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$L^k_{\mathbb{A}CP}$ LOGIC AND COMPLETENESS THEOREM

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Abstract. To $L_{\mathbb{A}P}$ logic we add a new type of CP-quantifiers and prove the completeness theorem for new logic $L^k_{\mathbb{A}CP}$. The new axioms result from the condition probability introduced by Kolmogorov, which explains the "k" letter in the name of the new logic.

INTRODUCTION

In this paper we will introduce the logic $L^k_{\mathbb{A}CP}$. This logic is similar to infinitary logic $L_{\mathbb{A}P}$ (see [2], [3]); Our logic will include a new types of quantifiers $CP\vec{x} \geqslant r$ and $CP\vec{x} \leqslant 0$ ($\vec{x}$ ia a finite sequence of variables). A model of this logic is also a classical model with a probability measure in the universe, such that each relation is measurable.

1. BASIC DEFINITION

Syntax. We assume that $\mathbb{A}$ is an admissible set such that $\mathbb{A} \subseteq HC$ and $\omega \in \mathbb{A}$.

Let L be a countable, Σ-definable set of finitary relation and constant symbols (no function symbols).

We need the following logical symbols:

(1) The parentheses (,).

(2) The variables $v_0,\ v_1,\ \ldots,\ v_n,\ \ldots, n \in \mathbb{N}$.

(3) The connectives $\neg$ and $\bigwedge$.

(4) The quantifiers

(i) $P\vec{x} \geqslant r$, where $r \in \mathbb{A} \cap [0,1]$

(ii) $CP\vec{x} \geqslant r$, where $r \in \mathbb{A} \cap [0,1]$

(iii) $CP\vec{x} \leqslant 0$

(5) The equality symbol $=$ (optional).

Definition 1.1. *The formulas of $L^k_{\mathbb{A}CP}$ are defined as follows:*

(1) An atomic formula of first-order logic is a formula of $L^k_{\mathbb{A}CP}$.

(2) If φ is a formula of $L^k_{\mathbb{A}CP}$, then $\neg\varphi$ is a formula of $L^k_{\mathbb{A}CP}$.

(3) If $\Phi \in \mathbb{A}$ is a set of formulas of $L^k_{\mathbb{A}CP}$ with only finitly many free variables, then $\bigwedge \Phi$ is a formula of $L^k_{\mathbb{A}CP}$.

(4) If φ is a formula of $L^k_{\mathbb{A}CP}$, then $(P\vec{x} \geqslant r)\varphi$ is a formula of $L^k_{\mathbb{A}CP}$.

(5) If φ and ψ are the formulas of $L^k_{\mathbb{A}CP}$, then $(CP\vec{x} \geqslant r)(\varphi \mid \psi)$ and $(CP\vec{x} \leq 0)(\varphi \mid \psi)$ are also formulas of logic $L^k_{\mathbb{A}CP}$.

We shall assume that $L^k_{\mathbb{A}CP} \subseteq \mathbb{A}$ and denote $L^k_{\mathbb{A}CP}$, where $\mathbb{A} = HC$, by $L_{\omega_1 CP}$. Thus, $L^k_{\mathbb{A}CP} = \mathbb{A} \cap L_{\omega_1 CP}$. All other syntactical notions are defined similarly as in the $L_{\mathbb{A}P}$ case.

Definition 1.2. *We shall use the following abbreviations:*

(1) $(P\vec{x} < r)\varphi$ *for* $\neg(P\vec{x} \geqslant r)\varphi$

(2) $(P\vec{x} \leqslant r)\varphi$ *for* $(P\vec{x} \geqslant 1 - r)\neg\varphi$

(3) $(P\vec{x} > r)\varphi$ *for* $\neg(P\vec{x} \geqslant 1 - r)\neg\varphi$

(4) $(CP\vec{x} < r)(\varphi \mid \psi)$ *for* $\neg(CP\vec{x} \geqslant r)(\varphi \mid \psi)$

(5) $(CP\vec{x} \leqslant r)(\varphi \mid \psi)$ *for* $(CP\vec{x} \geqslant 1 - r)(\neg\varphi \mid \psi)$ *where* $r \neq 0$

(6) $(CP\vec{x} > r)(\varphi \mid \psi)$ *for* $\neg(CP\vec{x} \leqslant r)(\varphi \mid \psi)$

(7) $(CP\vec{x} = r)(\varphi \mid \psi)$ *for* $(CP\vec{x} \geqslant r)(\varphi \mid \psi) \wedge (CP\vec{x} \leqslant r)(\varphi \mid \psi)$

(8) The connectives $\bigvee$, $\rightarrow$ *and* $\leftrightarrow$ *are defined as usual.*

Models. Let $\langle A,\ \mathcal{F},\ \mu\rangle$ be a probability space such that each singleton is measurable. Then, for each $n \in \mathbb{N}$, one shows that $\langle A,\ \mathcal{F}^{(n)},\ \mu^{(n)}\rangle$ is a probability space, where $\mathcal{F}^{(n)}$ is the σ-algebra generated by the measurable rectangles and the diagonal sets, and $\mu^{(n)}$ is the restriction of the completion of μ^n to $\mathcal{F}^{(n)}$.

