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ON THE EVALUATION OF THE DEGREE OF PROGRESS TO GOAL IN A MULTI-PURPOSE SELF-ORGANIZATION PROCESS

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Abstract. In this paper, we establish a result on the degree of progress to goal for a multi-purpose self-organization process. Our method of presentation is based on the convex combination of the various distance functions of the different self-organizing systems involved, the notions of the convexity of functions, triangle inequality, trace of a matrix, elementary idea of probability theory as well as the concept of regular curves. Just as in Olatinwo [8,9,10], the transition probabilities at various time intervals (including initial and final times) are also evaluated here, and then subsequently interpreted as the degrees of progress to goal at such time intervals.

Our result, as usual, is in agreement with the axiomatic properties of probability and it is a generalization of Theorem 2A of Olatinwo [8], Theorem 1 of Olatinwo [9] and Theorem 1 of Olatinwo [10].

1. INTRODUCTION

One of the powerful notions employed by Adeagbo-Sheikh [1] to explain the views of some notable thinkers such as Ashby [2] and Beer [3], in his model for self-organizing systems is the *distance function* $(g(t))$, where t is the time variable. Olatinwo [8] considered the degree of progress to goal at any stage during a self-organization process only for a self-organizing system. The distance function, $g(t)$, was employed in that paper, while Olatinwo [9] generalized those results in [8] with a distance function which is an implicit function of time (i.e. $g(\sum_{k=1}^{m} \alpha_k x_k(t)),\ \sum_{k=1}^{m} \alpha_k = 1$).

Given a finite set of self-organizing systems with each system self-organizing to a distinct desired state of affairs, is it possible for all these systems to interact and become a system that is self-organizing to a particularly desired state of affairs? This question was addressed in Olatinwo [10] by considering a convex combination of a finite number of the distance functions associated to a finite number of self-organizing systems. In other words, Olatinwo [10] contains results which are also generalizations of those in [8], but which are independent of those in Olatinwo [9].

In this paper, we shall establish a result on the degree of progress to goal for a multi-purpose self-organizing system. This result is a generalization of Theorem 2A of Olatinwo [8], Theorem 1 of Olatinwo [9] as well as Theorem 1 of Olatinwo [10]. We employ in the presentation of our result, the notions of convexity of a function, convex combination of functions, triangle inequality, trace of a matrix, regular curves as well as the elementary idea of probablity theory. It is found that the result obtained is in agreement with the axiomatic properties of probability. The result is pertinent because of its possible applications in diverse areas such as learning, adaptive control and pattern recognition systems. The theories of learning are available in literature and invariably employ statistical techniques. See Fu and Mendel [6] for detail.

However, we shall state the following Lemmas which are required in the sequel.

Lemma 1. (Olatinwo [10]) *Let $\{g_k(t)\}_{k=1}^{n}$ be a set of the distance functions for n different self-organization processes. Then, $\sum_{k=1}^{n} \lambda_k g_k(t)$ is a distance function for the resultant self-organization process, where $\lambda_1,\ \lambda_2, \ldots, \lambda_n \in [0,1]$ and $\sum_{k=1}^{n} \lambda_k = 1$.*

Proof. Let

$$u(t) = \sum_{k=1}^{n} \lambda_k g_k(t) = \lambda_1 g_1(t) + \lambda_2 g_2(t) + \ldots + \lambda_n g_n(t). \qquad (\star)$$

We now show that $u(t)$ is a distance function by showing that it satisfies all the properties of a distance function stated in Olatinwo [8].

We show that $u(t) > 0,\ t_0 \leq t < t_f < \infty$, where t_f is the final time for the completion of the self-organization process:
Since each $g_k(t),\ k = 1, 2, \ldots, n$, is a distance function, then each
$g_k(t) > 0,\ t_0 \leq t < t_f < \infty$, and so each $\lambda_k g_k(t) > 0,\ k = 1, 2, \ldots, n$, noting that each $\lambda_k > 0$. Hence, $u(t) > 0$.

We now show that $u'(t) < 0,\ t_0 \leq t < t_f < \infty$.

Differentiating $u(t)$ in $(\star)$ with respect to t yields

$$u'(t) = \lambda_1 g_1'(t) + \lambda_2 g_2'(t) + \ldots + \lambda_n g_n'(t) = \sum_{k=1}^{n} \lambda_k g_k'(t). \qquad (\star\star)$$

Since each $g_k(t)$, is a distance function, we have $g_k'(t) < 0$. Again, since each $\lambda_k > 0$, we have each $\lambda_k g_k'(t) < 0$. It follows from $(\star\star)$ that $u'(t) < 0$.

Using $(\star)$, we obtain
$u(t_f) = \lambda_1 g_1(t_f) + \lambda_2 g_2(t_f) + \ldots + \lambda_n g_n(t_f) = 0,\ \ t_0 < t_f < \infty$, since for the distance functions $g_k(t)$, we have $g_k'(t_f) = 0,\ k = 1, 2, \ldots, n$.

