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TRANSITIVE 3-GROUPS OF DEGREE $3^n (n = 2, 3)$

M. S. Audu[1], A. Afolabi[2] and E. Apine[1]

[1] *Department of Mathematics, Faculty of Natural Sciences, University of Jos, Jos, Nigeria*
[2] *Department of Mathematics, Faculty of Science, University of Lagos, Lagos, Nigeria*

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Abstract. In this paper we achieve a classification of transitive 3-groups of degrees 9 and 27. Other unique properties of these groups are discovered as a result.

INTRODUCTION

Let G be a group acting on a non-empty set Ω. The action of G on Ω is said to be transitive if for any α, β in Ω there exists some g in G such that $\beta = \alpha g$. In this case $|\Omega|$ is called the degree of G on Ω. In [4], M. S. Audu, determined the number of transitive p-groups of degree p^2 and in [10], E. Apine, achieved a classification of transitive and faithful p-groups (abelian and non-abelian) of degrees at most p^3 whose center is elementary abelian of rank two. In this paper, we determine, up to equivalence, the actual transitive p-groups (abelian and non-abelian) of degrees p^2 and p^3 for $p = 3$ and achieve a classification according to small degrees.

1. RESULTS

1.1 TRANSITIVE 3-GROUPS OF DEGREE $3^2 = 9$

Let G be a transitive 3-group of degree 3^2, then $|G| = 3^n$, n=1,2,3,4. Clearly, $n \neq 1$ and when $n = 2$, then $|G|$=9, G is essentially abelian and either $G \cong C_9$ or $G \cong C_3 \times C_3$. For transitivity,$|\alpha^G| = 9$, $|G_\alpha| = 1$, $\forall \alpha \in \Omega$ If $G \cong C_9$, then $G \cong G_{1,2} =< a >$, with generator, say, $a = (1,2,3,4,5,6,7,8,9)$. If $G \cong C_3 \times C_3$, then $G \cong G_{2,2} =< a, b : a^3 = 1, b^3 = 1, ab = ba >$ with generators, say, $a = (1,4,7)(2,5,8)(3,6,9)$ and $b = (1,2,3)(4,5,6)(7,8,9)$.

Clearly $G_{1,3}$ and $G_{2,2}$ are transitive on Ω and we have:

Lemma 1.1.1. *There are, up isomorphism, two transitive* 3*-groups of degree* 9 *and order* 9*, namely the abelian groups* $G_{1,2}$ *and* $G_{2,3}$ *described above.*

When n=3, then $|G|$=27 and for transitivity we must have $|\alpha^G$=9, $|G_\alpha|$=3,$\forall \alpha \in \Omega$.

Here G is non-abelian and we have the following possibilities for G: $G \cong G_{1,3} =< a, b : a^9 = 1,\ b^3 = 1, ab = ba^4 >$ or $G \cong G_{2,3} =< G_{2,2}, c >$ with c^3=1, $G_{2,2} \trianglelefteq G_{2,3}$.

Consider first $G_{1,3} =< a, b : a^9 = 1, b^3 = 1, ab = ba^4 >$ with a=(1,2,3,4,5,6,7,8,9), then, b=(2,5,8)(3,9,6) (obtained by a Gap-programme (see PROGRAMME 1)).

For the case $G_{2,3}$, we obtain a presentation as follows:
$G_{2,3} =< a, b, c : a^3 = 1, ab = ba, c^3 = 1, ac = cab, bc = cb >$, with, say, generators $a = (1,3,2)(4,6,5)(7,9,8)$, $b = (1,5,8)(3,4,7)(6,9,2)$ and $c = (2,9,6)(3,4,7)$ (obtained by a modification to PROGRAMME 1). Clearly the above groups are transitive on Ω and thus:

Lemma 1.1.2. *There are, up to isomorphism, two transitive* 3*-groups of degree* 9 *and order* 27*, namely the non-abelian groups* $G_{1,3}$ *and* $G_{2,3}$ *described above.*

When $n = 4$, $|G|$=81 and for transitivity, $|\alpha^G$, $|G_\alpha|$=9 $\forall \alpha \in \Omega$.

Thus G is non-abelian and the following are the possibilities for G: $G \cong G_{1,4} = < G_{1,3}, c >$, where c^3=1, $G_{1,3} \trianglelefteq G_{1,4}$ or G $\cong G_{2,4} =< G_{2,3}, d >$, where $d^3 = 1, G_{2,3} \trianglelefteq G_{2,4}$

For $G_{1,4}$, we have as a presentation:
$G_{1,4} =< a, b, c : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca^7 b, bc = cb >$, where a and b

are the same generators as those of $G_{1,3}$ and $c = (3, 6, 9)$.

For $G_{2,4}$, we have as a presentation:

$G_{2,4} =< a, b, c, d : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cab, bc = cb, d^3 = 1, ad = dac,$ $bd = db, cd = dc >$, with the same generators a, b, c as those of $G_{2,3}$ and d=(3,4,7). Here we notice that $G_{1,4} \cong G_{2,4}$. Thus we have:

Lemma 1.1.3. *There is, up to isomorphism, only one transitive* 3*-group of degree* 9 *and order* 81*, namely the non-abelian group* $G_{1,4}$ *described above.*

We summarize our findings into the table below:

	$\|G\| = 3^n$	Number of transitive abelian 3-group of degree 9 up to isomorphism	Number of transitive non-abelian 3-group of degree 9 up to isomorphism	Number of transitive 3-groups of degree 9 up to isomorphism
$n = 1$	3	0	0	0
$n = 2$	9	2	0	2
$n = 3$	27	0	2	2
$n = 4$	81	0	1	1
Total		2	3	5

Hence we have:

Proposition 1.1.4. *There are, up to isomorphism,* 5 *transitive* 3*-groups of degree* 3^2*,* 2 *of these are abelian and of the remaining* 3 *non-abelian,* 2 *are of exponent* 9 *and* 1 *is of exponent* 3.

1.2 TRANSITIVE 3-GROUPS OF DEGREE $3^3 = 27$

Let G be a transitive 3-group of degree 27, then $|G| = 3^n$, $n = 1, 2, \ldots, 13$. Clearly $n \neq 1$, $n \neq 2$. When $n = 3$, then $|G| = 27$ and for transitivity we must have $|\alpha^G|$=27, $|G_\alpha|$=1, $\forall \alpha \in \Omega$.

Assuming first G abelian, then either $G \cong C_{27}$ or $G \cong C_3 \times C_9$ or $G \cong C_3 \times C_3 \times C_3$. If $G \cong C_{27}$, then $G \cong G_{1,3} =< a >$, where we may take $a = (1, 2, \ldots, 27)$.