Definition 1.3. *A probability model for* L *is a structure*

$$\mathfrak{A} = \langle A,\ R_i^{\mathfrak{A}},\ c_j^{\mathfrak{A}},\ \mu\rangle_{i\in I,\ j\in J}$$

where $\langle A,\ R_i^{\mathfrak{A}},\ c_j^{\mathfrak{A}}\rangle_{i\in I, j\in J}$ *is a classical model,* μ *is a countably additive probability measure on* A *such that each singleton is measurable, each* n*-placed relation* $R_i^{\mathfrak{A}}$ *is* $\mu^{(n)}$*-measurable, and each* $c_j^{\mathfrak{A}} \in A$

Definition 1.4. *A graded probability model for* L *is a structure*

$$\mathfrak{A} = \langle A,\ R_i^{\mathfrak{A}},\ c_j^{\mathfrak{A}},\ \mu_n\rangle_{i\in I,\ j\in J,\ n\in N}$$

such that:

(1) $\langle A,\ R_i^{\mathfrak{A}},\ c_j^{\mathfrak{A}}\rangle_{i\in I,\ j\in J}$ *is a classical model;*

(2) Each μ_n *is a countably additive probability measure on* A^n*;*

(3) For all $m,\ n \in \mathbb{N}$, μ_{m+n} *is an extension of the product measure* $\mu_m \times \mu_n$*;*

(4) Each μ_n *is invariant under permutations, that is, whenever* π *is a permutation of* $\{1,\ 2,\ \ldots,\ n\}$ *and* $B \in dom(\mu_n)$*, if*

$$\pi B = \left\{\left(a_{\pi(1)},\ \ldots,\ a_{\pi(n)}\right) \mid (a_1,\ \ldots,\ a_n) \in B\right\},$$

then $\pi B \in dom(\mu_n)$ *and* $\mu_n(\pi B) = \mu_n(B)$*;*

(5) $\langle \mu_n | n \in \mathbb{N}\rangle$ *has the Fubini property: If* B *is* μ_{m+n}*-measurable, then*

(a) for each $\vec{x} \in A^m$*, the section* $B\vec{x} = \{\vec{y} \in A^n \mid (\vec{x}, \vec{y}) \in B\}$ *is* μ_n*-measurable;*

(b) the function $f(\vec{x}) = \mu_n(B\vec{x})$ *is* μ_m*-measurable;*

(c) $\int f(\vec{x})d\mu_m = \mu_{m+n}(B)$.

(6) Each atomic formula with n *free variables is measurable with respect to* μ_n.

Let $\mathfrak{A}$ be one of the models defined above and let ${}^n\mu$ denotes either $\mu^{(n)}$ or μ_n. The satisfaction relation is defined recursively in the same way as it was for $L_{\mathbb{A}P}$ except for the quantifier clause:
for $\varphi(\vec{y}) \in L^k_{\mathbb{A}CP}$ and $\vec{a} \in A^k$, $\psi(\vec{z}) \in L^k_{\mathbb{A}CP}$ and $\vec{c} \in A^l$

$$\mathfrak{A} \models (CP\vec{x} \geqslant r)\left(\varphi\left(\vec{x},\vec{y}\right)[\vec{a}] \mid \psi\left(\vec{x},\vec{z}\right)[\vec{c}]\right)$$

iff

$$\frac{{}^n\mu\left\{\vec{b} \in A^n \mid \mathfrak{A} \models \varphi\left[\vec{b},\vec{a}\right] \wedge \mathfrak{A} \models \psi\left[\vec{b},\vec{c}\right]\right\}}{{}^n\mu\left\{\vec{b} \in A^n \mid \mathfrak{A} \models \psi\left[\vec{b},\vec{c}\right]\right\}} \geq r$$

at condition that ${}^n\mu\left\{\vec{b} \in A^n \mid \mathfrak{A} \models \psi\left[\vec{b},\vec{c}\right]\right\} > 0$.

$$\mathfrak{A} \models (CP\vec{x} \leq 0)\left(\varphi\left(\vec{x},\vec{y}\right)[\vec{a}] \mid \psi\left(\vec{x},\vec{z}\right)[\vec{c}]\right)$$

iff

$${}^n\mu\left\{\vec{b} \in A^n \mid \mathfrak{A} \models \varphi\left[\vec{b},\vec{a}\right] \wedge \mathfrak{A} \models \psi\left[\vec{b},\vec{c}\right]\right\} = 0$$

and

$${}^n\mu\left\{\vec{b} \in A^n \mid \mathfrak{A} \models \psi\left[\vec{b},\vec{c}\right]\right\} > 0.$$

Remark. If we had ${}^n\mu\left\{\vec{b} \in A^n \mid \mathfrak{A} \models \psi\left[\vec{b},\vec{c}\right]\right\} = 0$, then we could claim that the formula is trivially satisfied in structure, analogous to the definition of condition probability being one.