Finally, we have using $(\star\star)$ and triangle inequality that

$$|u'(t)| = |\sum_{k=1}^{n} \lambda_k g_k'(t)| \leq \sum_{k=1}^{n} \lambda_k |g_k'(t)| < \infty,$$

since $\lambda_k > 0,\ |g_k'(t)| < \infty,\ k = 1, 2, \ldots, n,\ t_0 < t < t_f < \infty$.

Therefore, $u(t)$ is a distance function. This completes the proof of the Lemma.

Lemma 2. (Olatinwo [9, 10]) *Let $\delta(x)$ be continuous on $[a, b] \subset \Re$. Then, $\int_a^x ||\delta(u)||du$ is the length of a certain curve from a to x.*

Lemma 3. (Jensen's Inequality [5, 9]) *Let $f(x)$ be convex on (a, b), and $x_1, x_2, \ldots, x_m$ be m points of (a, b). Also, let $c_1, c_2, \ldots, c_m$ be nonnegative constants such that $\sum_{i=1}^{m} c_i = 1$. Then, $f(\sum_{i=1}^{m} c_i x_i) \leq \sum_{i=1}^{m} c_i f(x_i)$.*

If f is strictly convex and if additionally each $c_i > 0$, then equality holds if and only if $x_1 = x_2 = \ldots = x_m$.

2. MAIN RESULTS

We recall from Olatinwo [8, 9] that if X_k is the event that a self-organizing system attains a stage P_k at time t_k during self-organization process, then its probability is given by

$$Prob\,\{X_k\} = \frac{l(t_k)}{l(t_n)}, \quad k = 0, 1, 2, \ldots, n, \tag{1}$$

where $l(t_k)$ and $l(t_n)$ are the lengths of a curve $f(t)$ (see Bruce and Giblin [4] as well as Olatinwo [8, 9, 10]) defined at times t_k and t_n respectively by
$l(t_k) = \int_{t_0}^{t_k} ||f'(u)||du$ and $l(t_n) = \int_{t_0}^{t_n} ||f'(u)||du.$

If $f(t)$ is a distance function as defined in Adeagbo-Sheikh [1] and Olatinwo [8, 9, 10], then we obtain Theorem 2A of Olatinwo [8] from (1) on substituting for $l(t_k)$ and $l(t_n)$.

Suppose now that $f(t)$ is replaced by a matrix whose entries are distance functions or multiples of distance functions. Then, we have a multi-purpose self-organization process involving several self-organizing systems. The degree of progress to goal for a multi-purpose self-organization process is discussed in the following Theorem which is the main result in this paper.

Theorem. *Suppose that $[t_0, t_k]$ and $[t_0, t_n]$ are two given time intervals such that $[t_0, t_k] \subseteq [t_0, t_n] \subset \Re_+$. Let X_k be the event that a multi-purpose self-organizing system whose distance function is a matrix function $G(t)$ defined by*

$$G(t) = (\lambda_{ij} g_{ij}(\sum_{k=1}^{m} \alpha_k x_k(t)))_{s,s}\ ,\ \lambda_{ij} \geq 0, \quad \sum_{i=1}^{s} \lambda_{ii} = 1\ ,\quad \sum_{k=1}^{m} \alpha_k = 1\ ,\ \alpha_k \geq 0, \tag{2}$$

attains a stage P_k at time t_k during self-organization process. Then,

$$Prob\,\{X_k\} = \frac{\sum_{r=1}^{k} \int_{t_{r-1}}^{t_r} \left\{ \sum_{i=1}^{s} \lambda_{ii} \left[\sum_{k=1}^{m} \alpha_k || \frac{dg_{ii}}{dx_k} ||\ || \frac{dx_k}{du} || \right] \ -\ ||\delta(u)|| \right\} du}{\sum_{r=1}^{n} \int_{t_{r-1}}^{t_r} \left\{ \sum_{i=1}^{s} \lambda_{ii} \left[\sum_{k=1}^{m} \alpha_k || \frac{dg_{ii}}{dx_k} ||\ || \frac{dx_k}{du} || \right] \ -\ ||\delta(u)|| \right\} du}\ , \tag{3}$$

where $k \leq n, \quad k, n \in \{1, 2, 3, \ldots, \}$, $\delta(t)$ is a continuous function on $[t_0, t_n]$, $0 \leq \delta(t) \leq ||\delta(t)|| < \sum_{i=1}^{s} \lambda_{ii} \left[\sum_{k=1}^{m} \alpha_k || \frac{dg_{ii}}{dx_k} ||\ || \frac{dx_k}{du} || \right].$