If $G \cong C_3 \times C_9$, then $G \cong G_{2,3} =< a, b : a^9 = 1, b^3 = 1, ab = ba >$, with, say,
$a = (1,2,3,4,5,6,7,8,9)(10,11,12,13,14,15,16,17,18)(19,20,21,22,23,24,25,26,27)$, and
$b = (1,17,19)(2,18,20)(3,10,21)(4,11,22)(5,12,23)(6,13,24)(7,14,25)(8,15,26)(9,16,27)$.

If $G \cong C_3 \times C_3 \times C_3$, then $G \cong G_{3,3} =< a, b, c : a^3 = 1, b^3 = 1, c^3 = 1, ab = ba,$ $ac = ca, bc = cb >$, with, say,
$a = (1,4,7)(2,5,8)(3,6,9)(10,13,16)(11,14,17)(12,15,18)(19,22,25)(20,23,26)(21,24,27)$,
$b = (1,5,6)(2,3,7)(4,8,9)(10,14,15)(11,12,16)(13,17,18)(19,23,24)(20,21,25)(22,26,27)$,
$c = (1,13,26)(2,14,24)(3,15,19)(4,16,20)(5,17,27)(6,18,22)(7,10,23)(8,11,21)(9,12,25)$.

We next assume G non-abelian. Then the following are the possibilities for G:
$G \cong G_{4,3} =< a, b : a^9 = 1, b^3 = 1, ab = ba^4 >$ or $G \cong G_{5,3} =< K, c >$, with $c^3 = 1$, $K \cong C_3 \times C_3$, $K \trianglelefteq G_{6,3}$.

Taking $a = (1,2,3,4,5,6,7,8,9)(10,11,12,13,14,15,16,17,18)(19,20,21,22,23,24,25,26,27)$ and
b=(1,10,19)(2,14,26)(3,18,24)(4,13,22)(5,17,20)(6,12,27)(7,16,25)(8,11,23)(9,15,21) satisfy the requirement of $G_{4,3}$.

For $G_{5,3}$, we obtain a presentation as follow:
$G_{5,3} =< a, b, c : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cb, bc = ca^2b^2 >$ with generators:
$a = (1,4,7)(2,5,8)(3,6,9)(10,13,16)(11,14,17)(12,15,18)(19,22,25)(20,23,26)(21,24,27)$,
$b = (1,5,6)(2,3,7)(4,8,9)(10,14,15)(11,12,16)(13,17,18)(19,23,24)(20,21,25)(22,26,27)$,
$c = (1,10,19)(2,11,20)(3,13,23)(4,14,21)(5,12,22)(6,17,25)(7,15,26)(8,16,24)(9,18,27)$
We easily check that the above-named groups are transitive on Ω and we conclude:

Lemma 1.2.1. *There are, up to isomorphism, five transitive* 3*-groups of degree* 3^3 *and order* 27*, namely the groups* $G_{1,3}$ *(of exponent* 27*),* $G_{2,3}$ *and* $G_{4,3}$ *(of exponent* 9*) and* $G_{3,3}$ *and* $G_{5,3}$ *(of exponent* 3*) described above.*

When $n = 4$, then $|G| = 81$ and for transitivity we must have
$|\alpha^G| = 27$, $|G_\alpha| = 3$, $\forall \alpha \in \Omega$.

Thus G must not be abelian and we have the following possibilities for G:
$G \cong G_{1,4} =< a, b : a^{27} = 1, b^3 = 1, ab = ba^{10} >$ or $G \cong G_{2,4} =< G_{2,3}, c >$, with c^3=1, $G_{2,3} \trianglelefteq G_{2,4}$. or $G \cong G_{3,4} =< G_{3,3}, d >$ with d^3=1, $G_{3,3} \trianglelefteq G_{3,4}$ or $G_{4,4} =< G_{4,3}, c >$ with c^3=1, $G_{4,3} \trianglelefteq G_{4,4}$. or $G \cong G_{5,4} =< G_{5,3}, d >$ where d^3=1, $G_{5,3} \trianglelefteq G_{5,4}$ or $G \cong$

$G_{6,4} =< a,b : a^9 = 1,\ b^9 = 1, ab = ba^4 >$ or $G \cong G_{7,4} =< K,c >$, with c^9=1, $K \cong C_3 \times C_3$, $K \trianglelefteq G_{7,4}$. Of these groups only four satisfy the requirements for G, namely $G_{1,4}, G_{3,4}, G_{4,4}$ and $G_{5,4}$.

Now taking $a = (1, 2, \ldots, 27)$ and by an argument similar to the case $n = 3$, we get b=(1,19,10)(3,12,21)(4,22,13)(6,15,24)(7,25,16)(9,18,27).

For $G_{4,4}$, we have a presentation as follows:

$G_{4,4} =< a,b,c : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6b >$, where the generators a and b are the same for $G_{4,3}$ and

c=(1,4,7)(2,5,8)(3,6,9)(19,25,22)(20,26,23)(21,27,24)

(obtained by a Gap-programme(see PROGRAMME 2)). For $G_{3,4}$, we have a presentation as follow:

$G_{3,4} =< a,b,c : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = ca, bc = cb, d^3 = 1, ad = dab,$ $bd = db, cd = dc >$, where a, b and c are the same generators of $G_{3,3}$ and

d=(1,27,18)(2,24,14)(3,19,15)(4,21,11)(5,22,12)(6,26,16) (7,23,10)(8,20,12)(9,20,16).

For $G_{5,4}$, we have the presentation as follows:

$G_{5,4} =< a,b,c,d : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cb, bc = ca^2b^2, d^3 = 1, ad = da,$ $bd = db, cd = dc >$ when a, b, c are the same generators of $G_{6,3}$ and

$d = (1, 27, 11)(2, 19, 18)(3, 23, 13)(4, 21, 14)(5, 22, 12)(6, 26, 16)(7, 24, 17)(8, 25, 15)(9, 20, 10)$.