Theorem 1.1. (Fubini theorem.) *Let μ be a probability measure such that each singleton is measurable, and let $B \subseteq A^{m+n}$ be μ^{m+n} measurable. Then:*

(1) Every section $B\vec{x} = \{y \in A^n \mid (\vec{x},\vec{y}) \in B\}$ is $\mu^{(n)}$-measurable.

(2) The function $f(\vec{x}) = \mu^{(n)}(B\vec{x})$ is $\mu^{(m)}$-measurable.

(3) $\mu^{(m+n)}(B) = \int f(\vec{x})d\mu^{(m)}$.

Proof theory. We now give a list of axioms and rules of inference for $L^k_{\mathbb{A}CP}$. In what follows, φ, ψ are arbitrary formulas of $L^k_{\mathbb{A}CP}$, $\Phi \in \mathbb{A}$ is an arbitrary set of formulas of $L^k_{\mathbb{A}CP}$ and $r,\ s \in \mathbb{A} \cap [0,1]$.

Definition 1.5. *The axioms of the weak $L^k_{\mathbb{A}CP}$ are the following:*

(A_1) All axioms of $L_{\mathbb{A}}$ without quantifiers;

(A_2) Monotonicity: $(P\vec{x} \geqslant r)\varphi \to (P\vec{x} \geqslant s)\varphi$, where $r \geqslant s$

(A_3) (a) $(P\vec{x} \geqslant r)\varphi(\vec{x}) \to (P\vec{y} \geqslant r)\varphi(\vec{y})$

(b) $(CP\vec{x} \geqslant r)(\varphi(\vec{x}) \mid \psi(\vec{x})) \to (CP\vec{y} \geqslant r)(\varphi(\vec{y}) \mid \psi(\vec{y}))$

(A_4) $(P\vec{x} \geqslant 0)\varphi$

(A_5) Finite additivity:

(a) $(P\vec{x} \leqslant r)\varphi \ \wedge \ (P\vec{x} \leqslant s)\psi \ \to \ (P\vec{x} \leqslant r+s)(\varphi \vee \psi)$

(b) $(P\vec{x} \geqslant r)\varphi \ \wedge \ (P\vec{x} \geqslant s)\psi \ \wedge \ (P\vec{x} \leqslant 0)(\varphi \wedge \psi) \ \to \ (P\vec{x} \geqslant r+s)(\varphi \vee \psi)$

(A_6) The Archimedean property:

$$(P\vec{x} > r)\varphi \ \leftrightarrow \ \bigvee_{n\in\mathbb{N}} \left(P\vec{x} \geqslant r + \frac{1}{n}\right)\varphi$$

(A_7) $(CP\vec{x} \geqslant r)(\varphi|\psi) \ \wedge \ (P\vec{x} \geqslant s)\psi \ \to \ (P\vec{x} \geqslant r\cdot s)(\varphi \wedge \psi)$, where $s > 0$.

(A_8) $(P\vec{x} = 0)(\varphi \wedge \psi) \ \wedge \ (P\vec{x} > 0)\psi \ \leftrightarrow \ (CP\vec{x} \leqslant 0)(\varphi|\psi)$

(A_9) $\displaystyle\bigwedge_{r\in[0,1]\cap Q} [(P\vec{x} \geqslant r)\psi \ \to \ (P\vec{x} \geqslant r\cdot s)(\varphi \wedge \psi)] \ \to \ (CP\vec{x} \geqslant s)(\varphi|\psi)$

Definition 1.6. *The axioms for graded $L^k_{\mathbb{A}CP}$ consist of the axioms for weak $L^k_{\mathbb{A}CP}$ plus following set of Hoover's axioms:*

(B_1) Countable additivity:

$$\bigwedge_{\Psi\subseteq\Phi} (P\vec{x} \geqslant r) \bigwedge \Psi \ \to \ (P\vec{x} \geqslant r) \bigwedge \Phi$$

where Ψ ranges over the finite subset of Φ.

(B_2) Symmetry:

$$(Px_1 \ \ldots \ x_n \geqslant r)\varphi \ \leftrightarrow \ (Px_{\pi(1)} \ \ldots \ x_{\pi(n)} \geqslant r)\varphi$$

where π is a permutation of $\{1,\ 2,\ \ldots,\ n\}$.

