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ORTHOGONAL CONSTANT MAPPINGS IN ISOSCELES ORTHOGONAL SPACES

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Abstract. In this paper we introduce the notion of orthogonally constant mapping in an isosceles orthogonal space and establish stability of orthogonally constant mappings. As an application, we discuss the orthogonal stability of the Pexiderized quadratic equation $f(x+y)+g(x+y) = h(x)+k(y)$.

1. INTRODUCTION

We say a functional equation $(\mathcal{E})$ is *stable* if any function g satisfying the equation $(\mathcal{E})$ "approximately" is near to an exact solution of $(\mathcal{E})$.

The stability problem of functional equations originated from a question of Ulam [22] concerning algebra homomorphisms. In 1941, Hyers [10] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [19] extended the theorem of Hyers. The result of Rassias has provided a lot of influence in the development of what we now call *Hyers–Ulam–Rassias stability* of functional equations. During the last decades several stability problems of functional equations have been investigated in the spirit of Hyers–Ulam–Rassias. The reader is referred to [3, 6, 11, 13, 16, 18, 20] and references therein for more comprehensive information on stability of functional equations.

There are several concepts of orthogonality such as Birkhoff–James, Phythagorean, isosceles, Singer, Roberts, Diminnie, Carlsson, T-orthogonality, Rätz, etc, in an arbitrary real normed space $\mathcal{X}$, which can be regarded as generalizations of orthogonality in the inner product spaces, in general. These are of intrinsic geometric interest and have been studied by many mathematicians; see [4]. Among them we deal with isosceles orthogonality $\perp$. This notion was introduced by C. James [12] as follows: $x \perp y$ if and only if $\|x+y\| = \|x-y\|$. He proved that a Banach space $\mathcal{X}$ of three or more dimensions is an inner-product space if isosceles orthogonality in $\mathcal{X}$ is homogeneous in one variable, or equivalently, it is additive for biorthogonal pairs of vectors (see also [1, 2, 7]) . In this paper, a real normed space $\mathcal{X}$ endowed with the isosceles orthogonality $\perp$ is called an isosceles orthogonal space.

The orthogonally quadratic equation $f(x+y)+f(x-y) = 2f(x)+2f(y), \ \ x \perp y$ was first studied by Vajzović [23] where $\perp$ means the Hilbert space orthogonality. Later, Drljević [8], Fochi [9] and Szabó [21] generalized this result. An investigation of the orthogonal stability of the Pexiderized quadratic equation $f(x+y)+f(x-y) = 2g(x)+2h(y), \ \ x \perp y$ may be found in [15, 17].

In this paper we introduce the notion of orthogonally constant mapping in an isosceles orthogonal space and establish stability of orthogonally constant mappings. As an application, we use this notion in a natural fashion to discuss the orthogonal stability of the Pexiderized quadratic equation $f(x+y)+g(x+y) = h(x)+k(y)$.

Throughout this paper, $(\mathcal{X}, \perp)$ denotes an isosceles ortohgonal space and $\mathcal{Y}$ is a normed space. We also denote the even and odd parts of a given function ρ by $\rho^e(x) := \frac{\rho(x)+\rho(-x)}{2}$ and $\rho^o(x) := \frac{\rho(x)-\rho(-x)}{2}$, respectively.

2. ORTHOGONAL MAPPINGS

We start our work with the following definition.

Definition 2.1. *(i) A mapping* $c : \mathcal{X} \to \mathcal{Y}$ *is called orthogonally constant if* $c(x+y) = c(x-y)$ *for all* $x, y \in \mathcal{X}$ *with* $x \perp y$.

(ii) A mapping $f : \mathcal{X} \to \mathcal{Y}$ is called approximately orthogonally constant if there is a positive number $\varepsilon > 0$ such that $\|f(x+y) - f(x-y)\| \le \varepsilon$ for all $x, y \in \mathcal{X}$ with $x \perp y$.