We easily check that the above-named groups are transitive on Ω and we conclude:

Lemma 1.2.2. *There are, up to isomorphism, four transitive 3-groups of degree* 3^3 *and order 81, namely the non-abelian groups* $G_{1,4}$ *(exponent 27),* $G_{4,4}$ *(exponent 9),* $G_{3,4}$ *and* $G_{5,4}$ *(both of exponent 3) described above.*

When n=5, then$|G| = 243$ and for transitivity we must have $|\alpha^G|$=27, $|G_\alpha|$=9,$\forall \alpha \in \Omega$. Thus G must be non-abelian and we have the following possibilities for G:

$G \cong G_{1,5} =< G_{1,4}, c >$ with c^3=1, $G_{1,4} \trianglelefteq G_{1,5}$ or $G \cong G_{2,5} =< G_{4,4}, d >$ with d^3=1, $G_{4,4} \trianglelefteq G_{2,5}$ or $G \cong G_{3,5} =< G_{3,4}, d >$ with e^3=1, $G_{5,4} \trianglelefteq G_{4,5}$ or $G \cong G_{5,5} =< K,c >$ with c^3=1,

$K \cong C_{27} \times C_3$, $K \trianglelefteq G_{9,5}$ or $G \cong G_{6,5} =< G_{2,3}, c >$ with c^9=1,

$G_{2,3} \trianglelefteq G_{6,5}$ or $G \cong G_{7,5} =< K,c >$ with c^{27}=1,

$K \cong C_3 \times C_3$, $K \trianglelefteq G_{7,5}$ or $G \cong G_{8,5} =< K, d >$ where d^3=1,

$K \cong C_9 \times C_3 \times C_3$, $K \trianglelefteq G_{8,5}$ or $G \cong G_{9,5} =< K, d >$ where c^3=1,

$K \cong C_9 \times C_9$, $K \trianglelefteq G_{9,5}$ or $G \cong G_{10,5} =< K, d >$, where d^3=1,

$K \cong C_9 \times C_3 \times C_3$, $K \trianglelefteq G_{10,5}$ or $G \cong G_{11,5} =< G_{6,3}, d >$ where d^9=1,

$G_{6,3} \trianglelefteq G_{11,5}$ or $G \cong G_{12,5} =< K, d >$, where d^9=1,

$K \cong C_3 \times C_3 \times C_3$, $K \trianglelefteq G_{12,5}$ or $G \cong G_{13,5} =< K, d >$, where d^3=1,

$K \cong C_9 \times C_3 \times C_3$, $K \trianglelefteq G_{13,5}$.

For obvious reasons, only $G_{1,5}$, $G_{3,5}$, $G_{4,5}$ and $G_{2,5}$ satisfy the requirements for G.

For $G_{1,5}$, we obtain as a presentation:
$G_{1,5} =< a, b, c : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = cb, ac = ca^{10}b^2 >$, where a, b are the same generators of $G_{1,4}$ and c = (1,19,10)(4,22,13)(7,25,16) (obtained by a modification to PROGRAMME 2).

For $G_{3,5}$, we have a presentation as follows:
$G_{3,5} =< a, b, c, d, e : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = ca, bc = cb, d^3 = 1, ad = dab, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = eb^2c, de = eb^2d >$, where a, b, c, d are the same generators of $G_{3,4}$ and
e = (1,14,21)(2,12,22)(3,16,26)(4,17,24)(5,15,25)(6,10,20)(7,11,27)(8,18,19)(9,13,23).

For $G_{4,5}$, we have a presentation as follows:
$G_{4,5} =< a, b, c, d, e : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cb, bc = ca^2b^2, d^3 = 1, ad = da, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = ec, de = eab^2d >$, where a, b, c, d are the same generators of $G_{5,4}$ and
e = (1,3,8)(2,4,6)(5,7,9)(10,13,16)(11,14,17)(12,15,18)(19,23,24)(20,21,25)(22,26,27).

For $G_{2,5}$, we have:
$G_{2,5} =< a, b, c, d : a^9 = 1, b^3 = 1, ab = ba^4,\ c^3 = 1, ac = ca, bc = ca^6b, d^3 = 1, ad = da, bd = da^6bc^2, cd = da^6c >$, where the generators a, b, c are the same for $G_{4,4}$ and
d=(1,27,11)(2,19,12)(3,20,13)(4,21,14)(5,22,15)(6,23,16)(7,24,17) (8,25,18)(9,26,10).

Hence we have:

Lemma 1.2.3. *There are, up to isomorphism, four transitive 3-groups of degree* 3^3 *and order 243, namely the non-abelian groups* $G_{1,5}$ *(exponent 27),* $G_{2,5}$ *(of exponent 9),* $G_{3,5}$ *and* $G_{4,5}$ *(of exponent 3) described above.*

When n=6, then$G$ = 729 and for transitivity we must have $|\alpha^G|$=27, $G_\alpha|$=27, $\forall\alpha \in \Omega$. Thus G must be non-abelian and we have the following possibilities for G: $G \cong G_{1,6} = < G_{1,5}, d >$ with d^3=1, $G_{1,5} \trianglelefteq G_{9,6}$ or $G \cong G_{2,6} = < G_{2,5}, e >$ with e^3=1,

$G_{1,5} \trianglelefteq G_{2,6}$ or $G \cong G_{3,6} =< G_{3,5}, f >$ with f^3=1,

$G_{3,5} \trianglelefteq G_{3,6}$ or $G \cong G_{4,6} =< G_{4,5}, f >$ with f^3=1,

$G_{4,5} \trianglelefteq G_{4,6}$ or $G \cong G_{13,6} =< K, c >$ with c^{27}=1,

$K \cong C_9 \times C_3$, $K \trianglelefteq G_{13,6}$ or $G \cong G_{14,6} =< G_{4,3}, c >$ with c^{27}=1,

$G_{4,3} \trianglelefteq G_{14,6}$ or $G \cong G_{15,6} =< G_{3,3}, d >$ where d^{27}=1,

$G_{3,3} \trianglelefteq G_{15,6}$ or $G \cong G_{16,6} =< G_{5,2}, d >$ where d^{27}=1,

$G_{5,2} \trianglelefteq G_{16,6}$ or $G \cong G_{17,6} =< K, d >$ with d^3=1,

$K \cong C_9 \times C_9 \times C_3$, $K \trianglelefteq G_{17,6}$ or $G \cong G_{18,6} =< K, e >$ where e^9=1,

$K \cong C_3 \times C_3 \times C_3 \times C_3$, $K \trianglelefteq G_{18,6}$ or $G \cong G_{19,6} =< K, e >$ where e^3=1,

$K \cong C_9 \times C_3 \times C_3 \times C_3$, $K \trianglelefteq G_{19,6}$ or $G \cong G_{20,6} =< G_{3,4}, e >$, where e^9=1,

$G_{3,4} \trianglelefteq G_{20,6}$ or $G \cong G_{21,6} =< G_{5,4}, e >$ where e^9=1,

$G_{5,4} \trianglelefteq G_{21,6}$ or $G \cong G_{22,6} =< K, f >$, where f^3=1,

$K \cong C_3 \times C_3 \times C_3 \times C_3 \times C_3$, $K \trianglelefteq G_{22,6}$

It is readily seen that of the above groups, $G_{9,6}$, $G_{10,6}$,$G_{11,6}$ and $G_{12,6}$ are acceptable.