(B_3) Product independence:

$$(P\vec{x} \geqslant r)(P\vec{y} \geqslant s)\varphi \ \rightarrow \ (P\vec{x}\vec{y} \geqslant r \cdot s)\varphi$$

Definition 1.7. *The axioms for the full $L^k_{\mathbb{A}CP}$ consist of the axioms for graded $L^k_{\mathbb{A}CP}$ plus the following Keisler's axiom:*

(B_4) Product measurability:

$$(P\vec{x} \geqslant 1)(P\vec{y} > 0)(P\vec{z} \geqslant r)(\varphi(\vec{x}, \vec{z}) \ \leftrightarrow \ \varphi(\vec{y}, \vec{z}))$$

for each $r < 1$, where all variables in $\vec{x}$, $\vec{y}$, $\vec{z}$ are distinct.

Definition 1.8. *The rules of inference for all of the above logics are:*

(MP) Modus Ponens: $\dfrac{\varphi, \ \varphi \rightarrow \psi}{\psi}$

(C) Conjuction: $\dfrac{\varphi \rightarrow \psi, \ \psi \in \Phi}{\varphi \rightarrow \bigwedge \Phi}$

(G) Generalization: $\dfrac{\varphi \rightarrow \psi(\vec{x})}{\varphi \rightarrow (P\vec{x} \geqslant 1)\psi(\vec{x})}$

$\vec{x}$ is not free in φ.

Proposition 1.1. *The following are theorems of graded $L^k_{\mathbb{A}CP}$:*

(1) $\bigwedge_n \bigvee_m \left(P\vec{y} < \dfrac{1}{n}\right)\left(\left(P\vec{x} \geqslant r - \dfrac{1}{m}\right)\varphi(\vec{x}, \vec{y}) \wedge \neg(P\vec{x} \geqslant r)\varphi(\vec{x}, \vec{y})\right)$

(2) $\bigwedge_n \bigvee_{\Phi_0 \subseteq \Phi} \left(P\vec{x} < \dfrac{1}{n}\right)(\bigwedge \Phi_0(\vec{x}) \wedge \neg \bigwedge \Phi(\vec{x}))$, *where Φ is finite and $\bigwedge \Phi(\vec{x}) \in \mathbb{A}$.*

2. COMPLETENESS THEOREM

Consistency properties and weak models.

Definition 2.1. *A weak model for $L^k_{\mathbb{A}CP}$ is a structure*

$$\mathfrak{A} = \langle A, \ R_i^{\mathfrak{A}}, \ c_j^{\mathfrak{A}}, \ \mu_n \rangle_{i \in I, \ j \in J, \ n \in N}$$

such that $\langle A,\ R_i^{\mathfrak{A}},\ c_j^{\mathfrak{A}}\rangle_{i\in I, j\in J}$ *is a classical model, each* μ_n *is a finitely additive probability measure on* A^n *with each singleton measurable, and with the set*

$$\{\vec{c} \in A^n \mid \mathfrak{A} \models \varphi[\vec{a}, \vec{c}]\}$$

μ_n *measurable for each* $\varphi(\vec{x}, \vec{y}) \in L^k_{\mathbb{A}CP}$ *and each* $\vec{a} \in A$.

Let us introduce some convenient notation. $\varphi^\neg$ is defined as follows:

$\varphi^\neg = \neg\varphi,\ \varphi$ atomic $\qquad (\bigwedge_n \varphi_n)^\neg = \bigvee_n \neg\varphi_n$

$(\neg\varphi)^\neg = \varphi \qquad (\bigvee_n \varphi_n)^\neg = \bigwedge_n \neg\varphi_n$

$((P\vec{x} \geqslant r)\varphi)^\neg = (P\vec{x} > 1 - r)\neg\varphi$

$((CP\vec{x} \geqslant r)(\varphi|\psi))^\neg = (CP\vec{x} > 1 - r)(\neg\varphi|\psi)$

$((CP\vec{x} \leq 0)(\varphi|\psi))^\neg = (CP\vec{x} < 1)(\neg\varphi|\psi)$

We can suppose that $\mathbb{A}$ is a countable set. Let C be a countable set of new constant symbol, and let $K = L \cup C$. Then we form the logic $K^k_{\mathbb{A}CP}$ coresponding to K and we introduce a notion of a consistency property.

Definition 2.2. *A consistency property for* $L^k_{\mathbb{A}CP}$ *is a set* S *of countable sets* s *of sentences of* $K^k_{\mathbb{A}CP}$ *which satisfies the following conditions for each* $s \in S$*:*

(C_1) (Triviality rule) $\phi \in S$*;*

(C_2) (Consistency rule) either $\varphi \notin s$ *or* $\neg\varphi \notin s$*;*

(C_3) ($\neg$ − rule) If $\neg\varphi \in s$*, then* $s \cup \{\varphi^\neg\} \in S$*;*

(C_4) ($\bigwedge$ − rule) If $\bigwedge \Phi \in s$*, then for all* $\varphi \in \Phi,\ s \cup \{\varphi\} \in S$*;*

(C_5) ($\bigvee$ − rule) If $\bigvee \Phi \in s$*, then for some* $\varphi \in \Phi,\ s \cup \{\varphi\} \in S$*;*

