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THE SPECIAL FUNCTION ش , II

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Abstract. We describe a method for estimating the special function ش , in the complex cut plane $A = \mathbf{C} \setminus (-\infty, 0]$, with a Stieltjes transform, which implies that the function ش is *logarithmically completely monotonic*. To be complete, we find a nearly exact integral representation. At the end, we also establish that 1/ش(x) is a complete Bernstein function and we give the representation formula which is analogous to the Lévy-Khinchin formula.

1. INTRODUCTION

In [8] the author introduces a new special function, named with the Arabian letter[1] ش, and proves that this is logarithmically convex and completely monotonic for all the closed real intervals I_ℓ with $\ell = 1, 2, 3, \ldots$.

The explicit formula of the special function ش , in the discrete field, is:

$$\text{ش}[k, \Omega\ (I_\ell)] \quad = \quad \left(1 + \frac{1}{3k - \Omega\ (I_\ell)}\right)^{2k+1} \tag{1}$$

[1]The letter ش (shin) is the thirteenth letter of the Arabian alphabet.

where is always valid *the following boundary*[2]:

$$\text{ش}[k, \Omega\ (I_\ell - 1)] \prec 2 \prec \text{ش}[k, \Omega\ (I_\ell)] \quad \text{with } \Omega\ (I_\ell - 1) = \Omega\ (I_\ell) - 1\ ;\ \forall\ I_\ell\ ,\ k, \ell \in \mathbf{N}$$

The *auxiliary* integer function $\Omega(I_\ell)$, that really represents a growing "step function", is defined, for the intervals of 8 or 9 following values of k, in the following way:

- $\Omega\ (I_1) = 0$ for k=1,. . . ,8
- $\Omega\ (I_2) = 1$ for k=9,. . . ,16 ; $\Omega\ (I_3)$=2 for k=17,. . . ,25 ; $\Omega\ (I_4)$=3 for k=26,. . . ,34
- $\Omega\ (I_5) = 4$ for k=35,. . . ,43 ; $\Omega\ (I_6)$ =5 for k=44,. . . ,51 ; $\Omega\ (I_7)$=6 for k=52,. . . ,60
- $\Omega\ (I_8) = 7$ for k=61,. . . ,69; $\Omega\ (I_9)$ =8 for k=70,. . . ,78 ; $\Omega\ (I_{10})$=9 for k=79,. . . ,86
- $\Omega\ (I_{11}) = 10$ for k=87,. . . ,95 ; $\Omega\ (I_{12})$=11 for k=96,. . . ,104. ; etc.

Successively we give (Fig. 1) the graphs, related to the families of ش functions, that are $\text{ش}[k, \Omega\ (I_\ell)]$ and $\text{ش}[k, \Omega\ (I_\ell - 1)]$, or better, to the set of the arcs belonging to them, and to the auxiliary function $\Omega\ (I_\ell)$.

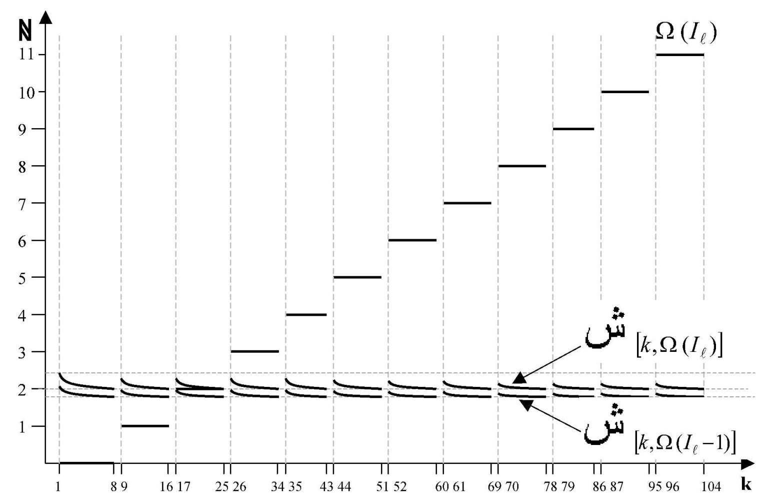


Fig. 1.

[2]The boundary can include the sign "=" if the integer variable k goes towards zero or the infinity.