For $G_{9,6}$, we have a presentation:

$G_{9,6} =< a, b, c, d : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = cb, ac = ca^{10}b^2, d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc >$, where a, b, c are the same for $G_{10,5}$ and

d = (1,10,19)(3,21,12)(4,22,13)(5,14,23)(8,26,17)(9,18,27)

For $G_{10,6}$, we have a presentation:

$G_{10,6} =< a, b, c, d, e, f : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = ca, bc = cb, d^3 = 1, ad = dab, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = eb^2c, de = eb^2d, f^3 = 1, af = fa, bf = fb, cf = fab^2e, ef = fabc^2e^2 >$, where a, b, c, d, e are the same generators of $G_{16,5}$ and

f = (1,3,8)(2,4,6)(5,7,9)(10,12,17)(11,13,15)(14,16,18)(19,20,27)(21,22,23) (24,25,26).

For $G_{11,6}$, we have a presentation as follows:

$G_{11,6} =< a, b, c, d, e, f : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cb, bc = ca^2b^2, d^3 = 1, ad = da, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = ec, de = eab^2d, f^3 = 1, af = fa, bf = fb, cf = fac, df = fde^2, ef = fe >$, where a, b, c, d, e are the same generators of $G_{5,4}$ and

$f = (1,5,6)(2,3,7)(4,8,9)(10,16,13)(11,17,14)(12,18,15)$.

We notice here that $G_{10,6}$ and $G_{11,6}$ aer non-isomorphic and are of exponent 9.

For $G_{12,6}$, we have a representation as follows:

$G_{12,6} =< a, b, c, d, e : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6b, d^3 = 1, ad = da, bd = da^6bc^2, cd = da^6c, e^3 = 1, ae = ea, be = eb, ce = ea^3c, de = ecd >$, where the generators a, b, c and d are the same for $G_{2,5}$ and

e=(1,10,19)(2,11,20)(3,12,21)(4,13,22)(5,14,23)(6,15,24)(7,16,25) (8,17,26)(9,18,27).

Clearly, $G_{12,6}$ is neither isomorphic to $G_{10,6}$ nor to $G_{11,6}$. Moreover, Gap-programme and computations in Sym(27) show that there are no transitive p-groups of degree p^3, exponent p and orders greater than and equal to 3^6. Hence we have:

Lemma 1.2.4. *There are, up to isomorphism, four transitive 3-groups of degree* 3^3 *and order 729, namely the non-abelian groups* $G_{9,6}$ *(of exponent 27),* $G_{11,6}, G_{12,6}$ *and* $G_{10,6}$ *(of exponent 9) described above.*

When n=7, then $|G| = 2187$ and for transitivity we must have

$|\alpha^G|$=27, $|G_\alpha|$=81, $\forall \alpha \in \Omega$.

Thus, G must be non-abelian and arguing in a fashion similar to the case n=6, we have the following five representations for G as follows:

$G_{1,7} =< a, b, c, d, e : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = cb, ac = ca^{10}b^2, d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc, e^3 = 1, ae = eab^2d, be = eb, ce = ec, de = ed >$, where a, b, c, d are the same generators of $G_{9,6}$ and

$e = (1,19,10)(2,20,11)(5,14,23)(7,16,25)$.

$G_{2,7} =< a, b, c, d, e, f, g : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cb, bc = ca^2b^2, d^3 = 1, ad = da, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = ec, de = eab^2d, f^3 = 1, af = fa, bf = fb, cf = fac, df = fde^2, ef = fe, g^3 = 1, ag = ga, bg = gb, cg = ga^2bce, dg = ga^2b^2de, eg = ge, fg = gf >$, where a, b, c, d, e, f are the same generators of $G_{11,6}$ and

$g = (1,4,7)(2,5,8)(3,6,9)(10,15,14)(11,16,12)(13,18,17)$.

$G_{3,7} =< a, b, c, d, e, f : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6b, d^3 = 1, ad = da, bd = da^6bc^2, cd = da^6c, e^3 = 1, ae = ea, be = eb, ce = ea^3c, de = ecd, f^3 = 1, af = fa^7b^2c^2e, bf = fa^3b^2ce^2, cf = fc, df = fd, ef = fbc >$, where a, b, c, d, e, are the same generators of $G_{12,6}$ and

$f = (1,4,7)(3,6,9)(10,13,16)(11,14,17)(20,23,26)(21,24,27)$.

$G_{4,7} =< a, b, c, d, e, f, g : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = ca, bc = cb, d^3 = 1, ad = dab, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = eb^2c, de = eb^2d, f^3 = 1, af = fa, bf = fb, cf = fab^2e, ef = fabc^2e^2, g^3 = 1, ag = gab^2c^2d^2, bg = gb, cg = gc, eg = ga^2cde, fg = gabdef >$, where a, b, c, d, e, f are the same generators of $G_{10,6}$ and

$g = (1,27,18)(2,24,14)(3,19,15)(4,21,12)(5,22,13)(6,26,17)(7,23,10) (8,25,16)(9,20,11)$.

$G_{5,7} =< a, b, c, d : a^9 = 1, b^9 = 1, ab = ba, c^9 = 1, ac = ca, bc = cb, d^3 = 1, ad = da, bd = dab^8c^4, cd = db^2 >$, with generators a, b, c and d given as:

a=(1, 2, . . . , 9)(10, 11, . . . , 18)(19, 20, . . . , 27), b=(1, 2, . . . , 9),

c=(10, 12, 14, 16, 18, 11, 13, 15, 17)

d=(1,27,11)(2,19,12)(3,20,13)(4,21,14)(5,22,15)(6,23,16)(7,24,17)(8,25,18) (9,26,10).

Now we easily see that $G_{3,7} \cong G_{4,7}$. Hence, we have:

Lemma 1.2.5. *There are, up to isomorphism, four transitive 3-groups of degree* 3^3 *and order 2187, namely the non-abelian groups* $G_{1,7}$, $G_{5,7}$ *(both of exponent 27),* $G_{2,7}$, $G_{3,7}$ *(both of exponent 9) described above.*

When $n = 8$, then$|G| = 6561$ and for transitivity we must have

$|\alpha^G| = 27$, $|G_\alpha| = 243$, $\forall \alpha \in \Omega$.