(C_6) (P - rule) If $(P\vec{x} > 0)\varphi(\vec{x}) \in s$*, then for some* $\vec{c} \in C,\ s \cup \{\varphi(\vec{c})\} \in S$*;*

(C_7) If $\varphi(\vec{x}) \in K^k_{\mathbb{A}CP}$ *is an axiom, then*

(a) $s \cup \{(P\vec{x} \geqslant 1)\varphi(\vec{x})\} \in S$*,*

(b) $s \cup \{\varphi(\vec{c})\} \in S$*, where* $\vec{c} \in C$*.*

Theorem 2.1. (Model Existence Theorem) *If* S *is a consistency property, then any* $s_0 \in S$ *has a weak model.*

Proof. Let $\varphi_0,\ \varphi_1,\ \varphi_2,\ \ldots$ be a enumeration of the sentences of $K^k_{\mathbb{A}CP}$. We shall construct a sequence $s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots$ of elements of S as follows. s_0 is given. Given s_n choose s_{n+1} to satisfy the following conditions:

(1) $s_n \subseteq s_{n+1}$.

(2) If $s_n \cup \{\varphi_n\} \in S$, then $\varphi_n \in s_{n+1}$.

(3) If $s_n \cup \{\varphi_n\} \in S,\ \varphi_n = \bigvee \Phi$, then for some $\theta \in \Phi,\ \theta \in s_{n+1}$.

(4) If $s_n \cup \{\varphi_n\} \in S,\ \varphi_n = (P\vec{x} > 0)\psi(\vec{x})$, then for some $\vec{c} \in C,\ \psi(\vec{c}) \in s_{n+1}$.

We now define a model $\mathfrak{A}$ of s_0. Let $s_\omega = \bigcup\limits_{n<\omega} s_n$. Let T be a set of constants of $K^k_{\mathbb{A}CP}$. For $c,\ d \in T$, let $c \sim d$ iff $c = d \in s_\omega$. Then, $\sim$ is an equivalence relation. Let $[c]$ denote the equivalence class of the constant c. Let $\mathfrak{A}$ have the universe set $A = \{[c] \mid c \in T\}$. If R is an n-placed relation symbol and $c_1,\ \ldots,\ c_n \in C$, then

$$\mathfrak{A} \models R([c_1],\ \ldots,\ [c_n]) \ \text{ iff } \ R(c_1,\ \ldots,\ c_n) \in s_\omega$$

Define μ_n on the subset of A^n definable by formulas of $L^k_{\mathbb{A}CP}$ with parameters from A, by

$$\mu_n\,\{\vec{a} \in A^n \mid \mathfrak{A} \models \varphi[\vec{a}, \vec{c}]\} \ = \ \sup\{r \mid (P\vec{x} \geqslant r)\varphi(\vec{x}, \vec{c}) \in s_\omega\}$$

Only difference in relation to the logic $L_{\mathbb{A}P}$ is that formula φ can also contain quantifiers $CP\vec{x} \geqslant r$, resulting in no change.

It is not difficult to show that everything is well-defined, μ_n's are finitely additive probability measures, and it is routine to check that

$$\mathfrak{A} \models \varphi\,[[c_1],\ \ldots,\ [c_n]] \ \text{ iff } \ \varphi(c_1,\ \ldots,\ c_n) \in s_\omega$$

Therefore $\mathfrak{A}$ is a weak model of s_ω, and hence a model of s_0. □

Theorem 2.2. (Weak Completeness Theorem) *A set T of sentences of $L^k_{\mathbb{A}CP}$ has a weak model if and only if T is consistent in weak $L^k_{\mathbb{A}CP}$.*

Proof. Let S be the set of all countable sets s of sentences of $K^k_{\mathbb{A}CP}$ such that only finitely many $c \in C$ occur in s and not $\vdash_{K^k_{\mathbb{A}CP}} \neg \bigwedge s$. We claim that S is a consistency property. We check that S satisfies (C_6).

Let $(P\vec{x} > 0)\varphi(\vec{x}) \in s$ but for all $\vec{c} \in C,\ s \cup \{\varphi(\vec{c})\} \notin S$. Take a $\vec{c} \in C$ which does not occur in s. Then $\vdash_{K^k_{\mathbb{A}CP}} \neg \bigwedge(s \cup \{\varphi(\vec{c})\})$, hence $\vdash_{K^k_{\mathbb{A}CP}} \neg (\bigwedge s \wedge \varphi(\vec{c}))$ and $\vdash_{K^k_{\mathbb{A}CP}} \bigwedge s \rightarrow \neg\varphi(\vec{c})$. Let $\vec{y}$ be a type of variables not occuring in s. Then replacing $\vec{c}$ by $\vec{y}$ in the proof and using axioms and rule (G) we get $\vdash_{K^k_{\mathbb{A}CP}} \bigwedge s \rightarrow \neg(P\vec{y} > 0)\varphi(\vec{y})$ and hence $\vdash_{K^k_{\mathbb{A}CP}} \neg \bigwedge s$. A contradiction. □

Graded Models.

