Kragujevac J. Math. 29 (2006) 157-164.

GENERALIZATION OF SOME RESULTS ON THE DEGREE OF PROGRESS TO GOAL IN SELF-ORGANIZATION PROCESS

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(Received June 7, 2005)

Abstract. The degree of progress to goal at any stage during self-organization process was considered in Olatinwo [8]. The transition probabilities at various time intervals(including initial and final times) were evaluated and then subsequently interpreted as the degrees of progress to goal at such time intervals. The distance function employed in that paper was time-dependent of explicit form, contained in Adeagbo-Sheikh [1].

In this paper, we generalize the results of Olatinwo [8] by considering a distance function which is an implicit function of time. We apply both the notions of Jensen's inequality and triangle inequality as well as the elementary ideas of the convex functions and the curve theory.

The results obtained are in agreement with the axiomatic properties of probability.

1. INTRODUCTION

Adeagbo-Sheikh [1] in his model for self-organizing systems employed the concepts of a *distance function*, $g(t)$, and that of a *controlled-disturbance function*, $h(g(t))$, (where t is the time variable) in explaining the views of some notable thinkers as Ashby [2] and Beer [3].

In Olatinwo [8], the transition probabilities at various time intervals were evaluated and then subsequently interpreted as the degrees of progress to goal at such time intervals.

The objective of this paper is to determine the level or degree of progress to goal at any stage during self-organization process by employing a distance function of the form $g(\sum_{k=1}^{m} \alpha_k x_k(t))$, with $\sum_{k=1}^{m} \alpha_k = 1$, rather than the distance function, $g(t)$, used in Olatinwo [8]. The results obtained are in agreement with the axiomatic properties of probability. Our results are established by using elementary concepts of the probability and the curve theory as well as the idea of convex functions (see [5] for detail).

The study becomes pertinent for its possible applications in diverse areas, especially in learning, adaptive control and pattern recognition systems. Literature abounds with the theories of learning and invariably use statistical techniques. See Fu and Mendel [6].

However, we shall require the following Lemmas in the sequel.

Lemma 1. (Jensen's Inequality [5]) *Let $f(x)$ be convex on (a, b), and $x_1, x_2, ..., x_m$ be m points of (a, b). Also, let $c_1, c_2, ..., c_m$ be nonnegative constants such that $\sum_{i=1}^{m} c_i = 1$. Then, $f(\sum_{i=1}^{m} c_i x_i) \leq \sum_{i=1}^{m} c_i f(x_i)$.*

If f is strictly convex and if additionally each $c_i > 0$, then equality holds if and only if $x_1 = x_2 = ... = x_m$.

The proof of Lemma 1 is contained in Cloud and Drachman [5].

Lemma 2. *Let $\delta(x)$ be continuous on $[a, b]$. Then, $\int_a^x ||\delta(u)||du$ is the length of a certain curve from a to x.*

Proof. Since $\delta(x)$ is continuous on $[a, b]$, there exists a differentiable function $\rho(x)$ on (a, b) such that

$$\rho(x) = \int_a^x \delta(u)du, x \in [a, b]. \tag{1}$$

Applying the Fundamental Theorem of integral calculus in eqn(1) yields

$$\rho'(x) = \frac{d}{dx}\int_a^x \delta(u)du = \delta(x),$$

from which we obtain

$$||\delta(x)|| = ||\rho'(x)||. \tag{2}$$

Integrating both sides of eqn(2) yields

$$\int_a^x ||\delta(u)||du = \int_a^x ||\rho'(u)||du.$$

Since $\int_a^x ||\rho'(u)||du$ is the length of the curve $\rho(x)$ from a to x, then we have that $\int_a^x ||\delta(u||du$ is indeed the length of a certain curve from a to x. □

2. MAIN RESULTS

According to Adeagbo-Sheikh [1], the distance function, $g(t)$, is the distance from the goal at any time satisfying the following properties:

$$(i)\ g(t) > 0, t_0 \leq t < t_n < \infty, \quad (ii)\ g'(t) < 0, t_0 < t < t_n < \infty,$$
$$(iii)\ g(t_n) = 0, t_0 < t_n < \infty, \quad (iv)\ |g'(t)| < \infty, t_0 < t < t_n < \infty,$$

where t_n is the final time. Our self-organizing system is considered to be in the sense of Ramon-Margalef (see Beer [3]).The property(ii) of $g(t)$ shows that it is (strictly)monotone decreasing. The successive stages $P_0, P_1, ..., P_n$ reached during self-organization process and the respective times $t_0, t_1, ..., t_n$ were shown by the graph (figure 2.1) in Olatinwo [8]. The system began to self-organize towards some desired state of affairs at time t_0 and the self-organization process reached completion at time t_n (i.e. $g(t_n) = 0$), see property (iii).