Proof. Each entry of the matrix $G(t)$ is well-defined as a distance function, since each $g_{ij}(\sum_{k=1}^{m}\alpha_k x_k(t))$ is a distance function (an implicit function of time variable t) according to Olatinwo [9] and each λ_{ij} is nonnegative. Suppose that

$$G'(t) = \frac{d}{dt}(\text{trace } G(t)),$$

where,

$$\text{trace } G(t) = \sum_{i=1}^{s}\lambda_{ii}\left[g_{ii}(\ \sum_{k=1}^{m}\alpha_k x_k(t)\ )\right]\ ,\ \sum_{i=1}^{s}\lambda_{ii} = 1. \tag{4}$$

Then, we have by Lemma 1 that trace $G(t)$, defined by (4) is a distance function. Therefore, (1) becomes

$$Prob\,\{X_k\} = \frac{\int_{t_0}^{t_k}||G'(u)||du}{\int_{t_0}^{t_n}||G'(u)||du} = \frac{\int_{t_0}^{t_k}||\frac{d}{du}\text{trace } G(u)||du}{\int_{t_0}^{t_n}||\frac{d}{du}\text{trace } G(u)||du}. \tag{5}$$

Applying Lemma 3 in (4) yields

$$\text{trace } G(t) = \sum_{i=1}^{s}\lambda_{ii}g_{ii}(\sum_{k=1}^{m}\alpha_k x_k(t)) \leq \sum_{i=1}^{s}\lambda_{ii}\left[\sum_{k=1}^{m}\alpha_k g_{ii}(x_k(t))\right]. \tag{6}$$

Differentiating (6) and taking the norms of both sides, as well as using the triangle inequality yield

$$\begin{aligned}
\left\|\frac{d}{dt}\left[\text{trace } G(t)\right]\right\| &\leq \left\|\frac{d}{dt}\sum_{i=1}^{s}\lambda_{ii}\left[\sum_{k=1}^{m}\alpha_k g_{ii}(x_k(t))\right]\right\| \\
&= \left\|\sum_{i=1}^{s}\lambda_{ii}\left[\sum_{k=1}^{m}\alpha_k \frac{d}{dt}g_{ii}(x_k(t))\right]\right\| \\
&\leq \left|\sum_{i=1}^{s}\lambda_{ii}\right|\left[\left|\sum_{k=1}^{m}\alpha_k\right|\left\|\frac{dg_{ii}}{dx_k}\frac{dx_k}{dt}\right\|\right] \\
&\leq \sum_{i=1}^{s}|\lambda_{ii}|\left[\sum_{k=1}^{m}|\alpha_k|\left\|\frac{dg_{ii}}{dx_k}\right\|\left\|\frac{dx_k}{dt}\right\|\right] \\
&= \sum_{i=1}^{s}\lambda_{ii}\left[\sum_{k=1}^{m}\alpha_k\left\|\frac{dg_{ii}}{dx_k}\right\|\left\|\frac{dx_k}{dt}\right\|\right],
\end{aligned} \tag{7}$$

since $\alpha_k, \lambda_{ii} \in \Re_+$.

Addition of $||\delta(t)||$ to the left-hand side of (7) yields

$$\left\|\frac{d}{dt}\left[\text{trace } G(t)\right]\right\| = \sum_{i=1}^{s}\lambda_{ii}\left[\sum_{k=1}^{m}\alpha_k\left\|\frac{dg_{ii}}{dx_k}\right\|\left\|\frac{dx_k}{dt}\right\|\right] - \|\delta(t)\|. \tag{8}$$

Using (8) in (5) yields

$$Prob\{X_k\} = \frac{\int_{t_0}^{t_k} \left\{\sum_{i=1}^{s} \lambda_{ii} \left[\sum_{k=1}^{m} \alpha_k \left\|\frac{dg_{ii}}{dx_k}\right\| \left\|\frac{dx_k}{du}\right\|\right] - \|\delta(u)\|\right\} du}{\int_{t_0}^{t_n} \left\{\sum_{i=1}^{s} \lambda_{ii} \left[\sum_{k=1}^{m} \alpha_k \left\|\frac{dg_{ii}}{dx_k}\right\| \left\|\frac{dx_k}{du}\right\|\right] - \|\delta(u)\|\right\} du}, \tag{9}$$

where we have by Lemma 2 that $\int_{t_0}^{t_i} ||\delta(u)||du, (i = k, n)$ are lengths of certain curves. Application of the fact that finite union of intervals can be split up into disjoint ones (see Kai Lai [7] and Olatinwo [8, 9, 10]) yields (3).

Remark 1. If $s = 1$, in this Theorem, then we obtain Theorem 1 of Olatinwo [9], while we deduce Theorem 2A of Olatinwo [8] with $m = s = 1, \delta(t) = 0$ and $x_k(t) = t$.

Remark 2. Theorem 1 of Olatinwo [10] is obtained when $m = 1, x_k(t) = t$.

Remark 3. Since the matrix $G(t)$ is of order s, then there are s^2 self-organizing systems involved in the multi-purpose self-organization process.

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