Let's extend the dependence of the integer step function $\Omega\ (I_l)$ to the real field and let's use the following definition:

$$\Omega\,(x) = \min\left\{k \in \mathbf{N}\colon\ \mathrm{S}_{k+1}\,(x) \geq 2\ \right\};\ x \in \mathbf{R}^{+} \text{ and where } \mathrm{S}_k\,(x) = \left(1 + \frac{1}{3x - k + 1}\right)^{2x+1}$$

By simple algebraic passages we have, by a more appropriate notation, due to Iverson, that:

$$\Omega\ (x) = \left\lceil 3x - \frac{1}{2^{\frac{1}{2x+1}} - 1} \right\rceil \tag{2}$$

where $\lceil\, x \rceil$ means the smallest integer, greater than x or equal to it.

Therefore the special function ش possesses the following explicit formula in the real field:

$$\text{ش}\,(x) = \left(1 + \frac{1}{3x - \Omega\,(x)}\right)^{2x+1} = \left(1 + \frac{1}{3x - \left\lceil 3x - \frac{1}{2^{\frac{1}{2x+1}} - 1} \right\rceil}\right)^{2x+1}$$

Extending the field of definition of the variable k to the real positive numbers, it's possible to notice that such function, being represented by the union of continuous arcs (all above the straight line of height 2, see Fig.1) is actually assimilable to a piecewise continuous function.

That being stated, an important subclass of completely monotonic functions consists of the Stieltjes transforms defined as the class of functions $f\colon\ (0, \infty) \rightarrow \mathrm{R}$ of the form:

$$f\,(x)\ = a + \int_0^\infty \frac{d\mu\ (t)}{x + t} \tag{3}$$

where $a \geq 0$ and $\mu\,(t)$ is a nonnegative measure on $[0, \infty\ )$ with $\int_0^\infty \frac{d\mu\ (t)}{1+t} \leq \infty$, see [2].

In the Addenda and Problems in ([1], p.127), it is stated that if a function f is holomorphic in the cut plane $A = \mathbf{C}\backslash\ (-\infty, 0]$ and satisfies the following conditions :

(i) $\Im f\,(z) \leq 0$ for $\Im\,(z) \succ 0$

(ii) $f(x) \geq 0$ for $x \succ 0$

then f is a Stieltjes transform.

2. THE REPRESENTATION AS A STIELTJES TRANSFORM

In ([8], §6) the author characterizes the holomorphy (**piecewise analytic**) of the special function ش(z) in the cut plane $B = \mathbf{C}\backslash \left[-2/3, -1/3\right]$ and proves a remarkable result that implies :

$$\lim_{|z|\to\infty} ش(z) = 2 \qquad (z \in B) \tag{4}$$

To prove that the harmonic function $\Im($ش$)$ satisfies $\Im$ش$(z) \leq 0$ for $\Im(z) \succ 0$, we use that maximum principle for subharmonic functions, that can be found in ([4], p. 20), and show that lim sup of $\Im($ش$)$ at all boundary points including infinity is less than or equal to 0.

From (4) we conclude that this is true at infinity.

How, for definition ش$(x) \succ 0$ for $x \succ 0$; these last statements imply the result (3).

The constant a in (3) is given by :

$$a = \lim_{x\to\infty} ش(x)$$

and therefore for the *fundamental* theorem of the special function ش we have, see ([8], §2):

$$a = \lim_{x\to\infty} ش(x) = 2$$

In (3) $\mu(t)$ is the limit in the vague topology of measures

$$d\mu(t) = \lim_{y\to 0^+} \quad -\frac{1}{\pi}\Im f(-t+iy)\, dt$$

For $z \in B = \mathbf{C}\backslash \left[-2/3, -1/3\right]$ we have in the close interval $[-1, 0]$:

$$ش(z) \;=\left(1+\frac{1}{3z+1}\right)^{2z+1} \;=\; \exp\left((2z+1)\cdot Log\left(1+\frac{1}{3z+1}\right)\right) \tag{5}$$

where *Log* denotes the principal branch of the logarithm.

Let $t \in \mathbf{R}$ and $z \in \mathbf{C}$ with $\Im(z) \succ 0$.