Thus G must be non-abelian and arguing in a fashion similar to case n=6, we have the following presentations for G:

$G_{1,8} =< a, b, c, d, e, f : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = bc, ac = ca^{10}b^2, d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc, e^3 = 1, ae = eab^2d, be = eb, ce = ec, de = ed, f^3 = 1, af = fa^{10}bcde^2, bf = fb, cf = fc, df = fd, , ef = fe >$, where a, b, c, d, e, are the same for $G_{1,7}$ and

f=(1,10,19)(3,12,21)(4,22,13)(6,24,15).

$G_{2,8} =< a, b, c, d, e, f, g, h : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cb, bc = ca^2b^2$, $d^3 = 1$, $ad = da, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = ec, de = eab^2d$, $af = fa, bf = fb$, $cf = face^2, df = fa^2de, ef = fe, g^3 = 1, ag = ga, bg = gb$, $cg = gce^2, dg = gabde^2, eg = ge$, $fg = gf, h^3 = 1, ah = ha, bh = hb, ch = hce^2g$, $dh = habdfg, eh = he, fh = hf, gh = hg >$, where a, b, c, d, e, f, g are the same generators for $G_{11,7}$ and h = (1,3,8)(2,4,8)(5,7,9).

$G_{3,8} =< a, b, c, d, e, f, g : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6b$, $d^3 = 1$, $ad = da, bd = da^6bc^2, cd = da^6c, e^3 = 1, ae = ea, be = eb, ce = ea^3c$, $de = ecd, f^3 = 1, af = fa^7b^2c^2e, bf = fa^3b^2ce^2, cf = fc, df = fd, ef = fbc$, $g^3 = 1, ag = gac, bg = ga^3ce$, $cg = gc, dg = ga^3bc^2de^2, eg = gb^2ce^2, fg = gf >$, where the generators a, b, c, d, e, f, are the same generators for $G_{3,7}$ and g = (1,4,7)(2,8,5)(19,22,25) (21,27,24).

We easily see that $G_{3,8} \cong G_{2,8}$. $G_{4,8} =< a, b, c, d, e : a^9 = 1, b^9 = 1, ab = ba, c^9 = 1$, $ac = ca, bc = cb, d^3 = 1, ad = da, bd = dab^8c^4, cd = db^2, e^3 = 1, ae = ea^4, be = eb^4$, $ce = ec^4, de = ea^6d >$, where the generators a, b, c and d are the same for $G_{1,7}$ and e=(2,5,8)(3,9,6)(11,14,17)(12,18,15)(20,23,26)(21,27,24).

Hence we have:

Lemma 1.2.6. *There are, up to isomorphism, three transitive* 3*-groups of degree* 3^3 *and order* 6561*, namely the non-abelian groups* $G_{1,8}$*,* $G_{4,8}$ *(both of exponent* 27*) and* $G_{3,8}$ *(of exponent* 9*) described above.*

When n=9, then $|G|$=19683 and for transitivity we must have
$|\alpha^G|=27, G_\alpha|=729$, $\forall\alpha \in \Omega$.

Thus, G must be non-abelian and arguing in a fashion similar to case n=6, we have as presentations for G:

$G_{1,9} =< a, b, c, d, e, f, g : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = bc, ac = ca^{10}b^2$, $d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc, e^3 = 1, ae = eab^2d, be = eb, ce = ec$, $de = ed, f^3 = 1$, $af = fa^{10}bcde^2, bf = fb, cf = fc, df = fd, ef = fe, g^3 = 1$, $ag = ga^{19}cd^2e^2f^2, bg = gb$, $cg = gc, dg = gd, eg = ge, fg = gf >$, where a, b, c, d, e, f are the same for $G_{1,8}$ and g=(1,10,19)(2,11,20)(9,18,27).

$G_{2,9} =< a, b, c, d, e, f, g, h : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6b$,

$d^3 = 1$, $ad = da, bd = da^6bc^2, cd = da^6c, e^3 = 1, ae = ea, be = eb, ce = ea^3c,$ $de = ecdf^3 = 1$, $af = fa^7b^2c^2e, bf = fa^3b^2ce^2, cf = fc, df = fd, ef = fbc, g^3 = 1,$ $ag = gac, bg = ga^3ce$, $cg = gc, dg = ga^3bc^2de^2, eg = gb^2ce^2, fg = gf, h^3 = 1,$ $ah = ha^3b^2c^2d^2, bh = ha^4b^2c^2dg$, $ch = hc, dh = hbde^2g^2, eh = ha^4bcdeg, hg = gh,$ $fh = hf >$, where the generators a, b, c, d, e, f, g are the same for $G_{3,8}$ and h = (1,4,7)(3,6,9)(12,18,15)(21,27,24).

$G_{3,9} =< a, b, c, d, e, f : a^9 = 1, b^9 = 1, ab = ba, c^9 = 1, ac = ca, bc = cb, d^3 = 1,$ $ad = da, bd = dab^8c^4, cd = db^2, e^3 = 1, ae = ea^4, be = eb^4, ce = ec^4, de = ea^6d, f^3 = 1,$ $af = fa^4b^3c^3, bf = fb^7, cf = fc, df = fa^3b^6c^6de^2, ef = fe >$, where the generators a, b, c, d, e, are the same for $G_{1,8}$ and f = (1,7,4)(2,5,8)(19,22,25)(20,26,23).

Hence, we have:

Lemma 1.2.7. *There are, up to isomorphism, three transitive 3-groups of degree* 3^3 *and order 19683, namely the non-abelian groups* $G_{1,9}$, $G_{3,9}$ *(both of exponent 27) and* $G_{2,9}$ *(of exponent 9) described above.*

When n=10, then $|G|$ = 59049 and for transitivity we must have $|\alpha_G|$=27, $|G_\alpha|$=2187.

Thus G must be non-abelian and arguing in a fashion similar to case n=6, we have the following presentations for G:

$G_{1,10} =< a, b, c, d, e, f, g, h : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = bc, ac = ca^{10}b^2,$ $d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc, e^3 = 1, ae = eab^2d, be = eb, ce = ec, de = ed,$ $f^3 = 1, af = fa^{10}bcde^2, bf = fb, cf = fc, df = fd, ef = fe, g^3 = 1, ag = ga^{19}cd^2e^2f^2,$ $bg = gb, cg = gc, dg = gd, eg = ge, fg = gf, h^3 = 1, ah = hac^2dg^2, bh = hb, ch = hc,$ $dh = hd, eh = he, fh = hf, gh = hg >$, where a, b, c, d, e, f, g are the same for $G_{1,9}$ and h=(1,10,19)(3,21,12)(4,22,13)(7,16,25).