Theorem 2.3. (Graded Completeness Theorem) *Every countable set T of sentences which is consistent in graded $L^k_{\mathbb{A}CP}$ has a graded model.*

Proof. Let $V(S)$ be a superstructure over S and $\mathbb{R} \cup A \subseteq S$. We suppose that a formula $\varphi(\vec{x}, \vec{a})$ with parameters from A, a weak model $\mathfrak{A}$ of T, and the relation $\vDash$ are represented by sets in $V(S)$. Then ${}^*\varphi(\vec{x}, \vec{a})$ and ${}^*\mathfrak{A}$ are sets in the nonstandard universe $V({}^*S)$, and ${}^*F$ is an internal relation. If the context is clear we write simply F.

$\langle({}^*A)^n, L(\mu_n)\rangle$ is a probability space by Loeb's theorem. The model

$$\langle {}^*A,\ {}^*R_i^{\mathfrak{A}},\ c_j^{\mathfrak{A}},\ L(\mu_n)\rangle_{i\in I,\ j\in J,\ n\in N}$$

is graded because of the fact that the weak model $\mathfrak{A}$ is a model for graded $L^k_{\mathbb{A}CP}$.

The main step in our proof is to show that for each $\varphi(\vec{x}) \in L^k_{\mathbb{A}CP}$ and $\vec{a} \in A$

$$\langle \mathfrak{A}, \mu_n\rangle \vDash \varphi[\vec{a}] \ \text{ iff } \ \langle {}^*\mathfrak{A}, L(\mu_n)\rangle \vDash \varphi[\vec{a}]$$

To prove this, we prove by induction on formulas that for $\varphi(\vec{x}, \vec{y}) \in L^k_{\mathbb{A}CP},\ \vec{a} \in A$

$$L(\mu_n)\left(\{\vec{e} \in ({}^*A)^n \mid \langle {}^*\mathfrak{A}, L(\mu_n)\rangle \vDash \varphi[\vec{e}, \vec{a}]\} \vartriangle \{\vec{e} \in ({}^*A)^n \mid {}^*\langle \mathfrak{A}, \mu_n\rangle \vDash {}^*\varphi[\vec{e}, \vec{a}]\}\right) = 0$$

The nontrivial steps in our induction are conjunction and quantification.

Case 1. $\varphi(\vec{x}) = \bigwedge_n \psi_n(\vec{x})$

Then ${}^*(\bigwedge_n \psi_n) = \bigwedge_{n\in {}^*\mathbb{N}} {}^*\psi_n \neq \bigwedge_{n\in\mathbb{N}} {}^*\psi_n$. By proposition 1.1.(2) we have:

$$t = L(\mu_n)\left\{\vec{e} \in ({}^*A)^n \mid {}^*\langle \mathfrak{A}, L(\mu_m)\rangle \vDash {}^*(\bigwedge{}_m \psi_m[\vec{e}]) \vartriangle \bigwedge{}_m {}^*\psi_m[\vec{e}]\right\} = 0$$

The introduction step follows by the triangle argument and the induction hypothesis:

$$L(\mu_n)\,\{\vec{e}\in({}^*A)^n \mid {}^*\langle\mathfrak{A},L(\mu_m)\rangle\vDash\ {}^*(\varphi[\vec{e}])\ \triangle\ \varphi[\vec{e}]\}$$
$$\leqslant t+\textstyle\sum_m L(\mu_n)\{\vec{e}\in({}^*A)^n \mid {}^*\langle\mathfrak{A},L(\mu_m)\rangle\vDash\ {}^*(\psi_m[\vec{e}])\ \triangle\ \psi_m[\vec{e}]\}=0$$

Case 2. $\varphi(\vec{x})=(P\vec{y}\geqslant r)\psi(\vec{x},\vec{y})$

Then we have ${}^*((P\vec{x}\geqslant r)\psi(\vec{x}))\ \leftrightarrow\ {}^*\mu_n\{\vec{a}\mid{}^*\psi[\vec{a}]\}\geqslant r$ and

$$((P\vec{x}\geqslant r)^*\psi(\vec{x}))\ \leftrightarrow\ (\forall n\in\mathbb{N})\left({}^*\mu_n\{\vec{a}\mid{}^*\psi[\vec{a}]\}\geqslant r-\frac{1}{n}\right)$$