If z tends to t , then for (1) and (5), results (with $\ell \in \mathbf{Z}$) [3]:

$$ش(z) \;=\; \begin{cases} \left(1+\frac{1}{3t-\Omega(I_\ell)}\right)^{2t+1} & \text{if } t \succ 0 \\ 2 & \text{if } t = 0 \\ \exp\left[(2t+1)\cdot\log\left(\left|\frac{3t+2}{3t+1}\right|\right) - k\cdot i\,\pi\,(2t+1)\right] & \text{with } k=1 \text{ if } -\frac{2}{3} \prec t \prec -\frac{1}{3} \\ \quad \text{and with } k=0 \text{ if } -1 \prec t \prec -\frac{2}{3} \text{ or if } -\frac{1}{3} \prec t \prec 0 & \\ 2 & \text{if } t = -1 \\ \left(1+\frac{1}{3t-\Omega(I_\ell)}\right)^{2t+1} & \text{if } t \prec -1 \end{cases}$$

In particular then we obtain , if y tends to 0^+, for[4] $t \in \mathbf{R}$:

$$-\frac{1}{\pi}\Im f(-t+iy) \;\rightarrow\; \begin{cases} 0 & \text{if } t \le 1/3 \text{ or } t \ge 2/3 \\ \frac{1}{2\pi}\frac{\left((3t-1)^2\right)^t}{\left((3t-2)^2\right)^t}\cdot\sin\,(2\pi t)\cdot\left\{\left|\frac{3t-2}{3t-1}\right| - \frac{3t-2}{3t-1}\right\} & \text{if } 1/3 \prec t \prec 2/3 \end{cases}$$

and using the identity (the Euler reflection formula):

$$\Gamma(\alpha)\cdot\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}$$

we are now in a position to determine the following nearly exact integral representation (Stieltjes transform):

$$ش(x) \;\approx 2+\frac{1}{2}\cdot\int_{1/3}^{2/3}\left\{\frac{1}{\Gamma(2\cdot t)\cdot\Gamma(1-2\cdot t)}\frac{\left((3t-1)^2\right)^t}{\left((3t-2)^2\right)^t}\left[\left|\frac{3t-2}{3t-1}\right| - \frac{3t-2}{3t-1}\right]\right\}\frac{dt}{(x+t)} \tag{6}$$

[3] $\mathbf{Z}$ denotes the relative integer set.

[4] In the discontinuity points 1/3 and 2/3 we respectively compute the limits of the real variable t on the left and on the right (see Fig. 2).

The approximation is essentially origined by neglecting the point of discontinuities of the first kind of the special function ش , in the real field, between an interval I_ℓ and the following $I_{\ell+1}$ as far as the interval $I\ [0, \infty\ )$.

Successively we give (Fig. 2: $x \to t$) the graphs (*red color*) related to the function[5] $\dot{\mu}(t)$ in the real interval $I\,[-1, 1]$: for $t = 1/2 \Rightarrow \dot{\mu}(t) = 0$ and this point is a *flex point* with oblique tangent.

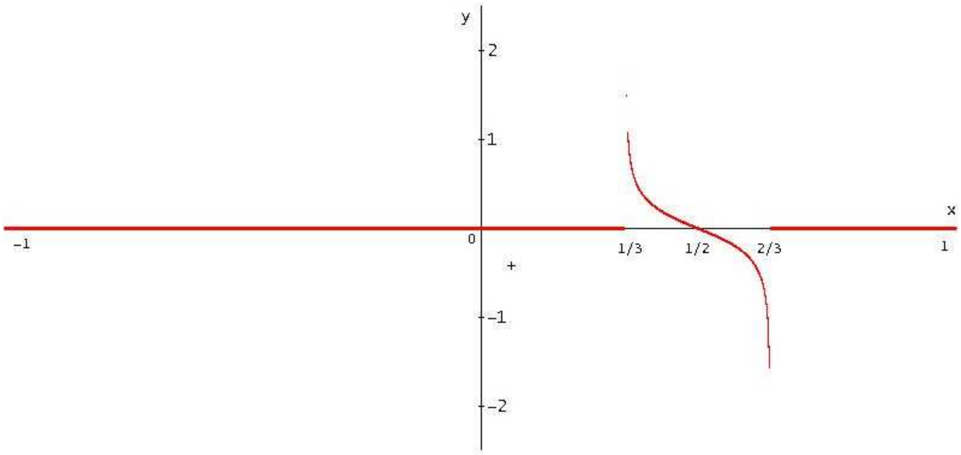


Fig. 2: The graph of $\dot{\mu}(t)$

That being stated, we denote the set of completely monotonic functions with $\mathcal{C}$.