$G_{3,10} =< a, b, c, d, e, f, g, h, k : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6b,$ $d^3 = 1, ad = da, bd = da^6bc^2, cd = da^6c, e^3 = 1, ae = ea, be = eb, ce = ea^3c, de = ecd,$ $f^3 = 1, af = fa^7b^2c^2e, bf = fa^3b^2ce^2, cf = fc, df = fd, ef = fbc, g^3 = 1, ag = gac,$ $bg = ga^3ce, cg = gc, dg = ga^3bc^2de^2, eg = gb^2ce^2, fg = gf, h^3 = 1, ah = ha^3b^2c^2d^2,$ $bh = ha^4b^2c^2dg, ch = hc, dh = hbde^2g^2, eh = ha^4bcdeg, hg = gh, fh = hf, k^3 = 1,$ $ak = kd^2e^2g, bk = ka^4bcdef, ck = kc, dk = kcdf^2g^2, ek = ka^7de^2f, fk = kf, gk = kg,$

$hk = kh >$, where the generators a, b, c, d, e, f, g, h are the same for $G_{3,9}$ and k=(1,4,7)(10,13,16)(12,18,15)(21,27,24).

$G_{2,10} =< a, b, c, d, e, f, g : a^9 = 1, b^9 = 1, ab = ba, c^9 = 1, ac = ca, bc = cb, d^3 = 1, ad = da, bd = dab^8c^4, cd = db^2, e^3 = 1, ae = ea^4, be = eb^4, ce = ec^4, de = ea^6d, f^3 = 1, af = fa^4b^3c^3, bf = fb^7, cf = fc, df = fa^3b^6c^6de^2, ef = fe, ag = gab^3, bg = gb^4, cg = gc, dg = gb^3c^6df^2, eg = ge, fg = gf >$, where the generators a, b, c, d, e, f are the same for $G_{1,9}$ and g=(1,7,4)(3,6,9).

Hence we have:

Lemma 1.2.8. *There are, up to isomorphism, three transitive* 3*-groups of degree* 3^3 *and order* 59049*, namely the non-abelian groups* $G_{1,10}$*,* $G_{2,10}$ *(of exponent* 27*) and* $G_{3,10}$ *(of exponent* 9*) described above.*

When n=11, then $|G|$=177147 and for transitivity we must have
$|\alpha^G|$=27, $|G_\alpha|$=6561 $\forall \alpha \in \Omega$.

Thus G must be non-abelian and arguing in a fashion similar to case n=6, we have the following presentations for G as follows:

$G_{1,11} =< a, b, c, d, e, f, g, h, k : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = bc, ac = ca^{10}b^2, d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc, e^3 = 1, ae = eab^2d, be = eb, ce = ec, de = ed, f^3 = 1, af = fa^{10}bcde^2, bf = fb, cf = fc, df = fd, ef = fe, g^3 = 1, ag = ga^{19}cd^2e^2f^2, bg = gb, cg = gc, dg = gd, eg = ge, fg = gf, h^3 = 1, ah = hac^2dg^2, bh = hb, ch = hc, dh = hd, eh = he, fh = hf, gh = hg, k^3 = 1, ak = kab^2d^2e^2fgh, bk = kb, ck = kc, dk = kd, ek = ke, fk = kf, gk = kg, hk = kh >$, where a, b, c, d, e, f, g, h are the same for $G_{1,10}$ and k=(1,10,19)(5,14,23)(6,24,15)(8,17,26).

$G_{3,11} =< a, b, c, d, e, f, g, h, k, m : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6b, d^3 = 1, ad = da, bd = da^6bc^2, cd = da^6c, e^3 = 1, ae = ea, be = eb, ce = ea^3c, de = ecd, f^3 = 1, af = fa^7b^2c^2e, bf = fa^3b^2ce^2, cf = fc, df = fd, ef = fbc, g^3 = 1, ag = gac, bg = ga^3ce, cg = gc, dg = ga^3bc^2de^2, eg = gb^2ce^2, fg = gf, h^3 = 1, ah = ha^3b^2c^2d^2, bh = ha^4b^2c^2dg, ch = hc, dh = hbde^2g^2, eh = ha^4bcdeg, hg = gh, fh = hf, k^3 = 1, ak = kd^2e^2g, bk = ka^4bcdef, ck = kc, dk = kcdf^2g^2, ek = ka^7de^2f, fk = kf, gk = kg, hk = kh, m^3 = 1, am = ma^8cdefgh^2, bm = ma^4bdef^2g^2h^2k^2, cm = mc, dm = mbc^2de^2fg^2h^2k, em = mae^2de^2f^2g^2k^2,$

$fm = mf, gm = mg, hm = mh, km = mk >$, where the generators $a, b, c, d, e, f, g, h, k$ are the same for $G_{3,10}$ and m=(1,4,7)(10,13,16)(11,14,17)(12,15,18).

$G_{2,11} =< a, b, c, d, e, f, g, h : a^9 = 1, b^9 = 1, ab = ba, c^9 = 1, ac = ca, bc = cb, d^3 = 1, ad = da, bd = dab^8c^4, cd = db^2, e^3 = 1, ae = ea^4, be = eb^4, ce = ec^4, de = ea^6d, f^3 = 1, af = fa^4b^3c^3, bf = fb^7, cf = fc, df = fa^3b^6c^6de^2, ef = fe, ag = gab^3, bg = gb^4, cg = gc, dg = gb^3c^6df^2, eg = ge, fg = gf, h^3 = 1, ah = hab^6c^3e, bh = hb, ch = ha^6b^3c^7e^2fg^2, dh = ha^6c^3def^2, he = eh, hf = fh, hg = gh >$, where the generators a, b, c, d, e, f, g are the same for $G_{1,10}$ and h = (1,7,4)(2,8,5)(12,15,18)(20,23,26).

Hence we have:

Lemma 1.2.9. *There is, up to isomorphism, three transitive* 3*-groups of degree* 3^3 *and order* 177147*, namely the non-abelian groups* $G_{1,11}$*,* $G_{2,11}$ *(of exponent* 27*) and* $G_{3,11}$ *(of exponent* 9*) described above.*

When n=12, then $|G| = 531441$ and for transitivity we must have $|\alpha^G| = 27$, $|G_\alpha| = 19683, \forall\alpha \in \Omega$.

