g(x)^\delta g(t)^{(p-1)(1+\delta)} f(t)^p dg(t)\right]^{\frac{1}{p}} \\ &\quad= (-\delta)^{\frac{1-p}{p}}\left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{\frac{p-1}{p}}\left\{\int_a^x g(x)^\delta g(t)^{(p-1)(1+\delta)} f(t)^p dg(t)\right\}^{\frac{1}{p}}. \end{aligned}$$

Hence the inequality (1.3) becomes

$$g(x)^{\frac{\delta}{p}}\left(\int_a^x f(t)dg(t)\right)$$
$$\leq (-\delta)^{\frac{(1-p)}{p}}\left[g(x)^{-\delta}-g(a)^{-\delta}\right]^{\frac{(p-1)}{p}}\left[g(x)^{\delta}\int_a^x g(t)^{(p-1)(1+\delta)}f(t)^p dg(t)\right]^{\frac{1}{p}}.$$

For $q \geq p$ we have

$$g(x)^{\frac{\delta q}{p}}\left(\int_a^x f(t)dg(t)\right)^q$$
$$\leq (-\delta)^{\frac{q(1-p)}{p}}\left[g(x)^{-\delta}-g(a)^{-\delta}\right]^{\frac{q(p-1)}{p}} g(x)^{\frac{\delta q}{p}}\left[\int_a^x g(t)^{(p-1)(1+\delta)}f(t)^p dg(t)\right]^{\frac{q}{p}}.$$

Integrating both sides with respect to $g(x)$ and then raising both sides to power $\frac{p}{q}$ yields

$$\left[\int_a^b g(x)^{\frac{\delta q}{p}}\left[\int_a^x f(t)dg(t)\right]^q dg(x)\right]^{\frac{p}{q}}$$
$$\leq \left[(-\delta)^{\frac{q(1-p)}{p}}\int_a^b\left(g(x)^{-\delta}-g(a)^{-\delta}\right)^{\frac{q(1-p)}{p}} g(x)^{\frac{\delta q}{p}} \right. \tag{2.4}$$
$$\left. \times\left(\int_a^x g(t)^{(p-1)(1+\delta)}f(t)^p dg(t)\right)^{\frac{q}{p}} dg(x)\right]^{\frac{p}{q}}.$$

Applying (2.3) to the right-hand side we obtain

$$\left[\int_a^b\left(g(x)^{-\delta}-g(a)^{-\delta}\right)^{\frac{q(p-1)}{p}} g(x)^{\frac{\delta q}{p}}\left\{\int_a^x g(t)^{(p-1)(1+\delta)}f(t)^p dg(t)\right\}^{\frac{q}{p}} dg(x)\right]^{\frac{p}{q}}$$
$$\leq \left[\int_a^b g(t)^{(p-1)(1+\delta)}f(t)^p\left\{\int_t^b\left(g(x)^{-\delta}-g(a)^{-\delta}\right)^{\frac{q(p-1)}{p}} g(x)^{\frac{\delta q}{p}}dg(x)\right\}\right]^{\frac{p}{q}} dg(t)$$
$$\leq \left(g(b)^{-\delta}-g(a)^{-\delta}\right)^{\frac{q(p-1)}{p}}\int_a^b g(t)^{(p-1)(1+\delta)}f(t)^p\left[\int_t^b g(x)^{\frac{\delta q}{p}}dg(x)\right]^{\frac{p}{q}} dg(t),$$

since $\delta < 0$

$$= \left(\frac{p}{p+\delta q}\right)^{\frac{p}{q}}\left(g(b)^{-\delta}-g(a)^{-\delta}\right)^{\frac{q(p-1)}{p}}$$
$$\times\int_a^b g(x)^{(p-1)(1+\delta)}f(x)^p\left[g(b)^{\frac{p+\delta q}{p}}-g(x)^{\frac{p+\delta q}{p}}\right]^{\frac{p}{q}} dg(x),$$
$$\leq C(a,b,p,q,\delta)^p\int_a^b g(x)^{(p-1)(1+\delta)}f(x)^p dg(x).$$

i.e. we have shown that

$$\left[\int_a^b \left(g(x)^{-\delta} - g(a)^{-\delta}\right)^{\frac{q(p-1)}{p}} g(x)^{\frac{\delta q}{p}} \left\{\int_a^x g(t)^{(p-1)(1+\delta)} f(t)^p dg(t)\right\}^{\frac{q}{p}} dg(x)\right]^{\frac{p}{q}}$$
$$\leq \left(\frac{p}{p+\delta q}\right)^{\frac{p}{q}} \left(g(b)^{-\delta} - g(a)^{-\delta}\right)^{\frac{q(p-1)}{p}}$$
$$\times \int_a^b g(x)^{(p-1)(1+\delta)} f(x)^p \left[g(b)^{\frac{p+\delta q}{p}} - g(x)^{\frac{p+\delta q}{p}}\right]^{\frac{p}{q}} dg(x),$$

in which reduces to

$$\left[\int_a^b g(x)^{\frac{\delta q}{p}} \left(\int_a^x f(t)dg(t)\right)^q dg(x)\right]^{\frac{1}{q}} \leq C(a,b,p,q,\delta) \left[\int_a^b g(x)^{(p-1)(1+\delta)} f(x)^p dg(x)\right]^{\frac{1}{p}}.$$

This completes the proof of the theorem. □

Remark. When $g(x) = x$, inequality (2.1) reduces to the form (1.1) with $|u(x)| = \int_a^x f(t)dt$, $|u'(x)| = f(x)$, $w(x) = x^{\frac{\delta q}{p}}$ and $v(x) = x^{(p-1)(1+\delta)}$.

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