Now, we also recall that a function f: $]0, \infty[\ \to\ ]0, \infty[$ is said to be logarithmically completely monotonic [5], if it is C^∞ and

$$(-1)^k \cdot [\log f(x)]^{(k)} \geq 0 \qquad \text{for } k = 1, 2, 3, ...$$

To simplify we denote the class of logarithmically completely monotonic functions by $\mathcal{L}$ and the set of Stieltjes transforms by $\mathcal{S}$.

In order to prove that the special function ش(x) is logarithmically completely monotonic, we need the following **lemma:**

$$\mathcal{S} \setminus \{0\} \subset \mathcal{L}$$

[5] $\dot{\mu}(t) = \frac{d\mu(t)}{dt}$

This lemma is a consequence of the following result, established by Horn [6], that allows also to characterize the class of logarithmically completely functions as the infinitely divisible completely monotonic functions:

Theorem 1. *For a function* $f\colon\,]0,\infty[\, \to\,]0,\infty[$ *the following are equivalent:*

(i) $f \in \mathcal{L}$; (ii) $f^{\alpha} \in \mathcal{C}$ for all $\alpha \succ 0$ and $\alpha \in \mathbf{R}$; (iii) $\sqrt[n]{f} \in \mathcal{C}$ for all $n = 1, 2, 3...$

In fact, let $f \in \mathcal{S}$ ($\mathcal{S} \subset \mathcal{C}$) and non-zero and let $\alpha \succ 0$, by Theorem 1 it is immediate to prove that $f^{\alpha} \in \mathcal{C}$.

Now, writing $\alpha = n + a$ with $n = 0, 1, 2, ...$ and $0 \leq a \prec 1$ we have $f^{\alpha} = f^{n} \cdot f^{a}$, and using the stability of $\mathcal{C}$ under multiplication and that $f^{a} \in S \Rightarrow \quad S \setminus \{0\} \subset \mathcal{L}$.

In conclusion for (6) also the special function ش$(x) \in \mathcal{L}$.

3. THE CLASS OF BERNSTEIN FUNCTIONS

There is an important relation between the set $\mathcal{S}$ of Stieltjes transforms and the class $\mathcal{B}$ of Bernstein functions.

We recall that a function $f : (0, \infty) \to [0, \infty)$ is called a Bernstein function, if f has derivatives of all orders and f' is completely monotonic.

Now, if f is non-zero Stieltjes transform, then $1/f$ is a Bernstein function ([3], Prop. 1.3).

The special function ش$(x) \in S \setminus \{0\}$ and this fact implies that $1/$ش(x) is a Bernstein function.

In addition, using the identity :

$$1\Big/\text{ش}(x) = x\Big/x \cdot \text{ش}(x)$$

and remembering the following definition:

A Bernstein function ϕ is called a special Bernstein function if the function $\lambda/\phi(\lambda)$ is also a Bernstein function.

we can conclude that $x \cdot$ ش(x) is a special Bernstein function.

The family of special Bernstein functions is very large, and it contains in particular the family of complete Bernstein functions (also known as *operator-monotone functions*, see [7], for instance).

Recall that a function $\phi : (0, \infty) \to \Re$ is called a complete Bernstein function if there exits a Bernstein function η such that :

$$\phi(\lambda) = \lambda^2 L\left[\eta(\lambda)\right] \quad , \quad \lambda \succ 0$$

where L stands for the Laplace transform.

Now, using the main results about the special function ش(x) ([8], §5 and §6) it is immediate to establish that $x \cdot$ ش(x) is a complete Bernstein function.

Note also that a function $f(x)$ is called a complete Bernstein function if, and only if,

$$f(x) = a + bx + \int_{0+}^{\infty} \frac{x}{t+x} \rho(dt) \tag{7}$$

where $a, b \geq 0$ and ρ is a Radon measure on $(0, \infty)$ such that $\int_{0+}^{\infty} (1/(1+t))\rho(dt) < \infty$.

From this one, we may deduce that the function $x \to f(x)/x$ is a Stieltjes transform [2].

This result was actually already obtained with the representation (6) of Stieltjes of the special function ش.

At the end, recall that the following conditions are equivalent:

(i) ϕ is a complete Bernstein function;

(ii) $\lambda/\phi(\lambda)$ is a complete Bernstein function.