Thus G must be non-abelian and, arguing in a fashion similar to case n=6, we have the following presentations for G:

$G_{1,12} =< a, b, c, d, e, f, g, h, k, m : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = bc, ac = ca^{10}b^2, d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc, e^3 = 1, ae = eab^2d, be = eb, ce = ec, de = ed, f^3 = 1, af = fa^{10}bcde^2, bf = fb, cf = fc, df = fd, ef = fe, g^3 = 1, ag = ga^{19}cd^2e^2f^2, bg = gb, cg = gc, dg = gd, eg = ge, fg = gf, h^3 = 1, ah = hac^2dg^2, bh = hb, ch = hc, dh = hd, eh = he, fh = hf, gh = hg, k^3 = 1, ak = kab^2d^2e^2fgh, bk = kb, ck = kc, dk = kd, ek = ke, fk = kf, gk = kg, hk = kh, m^3 = 1, am = ma^{16}bcdefg^2h, bm = mb, cm = mc, dm = ma^9bd, em = ma^{18}bc, fm = mb^2cf, gm = mcd^2eg, hm = ma^{18}b^2c^2f^2h, km = mbcde^2k >$, where $a, b, c, d, e, f, g, h, k$ are the same for $G_{1,11}$ and m = (1,10,19)(2,17,5)(3,6,27)(7,16,25)(8,23,20)(9,12,15)(11,26,14)(18,21,24).

$G_{3,12} =< a, b, c, d, e, f, g, h, k, m, n : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6b, d^3 = 1, ad = da, bd = da^6bc^2, cd = da^6c, e^3 = 1, ae = ea, be = eb, ce = ea^3c, de = ecd, f^3 = 1, af = fa^7b^2c^2e, bf = fa^3b^2ce^2, cf = fc, df = fd,$

$ef = fbc, g^3 = 1, ag = gac, bg = ga^3ce, cg = gc, dg = ga^3bc^2de^2, eg = gb^2ce^2, fg = gf, h^3 = 1, ah = ha^3b^2c^2d^2, bh = ha^4b^2c^2dg, ch = hc, dh = hbde^2g^2, eh = ha^4bcdeg, hg = gh, fh = hf, k^3 = 1, ak = kd^2e^2g, bk = ka^4bcdef, ck = kc, dk = kcdf^2g^2, ek = ka^7de^2f, fk = kf, gk = kg, hk = kh, m^3 = 1, am = ma^8cdefgh^2, bm = ma^4bdef^2g^2h^2k^2, cm = mc, dm = mbc^2de^2fg^2h^2k, em = mae^2de^2f^2g^2k^2, fm = mf, gm = mg, hm = mh, km = mk, n^3 = 1, an = nab^2d^2e, bn = nbcdf^2g, cn = na^6bc^2e^2, dn = nd, en = na^3bc^2df^2g, fn = nf, gn = nbce^2fg, hn = na^6b^2cef^2gk, kn = na^6b^2cef^2gh^2k^2, mn = nbe^2f^2h^2km >$, where the generators a, b, c, d, e, f, g, h, k, m are the same for $G_{3,11}$ and
n=(1,11,27)(2,25,15)(4,14,21)(5,19,18)(7,17,24)(8,22,12).

$G_{2,12} =< a, b, c, d, e, f, g, h, k : a^9 = 1, b^9 = 1, ab = ba, c^9 = 1, ac = ca, bc = cb, d^3 = 1, ad = da, bd = dab^8c^4, cd = db^2, e^3 = 1, ae = ea^4, be = eb^4, ce = ec^4, de = ea^6d, f^3 = 1, af = fa^4b^3c^3, bf = fb^7, cf = fc, df = fa^3b^6c^6de^2, ef = fe, ag = gab^3, bg = gb^4, cg = gc, dg = gb^3c^6df^2, eg = ge, fg = gf, h^3 = 1, ah = hab^6c^3e, bh = hb, ch = ha^6b^3c^7e^2fg^2, dh = ha^6c^3def^2, he = eh, hf = fh, hg = gh, ak = ka^4c^3f, bk = kb^7g^2, ck = kc, dk = kc^6de^2fgh^2, ek = ke, fk = kf, gk = kg, hk = kh >$, where the generators a, b, c, d, e, f, g, h are the same for $G_{1,11}$ and k = (2,8,5)(21,24,27).

Hence we have:

Lemma 1.2.10. *There are, up to isomorphism, three transitive* 3*-groups of degree* 3^3 *and order* 531441*, namely the non-abelian groups* $G_{1,12}$*,* $G_{2,12}$ *(of exponent* 27*) and* $G_{3,12}$ *(of exponent* 9*) described above.*

When n=13, then $|G|$=1594323 and for transitivity we must have
$|\alpha^G|$=27, $|G_\alpha|$=19683, $\forall \alpha \in \Omega$.

Thus G must be non-abelian and arguing in a fashion similar to case n=6, we have the following presentations for G:

$G_{1,13} =< a, b, c, d, e, f, g, h, k, m, n : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = bc, ac = ca^{10}b^2, d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc, e^3 = 1, ae = eab^2d, be = eb, ce = ec, de = ed, f^3 = 1, af = fa^{10}bcde^2, bf = fb, cf = fc, df = fd, ef = fe, g^3 = 1, ag = ga^{19}cd^2e^2f^2, bg = gb, cg = gc, dg = gd, eg = ge, fg = gf, h^3 = 1, ah = hac^2dg^2,$

$bh = hb, ch = hc, dh = hd, eh = he, fh = hf, gh = hg, k^3 = 1, ak = kab^2d^2e^2fgh,$
$bk = kb, ck = kc, dk = kd, ek = ke, fk = kf, gk = kg, hk = kh, m^3 = 1,$
$am = ma^{16}bcdefg^2h, bm = mb, cm = mc, dm = ma^9bd, em = ma^{18}bc,$
$fm = mb^2cf, gm = mcd^2eg, hm = ma^{18}b^2c^2f^2h, km = mbcde^2k, n^3 = 1,$
$an = na^{16}cd^2fh^2km^2, bn = nb, cn = nc, dn = nb^2cd, en = ne, fn = nb^2cf,$
$gn = na^9c^2defg, hn = na^{18}b^2c^2d^2e^2f^2h, kn = na^{18}bd^2e^2f^2k, mn = na^{18}b^2d^2fm >$,
where $a, b, c, d, e, f, g, h, k, m$ and n are the same for $G_{1,12}$ and
$n = (1,10,19)(12,24,27)(6,9,21)(3,15,18)(8,26,17)(7,25,16)(5,23,14)$.