By the triangle argument:

$$L(\mu_n)\,\{\vec{e}\in({}^*A)^n \mid {}^*\langle\mathfrak{A},L(\mu_m)\rangle\vDash\ {}^*(\varphi[\vec{e}])\ \triangle\ \varphi[\vec{e}]\}$$
$$\leqslant\ L(\mu_n)\,\{\vec{e}\in({}^*A)^n \mid {}^*\langle\mathfrak{A},L(\mu_m)\rangle\vDash\ {}^*((P\vec{y}\geqslant r)\psi(\vec{e},\vec{y}))\ \triangle\ (P\vec{y}\geqslant r)\ {}^*\psi(\vec{e},\vec{y})\}$$
$$+L(\mu_n)(\{\vec{e}\in({}^*A)^n \mid {}^*\langle\mathfrak{A},L(\mu_m)\rangle\vDash\ (P\vec{y}\geqslant r)\ {}^*\psi(\vec{e},\vec{y})\}$$
$$\triangle\ \{\vec{e}\in({}^*A)^n \mid \langle{}^*\mathfrak{A},L(\mu_m)\rangle\vDash\ (P\vec{y}\geqslant r)\psi(\vec{e},\vec{y})\})$$

The first term is 0 by proposition 1.1.(1). By applying the induction hypothesis:

$$L(\mu_{n+m})(\{(\vec{e},\vec{c})\in({}^*A)^{n+m} \mid {}^*\langle\mathfrak{A},\mu_k\rangle\vDash\ {}^*\psi[\vec{e},\vec{c}]\}$$
$$\triangle\ \{(\vec{e},\vec{c})\in({}^*A)^{n+m} \mid \langle{}^*\mathfrak{A},L(\mu_k)\rangle\vDash\psi[\vec{e},\vec{c}]\})=0$$

So, for all $\vec{e}$'s but a set of $L(\mu_n)$-measure 0 we have:

$$L(\mu_m)(\{\vec{c}\in({}^*A)^m \mid {}^*\langle\mathfrak{A},\mu_k\rangle\vDash\ {}^*\psi[\vec{e},\vec{c}]\}\ \triangle\ \{\vec{c}\in({}^*A)^m \mid \langle{}^*\mathfrak{A},L(\mu_k)\rangle\vDash\psi[\vec{e},\vec{c}]\})=0$$

So, for all $\vec{e}$'s but a set of $L(\mu_n)$-measure 0 we have:

$$L(\mu_m)(\{\vec{c}\in({}^*A)^m \mid {}^*\langle\mathfrak{A},\mu_k\rangle\vDash\ {}^*\psi[\vec{e},\vec{c}]\}\geqslant r$$

iff

$$L(\mu_m)(\{\vec{c}\in({}^*A)^m \mid \langle{}^*\mathfrak{A},\mu_k\rangle\vDash\ \psi[\vec{e},\vec{c}]\}\geqslant r$$

Hence the second term in the inequality is 0.

Case 3. CP - quantification

We should, in fact, only prove that

$$\langle\mathfrak{A},\mu_n\rangle\vDash(CP\vec{x}\geqslant r)(\varphi(\vec{x},\vec{b})\mid\psi(\vec{x},\vec{c}))$$

iff

$$\langle {}^*\mathfrak{A}, L(\mu_n)\rangle \vDash (CP\vec{x} \geqslant r)(\varphi(\vec{x},\vec{b}) \mid \psi(\vec{x},\vec{c})), \quad \vec{b},\vec{c} \in A$$

Then we will have

$$\langle \mathfrak{A}, \mu_n\rangle \vDash (CP\vec{x} \geqslant r)(\varphi(\vec{x},\vec{b}) \mid \psi(\vec{x},\vec{c}))$$

iff

$$\frac{\mu_n\{\vec{a} \mid \langle \mathfrak{A}, \mu_m\rangle \vDash \varphi[\vec{a},\vec{b}] \wedge \langle \mathfrak{A}, \mu_m\rangle \vDash \psi[\vec{a},\vec{c}]\}}{\mu_n\{\vec{a} \mid \langle \mathfrak{A}, \mu_m\rangle \vDash \psi[\vec{a},\vec{c}]\}} \geqslant r$$

iff

$$\frac{L(\mu_n)\{\vec{a} \mid \langle {}^*\mathfrak{A}, \mu_m\rangle \vDash \varphi[\vec{a},\vec{b}] \wedge \langle {}^*\mathfrak{A}, L(\mu_m)\rangle \vDash \psi[\vec{a},\vec{c}]\}}{L(\mu_n)\{\vec{a} \mid \langle {}^*\mathfrak{A}, L(\mu_m)\rangle \vDash \psi[\vec{a},\vec{c}]\}} \geqslant r$$

iff

$$\langle {}^*\mathfrak{A}, L(\mu_n)\rangle \vDash (CP\vec{x} \geqslant r)(\varphi(\vec{x},\vec{b}) \mid \psi(\vec{x},\vec{c}))$$

because of:

$$\begin{aligned}\mu_n\{\vec{a} \mid \langle \mathfrak{A}, \mu_m\rangle \vDash \varphi[\vec{a},\vec{b}] \wedge \langle \mathfrak{A}, \mu_m\rangle \vDash \psi[\vec{a},\vec{c}]\}\\ = L(\mu_n)\{\vec{a} \mid \langle {}^*\mathfrak{A}, \mu_m\rangle \vDash \varphi[\vec{a},\vec{b}] \wedge \langle {}^*\mathfrak{A}, L(\mu_m)\rangle \vDash \psi[\vec{a},\vec{c}]\}\end{aligned}$$