This result implies also that the function $1/$ش(x) is a complete Bernstein function and, remembering the standard form (7) and that the functions $1/\left[x\cdot ش(x)\right]$ and $1/ش\left(\frac{1}{x}\right)$ are Stieltjes transforms, it is easy and immediate to estabilish that the constant a (*killing rate*) and b (*drift coefficient*) are given by:

$$a=\lim_{x\to\infty}1/ش\left(\frac{1}{x}\right)=1/ش(0)\ =\frac{1}{2}$$

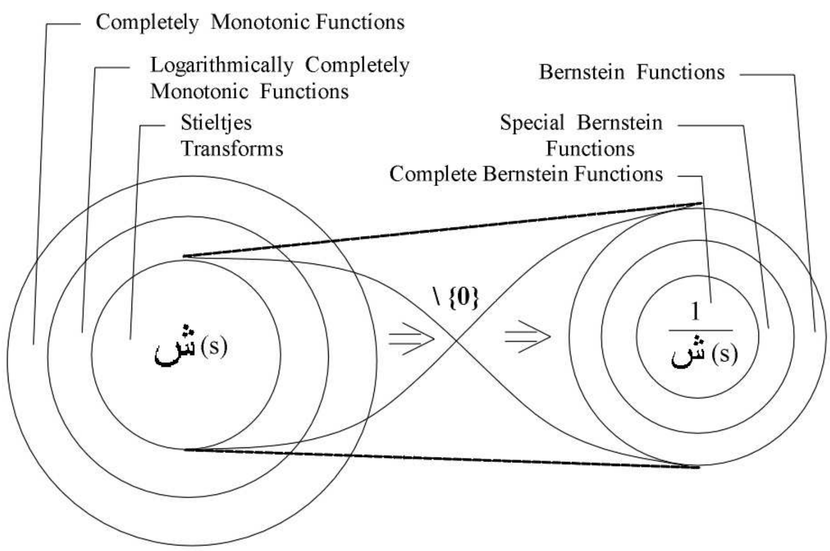
$$b=\lim_{x\to\infty}1/\left[x\cdot ش(x)\right]=0$$

and the following representation formula which is analogous to the Lévy-Khinchin formula:

$$1/ش(x)\ \approx\frac{1}{2}+\frac{1}{2}\cdot\int\limits_{0+}^{\infty}\left\{\frac{1}{\Gamma\left(2\cdot t\right)\cdot\Gamma\left(1-2\cdot t\right)}\frac{\left((3t-2)^2\right)^{\ t}}{\left((3t-1)^2\right)^{\ t}}\left[\left|\frac{3t-1}{3t-2}\right|-\frac{3t-1}{3t-2}\right]\right\}\frac{x\cdot t}{(t+x)}\,dt$$

For the interplay between complete Bernstein functions and Stieltjes transforms we refer also to [9].

Finally, with an Euler-Venn diagram, we give the most important analytic properties of the special function ش .



Completely Monotonic Functions ∼ vs ∼ Bernstein Functions

References

[1] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis* English translation, Oliver and Boyd, Edinburgh (1965).

[2] C. Berg, G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Ergebnisse der Math. **87**, Springer-Verlag, Berlin-Heidelberg-New York (1975).

[3] C. Berg, H. L. Pedersen, *A completely monotone function related to the gamma function*, J. Comp. Appl. Math., **133** (2001), 219–230.

[4] J. L. Doob, *Classical Potential Theory and its Probabilistic Counterpart*, Springer-Verlag, New York (1984).

[5] Fenq QI, Bai-Ni Guo, Chao-Ping Chen, *Complete monotonicities of function involving the Gamma and Digamma Functions*, RGMIA Res. Rep. Coll. **7**, No. 1 (2004), Art. 6.

[6] R. A. Horn, *On infinitely divisible matrices, kernels and functions*, Z. Wahrscheinlich- keitstheorie und Verw. Geb., **8** (1976), 219–230.

[7] N. Jacob, *Pseudo Differential Operators and Markov Processes,* Vol. **1**, Imperial College Press, London (2001).

[8] A. Ossicini, *The special function* ش, Kragujevac J. Math., **27** (2005), 63–90.

[9] R. L. Schilling, *Subordination in the sense of Bochner and a related functional calculus,* J. Aust. Math. Soc. Ser. A **64** (1998), 368–396.