Lemma 2.2. *Let $c : \mathcal{X} \to \mathcal{Y}$ be an orthogonally constant mapping. If $x, y \in \mathcal{X}$ and $\|x\| = \|y\|$, then $c(x) = c(y)$.*

Proof. Let $\|x\| = \|y\|$. Setting $h = \frac{x+y}{2}$ and $k = \frac{x-y}{2}$. Then $h \perp k$ and so $c(x) = c(h+k) = c(h-k) = c(y)$. □

Proposition 2.3. *Suppose that $c : \mathcal{X} \to \mathcal{Y}$ is an orthogonally constant mapping. Then there is a mapping $g : \mathbb{R} \to \mathcal{Y}$ such that $c(x) = g(\|x\|)$ for each $x \in \mathcal{X}$.*

Proof. Let $x_0 \neq 0$ be a fixed element of $\mathcal{X}$. Define $g : \mathbb{R} \to \mathcal{Y}$ by $g(r) = c(\frac{rx_0}{\|x_0\|})$. Then, by Lemma 2.2, we have

$$g(\|x\|) = c(\frac{\|x\|x_0}{\|x_0\|}) = c(x),$$

since $\left\|\frac{\|x\|x_0}{\|x_0\|}\right\| = \|x\|$. □

Proposition 2.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be an approximately orthogonally constant mapping such that*

$$\|f(x+y) - f(x-y)\| \le \varepsilon$$

for some $\varepsilon > 0$ and for all $x, y \in \mathcal{X}$ with $x \perp y$. Then there is an orthogonally constant mapping $c : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - c(x)\| \le \varepsilon,$$

for all $x \in \mathcal{X}$.

Proof. Let $x_0 \neq 0$ be a fixed element of $\mathcal{X}$. Define $c : \mathcal{X} \to \mathcal{Y}$ by $c(x) = f(\frac{x_0\|x\|}{\|x_0\|})$. Then c is an orthogonally constant mapping since $x \perp y$ implies that $\|x + eta\| = \|x - y\|$ and so $f(\frac{x_0\|x+y\|}{\|x_0\|}) = f(\frac{x_0\|x-y\|}{\|x_0\|})$, whence $c(x+y) = c(x-y)$. In addition

$$\|f(x) - c(x)\| = \left\|f(x) - f\left(\frac{x_0\|x\|}{\|x_0\|}\right)\right\| \le \varepsilon,$$

since $\frac{1}{2}(x + \frac{x_0\|x\|}{\|x_0\|}) \perp \frac{1}{2}(x - \frac{x_0\|x\|}{\|x_0\|})$. □

3. APPLICATION TO PEXIDERIZED QUADRATIC EQUATION

The problem of orthogonal stability of the quadratic equation $f(x+y)+f(x+y) = 2f(x)+2f(y)$ was discussed in [17]. Towards a satisfactory study of the orthogonal stability of the Pexiderized quadratic equation $f(x+y)+g(x+y) = h(x)+k(y)$ we use the notion of approximately orthogonally constant in a natural fashion as follows.

An approximately orthogonally Cauchy mapping is a mapping $f : \mathcal{X} \to \mathcal{Y}$ for which there is a positive number $\varepsilon > 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon,$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. By an approximately orthogonally quadratic mapping we mean a mapping $f : \mathcal{X} \to \mathcal{Y}$ for which there is a positive number $\varepsilon > 0$ such that

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \varepsilon,$$

for all $x, y \in \mathcal{X}$ with $x \perp y$.

Theorem 3.1. *Suppose that $\mathcal{X}$ is a isosceles orthogonal space with an orthogonal relation $\perp$ and $\mathcal{Y}$ is a Banach space. Let the mappings $f, g, h, k : \mathcal{X} \to \mathcal{Y}$ satisfy $f(0) = g(0) = h(0) = k(0) = 0$ and the following inequality*

$$\|f(x+y) + g(x-y) - h(x) - k(y)\| \le \varepsilon, \tag{3.1}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. Then f is a linear combination of an approximately orthogonally Cauchy mapping, an approximately orthogonally quadratic mapping and an approximately orthogonally constant mapping. The same is true for g, as well.