$G_{3,13} =< a, b, c, d, e, f, g, h, k, m, n, p : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca,$
$bc = ca^6b, d^3 = 1, ad = da, bd = da^6bc^2, cd = da^6c, e^3 = 1, ae = ea, be = eb,$
$ce = ea^3c, de = ecd, f^3 = 1, af = fa^7b^2c^2e, bf = fa^3b^2ce^2, cf = fc, df = fd,$
$ef = fbc, g^3 = 1, ag = gac, bg = ga^3ce, cg = gc, dg = ga^3bc^2de^2, eg = gb^2ce^2,$
$fg = gf, h^3 = 1, ah = ha^3b^2c^2d^2, bh = ha^4b^2c^2dg, ch = hc, dh = hbde^2g^2,$
$eh = ha^4bcdeg, hg = gh, fh = hf, k^3 = 1, ak = kd^2e^2g, bk = ka^4bcdef, ck = kc,$
$dk = kcdf^2g^2, ek = ka^7de^2f, fk = kf, gk = kg, hk = kh, m^3 = 1, am = ma^8cdefgh^2,$
$bm = ma^4bdef^2g^2h^2k^2, cm = mc, dm = mbc^2de^2fg^2h^2k, em = mae^2de^2f^2g^2k^2,$
$fm = mf, gm = mg, hm = mh, km = mk, n^3 = 1, an = nab^2d^2e, bn = nbcdf^2g,$
$cn = na^6bc^2e^2, dn = nd, en = na^3bc^2df^2g, fn = nf, gn = nbce^2fg, hn = na^6b^2cef^2gk,$
$kn = na^6b^2cef^2gh^2k^2, mn = nbe^2f^2h^2km, n^3 = 1, ap = pa^7bc^2de^2f^2n, bp = pceg^2h^2kn^2,$
$cp = pa^3cf^2, dp = pd, ep = pc^2ef^2g^2h^2kn^2, fp = pf, gp = pa^6fg, hp = pb^2efg^2h^2k^2,$
$kp = pb^2efg^2h, np = pn >$, where the generators $a, b, c, d, e, f, g, h, k, m, n$ are the same for $G_{3,12}$ and
p=(1,4,7)(2,12,19)(5,15,22)(8,18,25)(11,14,17)(21,24,27).

$G_{2,13} =< a, b, c, d, e, f, g, h, k, l : a^9 = 1, b^9 = 1, ab = ba, c^9 = 1, ac = ca,$
$bc = cb, d^3 = 1, ad = da, bd = dab^8c^4, cd = db^2, e^3 = 1, ae = ea^4, be = eb^4, ce = ec^4,$
$de = ea^6d, f^3 = 1, af = fa^4b^3c^3, bf = fb^7, cf = fc, df = fa^3b^6c^6de^2, ef = fe,$
$ag = gab^3, bg = gb^4, cg = gc, dg = gb^3c^6df^2, eg = ge, fg = gf, h^3 = 1, ah = hab^6c^3e,$
$bh = hb, ch = ha^6b^3c^7e^2fg^2, dh = ha^6c^3def^2, he = eh, hf = fh, hg = gh,$
$ak = ka^4c^3f, bk = kb^7g^2, ck = kc, dk = kc^6de^2fgh^2, ek = ke, fk = kf, gk = kg,$
$hk = kh, l^3 = 1, al = la^7b^6c^6f^2g^2, bl = lb, cl = lc, dl = lb^6defghk, el = le,$

$fl = lf, gl = lg, hl = lh, kl = lk >$, where the generators a, b, c, d, e, f, g, k are the same for $G_{1,12}$ and l=(21,27,24).

Hence we have:

Lemma 1.2.11. *There are, up to isomorphism, three transitive 3-groups of degree* 3^3 *and order 1594323, namely the non-abelian groups* $G_{1,13}$, $G_{2,13}$ *(of exponent 27) and* $G_{3,13}$ *(of exponent 9) described above.*

We summarize our findings:

	$\|G\| = 3^n$	Number of transitive abelian 3-group of degree 27 up to isomorphism	Number of transitive non-abelian 3-group of degree 27 up to isomorphism	Number of transitive 3-groups of degree 27 up to isomorphism
n=1	3	0	0	0
n=2	9	0	0	0
n=3	27	3	2	5
n=4	81	0	4	4
n=5	243	0	4	4
n=6	729	0	4	4
n=7	2187	0	4	4
n=8	6561	0	3	3
n=9	19683	0	3	3
n=10	59049	0	3	3
n=11	177147	0	3	3
n=12	531441	0	3	3
n=13	1594323	0	3	3
Total		3	36	39

We may state:

Proposition 1.2.12. *There are, up to isomorphism, 39 transitive 3 - groups of degree* 3^3*, three of these are abelian. Of the remaining 36 non - abelian, 17 are of exponent 27, 13 are of exponent 9 and 6 are of exponent 3.*

PROGRAMME 1:	PROGRAMME 2:
gap>s8:=Group((1,2),(1,2,3,4,5,6,7,8));; gap> a:=(1,2,3,4,5,6,7,8);; b:=(1,7)(3,5)(4,8);; gap > h:=Subgroup(s8,[a,b]);; gap > diff:= Difference(s8,h);; gap > req:= [];; gap > for c in diff do > if c^2=() then > if b^c=b then > if a^c=a^7 then > Add(req,c); > fi; > fi; > fi; > od; gap>req; [(1,3)(4,8)(5,7),(1,7)(2,6)(3,5)]	gap>s8:=SymmetricGroup(8);; gap> a:=(1,2,3,4,5,6,7,8);; b:=(1,7)(3,5)(4,8);;c:=(1,3)(4,8)(5,7);; gap>H:=Subgroup(s8,[a,b,c]);; gap > diff:=Difference(s8,H);; gap > req:=[];; gap > for r in diff do > if r^2=() then > if Order(s8,r)<>4 then > if Order(s8,r)<>8 then > if a^r in H then > if b^r in H then > if c^r in H then > if Size(Subgroup(s8,[a,b,c,r]))=64 then > Add(req,r); > fi; > fi; > fi; > fi; > fi; > fi; > fi; > fi; > od; gap>req; [(3,7)(4,8),(2,6)(3,7),(1,2)(3,4)(5,6)(7,8) (1,3)(2,4)(5,7)(6,8),(1,3)(2,8)(4,6)(5,7), (1,4)(2,7)(3,6)(5,8),(1,5)(4,8),(1,5)(2,6), ((1,6)(2,5)(3,8)(4,7),(1,7)(2,4)(3,5)(6,8), (1,7)(2,8)(3,5)(4,6),(1,8)(2,3)(4,5)(6,7)]

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