since $\langle \mathfrak{A}, \mu_n\rangle \vDash (P\vec{x} \geqslant t)\varphi(\vec{x},\vec{b}) \Leftrightarrow \langle {}^*\mathfrak{A}, L(\mu_n)\rangle \vDash (P\vec{x} \geqslant t)\varphi(\vec{x},\vec{b})$
and $\langle \mathfrak{A}, \mu_n\rangle \vDash (P\vec{x} \geqslant t)\psi(\vec{x},\vec{b}) \Leftrightarrow \langle {}^*\mathfrak{A}, L(\mu_n)\rangle \vDash (P\vec{x} \geqslant t)\psi(\vec{x},\vec{b})$

We only have to prove that all this is also true for the formula with quantifier $(CP\vec{x} \leqslant 0)$

$$\langle \mathfrak{A}, \mu_n\rangle \vDash (CP\vec{x} \leqslant 0)(\varphi|\psi) \qquad \text{iff} \qquad \langle {}^*\mathfrak{A}, L(\mu_n)\rangle \vDash (CP\vec{x} \leqslant 0)(\varphi|\psi)$$

This is direct consequence of case 2 and axiom (A_8).

Probability Models.

Lemma 2.1. *Let λ, ν and μ be probability measures on A, B and $A \times B$ such that $\lambda \times \nu \subseteq \mu$. Let T be μ-measurable . Then, for each $\varepsilon > 0$, there is a finite union M of $\lambda \times \nu$-measurable rectangles such that $\mu(T \vartriangle M) < \varepsilon$ iff there is a $\lambda \times \nu$-measurable set N such that $\mu(T \vartriangle N) = 0$.*

Lemma 2.2. (Rectangle Approximation Lemma for $L_{\mathbb{A}P}$ logic) *Let $\mathfrak{A}$ be a graded probability structure satisfying axiom (B_4). Then for each $\varepsilon > 0$ and a* *formula* $\varphi(\vec{x})$ *of* $L_{\mathbb{A}P}$*, there are finitely many formulas* $\psi_{ij}(\vec{y}, x_j)$*, where* $i = 1, \ldots, m$ *and* $j = 1, \ldots, n$ *such that*

$$\mathfrak{A} \models (P\vec{y} > 0)(P\vec{x} > 1 - \varepsilon)(\varphi(\vec{x}) \leftrightarrow \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \psi_{ij}(\vec{y}, x_j))$$

The lemma says that any definable set $\varphi(\vec{x})$ in $\mathfrak{A}$ can be approximated within ε by a finite union of definable rectangles and this can be done uniformly in parameters $\vec{y}$ from a set of positive measure. In the proof, axiom (B_4) is used n times.

Theorem 2.4. (Soundness and Completeness Theorem for full $L^k_{\mathbb{A}CP}$) *A set of sentences* T *of the full* $L^k_{\mathbb{A}CP}$ *has a probability model if and only if* T *is consistent in the full* $L^k_{\mathbb{A}CP}$.

Sketch of the proof. Soundness of our logic is the consequence of the soundness of logic $L_{\mathbb{A}P}$ since the axiom (A_7), (A_8) and (A_9) relate to the properties of condition probability.

Let us prove the second part of the theorem. Since T is consistent to axiom (B_4), we immediately get Rectangle Approximation Lemma for $L^k_{\mathbb{A}CP}$ logic, i.e. we extend the original version of lemma also to the case when formula φ can contain CP-quantifiers.

We use "new" Rectangle Approximation Lemma in order to find an ordinary probability model $\mathfrak{B}$ such that $\mathfrak{B}$ is $L^k_{\mathbb{A}CP}$-equivalent to $\mathfrak{A}$ ($\mathfrak{A}$ is graded model for T which we have). Models $\mathfrak{A}$ and $\mathfrak{B}$ have the same universe, constants and measures. For each $R^{\mathfrak{A}}$ and $\varepsilon > 0$ there is a finite union M of μ^n-measurable rectangles such that $\mu_n(M \vartriangle R^{\mathfrak{A}}) < \varepsilon$. Then, by lemma 7.1. there is a μ^n-measurable (and also $\mu^{(n)}$-measurable) $R^{\mathfrak{B}}$ such that $\mu_n(R^{\mathfrak{A}} \vartriangle R^{\mathfrak{B}}) = 0$.

By induction on φ we can show that

$$\mathfrak{A} \models \varphi[\vec{a}] \quad \text{iff} \quad \mathfrak{B} \models \varphi[\vec{a}]$$

for μ_n-almost all φ. It follows that $\mathfrak{B} \models T$. □

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