Proof. Set $\ell(x) = \frac{h(x)+k(x)}{2}$. If $x \perp y$ then $-x \perp -y$, hence we can replace x by $-x$ and y by $-y$ in (3.1) to obtain

$$\|f(-x-y) + g(-x+y) - h(-x) - k(-y)\| \le \varepsilon. \tag{3.2}$$

By virtue of triangle inequality and (3.1) and (3.2) we have

$$\|f^o(x+y) + g^o(x-y) - h^o(x) - k^o(y)\| \le \varepsilon, \tag{3.3}$$

$$\|f^e(x+y)+g^e(x-y)-h^e(x)-k^e(y)\| \le \varepsilon, \tag{3.4}$$

for all $x, y \in \mathcal{X}$.

Let $x \perp y$. Then $y \perp x$, and by (3.3)

$$\|f^o(x+y)-g^o(x-y)-h^o(y)-k^o(x)\| \le \varepsilon. \tag{3.5}$$

It follows from (3.3) and (3.5) that

$$\begin{aligned} &\|2f^o(x+y)-h^o(x)-k^o(x)-h^o(y)-k^o(y)\| \\ &\qquad \le \|f^o(x+y)+g^o(x-y)-h^o(x)-k^o(y)\| \\ &\qquad\quad +\|f^o(x+y)-g^o(x-y)-h^o(y)-k^o(x)\| \\ &\qquad \le 2\varepsilon. \end{aligned} \tag{3.6}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. In particular, for arbitrary x and $y = 0$ we get

$$\|2f^o(x)-h^o(x)-k^o(x))\| \le 2\varepsilon. \tag{3.7}$$

By (3.6) and (3.7), we have

$$\begin{aligned} \|f^o(x+y)-f^o(x)-f^o(y)\| &\le \tfrac{1}{2}\|2f^o(x+y)-h^o(x)-k^o(x)-h^o(y)-k^o(y)\| \\ &\quad +\tfrac{1}{2}\|2f^o(x)-h^o(x)-k^o(x))\| \\ &\quad +\tfrac{1}{2}\|2f^o(y)-h^o(y)-k^o(y))\| \\ &\le 3\varepsilon \end{aligned} \tag{3.8}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. Hence f^o is approximately orthogonally Cauchy mapping. It follows from (3.8) and (3.7) that ℓ^0 is an approximately orthogonally Cauchy mapping. Since $x \perp y$ implies that $x \perp -y$, it follows from (3.3) that

$$\|g^o(x+y)+f^o(x+y)-h^o(x)-(-k)^o(y)\| \le \varepsilon,$$

hence by the same reasoning as above, we conclude that g^o is also approximately orthogonally Cauchy mapping.

Now, putting $x = 0$ in (3.4) we get

$$\|f^e(y)+g^e(-y)-k^e(y)\| \le \varepsilon,$$

and putting $y = 0$ in (3.4) we obtain

$$\|f^e(x) + g^e(x) - h^e(x)\| \leq \varepsilon.$$

Thus we have

$$\|f^e(x+y) + g^e(x-y) - (f^e(x) + g^e(x)) - (f^e(y) + g^e(y))\| \leq 3\varepsilon,$$

or equivalently,

$$\|f^e(x+y) + g^e(x-y) - f^e(x) - g^e(x) - f^e(y) - g^e(y)\| \leq 3\varepsilon.$$

Now let $u = f^e + g^e$ and $v = f^e - g^e$. Then

$$\begin{aligned} &\|u(x+y) + u(x-y) - 2u(x) - 2u(y)\| \\ &\qquad = \|f^e(x+y) + g^e(x+y) + f^e(x-y) + g^e(x-y) \\ &\qquad\quad -2f^e(x) - 2g^e(x) - 2f^e(y) - 2g^e(y)\| \\ &\qquad \leq 6\varepsilon, \end{aligned}$$

and

$$\begin{aligned} \|v(x+y) - v(x-y)\| &= \|f^e(x+y) - g^e(x+y) - f^e(x-y) + g^e(x-y)\| \\ &\leq 6\varepsilon, \end{aligned}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. Thus u is an approximately orthogonally quadratic mapping, v is an approximately orthogonally constant mapping, $f^e = \frac{u+v}{2}$ and $g^e = \frac{u-v}{2}$. Hence $f = f^o - \frac{1}{2}u - \frac{1}{2}v$ and $g = g^o - \frac{1}{2}u + \frac{1}{2}v$. □

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