Recall that the length $l(t)$ of a curve $f(t)$ (see Bruce and Giblin [4]) is given by

$$l(t) = \int_{t_0}^t ||f'(u)||du, \tag{3}$$

$l(t_0) = \int_{t_0}^{t_0} ||f'(u)du = 0$ and $l(t_n) = \int_{t_0}^{t_n} ||f'(u)||du > 0$.

We assume that the curve $f(t)$ is regular.

Remark 1. Throughout, we shall take our distance function as

$$g(\sum_{k=1}^{m} \alpha_k x_k(t)), \tag{*}$$

where $\sum_{k=1}^{m} \alpha_k = 1, \alpha_k \in [0,1], k = 1, 2, ..., m.$

Definition. *Let X_k be the event that a self-organizing system attains a stage P_k at time t_k during self-organization process. Then, the probability of this event is given by*

$$Prob\{X_k\} = \frac{l(t_k)}{l(t_n)}, k = 0, 1, 2, ..., n. \tag{4}$$

The following are the main results.

Theorem 1. *Suppose that $[t_0, t_k]$ and $[t_0, t_n]$ are two given time intervals such that $[t_0, t_k] \subseteq [t_0, t_n]$. Let X_k be the event that the self-organizing system whose distance function satisfies (*) attains a stage P_k at time t_k during self-organization process. Then,*

$$Prob\{X_k\} = \frac{\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (\sum_{j=1}^{m} \alpha_j || \frac{dg}{dx_j} |||| \frac{dx_j}{du} || - ||\delta(u)||) du}{\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (\sum_{j=1}^{m} \alpha_j || \frac{dg}{dx_j} |||| \frac{dx_j}{du} || - ||\delta(u)||) du}, \tag{5}$$

where $k \leq n$, $k, n \in \{1, 2, 3, ...\}$, $\delta(t)$ is continuous on $[t_0, t_n]$, $0 \leq \delta(t) \leq ||\delta(t)|| < \sum_{j=1}^{m} \alpha_j || \frac{dg}{dx_j} |||| \frac{dx_j}{dt} ||$.

Proof. X_k is the event that the self-organizing system attains a stage P_k at time t_k, $k = 0, 1, 2, ..., n$.

Let

$$f(t) = g(\alpha_1 x_1(t) + \alpha_2 x_2(t) + ... + \alpha_m x_m(t)), \sum_{j=1}^{m} \alpha_j = 1.$$

Applying Jensen's inequality to the function $f(t)$, we have

$$\begin{aligned} f(t) &= g(\alpha_1 x_1(t) + \alpha_2 x_2(t) + ... + \alpha_m x_m(t)) \\ &\leq \alpha_1 g(x_1(t)) + \alpha_2 g(x_2(t)) + ... + \alpha_m g(x_m(t)). \end{aligned} \tag{6}$$

Differentiating both sides of (6) with respect to t yields

$$f'(t) \leq \alpha_1 \frac{dg}{dx_1} \frac{dx_1}{dt} + \alpha_2 \frac{dg}{dx_2} \frac{dx_2}{dt} + ... + \alpha_m \frac{dg}{dx_m} \frac{dx_m}{dt} = \sum_{j=1}^{m} \alpha_j \frac{dg}{dx_j} \frac{dx_j}{dt}.$$

We have by the triangle inequality and one other norm property that

$$\begin{aligned} ||f'(t)|| &\leq ||\sum_{j=1}^{m}\alpha_j\frac{dg}{dx_j}\frac{dx_j}{dt}|| \leq \sum_{j=1}^{m}||\alpha_j\frac{dg}{dx_j}\frac{dx_j}{dt}|| = \sum_{j=1}^{m}|\alpha_j|||\frac{dg}{dx_j}||||\frac{dx_j}{dt}|| \\ &= \sum_{j=1}^{m}\alpha_j||\frac{dg}{dx_j}||||\frac{dx_j}{dt}||. \end{aligned} \tag{7}$$

Addition of $||\delta(t)||$ to the left-hand side of (7) yields

$$||f'(t)|| = \sum_{j=1}^{m}\alpha_j||\frac{dg}{dx_j}||||\frac{dx_j}{dt}|| - ||\delta(t)||. \tag{8}$$

Applying (8) in (4) yields

$$Prob\{X_k\} = \frac{\int_{t_0}^{t_k}||f'(u)||du}{\int_{t_0}^{t_n}||f'(u)||du} = \frac{\int_{t_0}^{t_k}(\sum_{j=1}^{m}\alpha_j||\frac{dg}{dx_j}||||\frac{dx_j}{du}|| - ||\delta(u)||)du}{\int_{t_0}^{t_n}(\sum_{j=1}^{m}\alpha_j||\frac{dg}{dx_j}||||\frac{dx_j}{du}|| - ||\delta(u)||)du}, \tag{9}$$

where we have by Lemma 2 that, $\int_{t_0}^{t_i}||\delta(u)||du, (i = k, n)$ are lengths of certain curves. Application of the fact that finite union of intervals can be split up into disjoint ones (see Kai Lai [7] and Olatinwo [8]) yields (5).

This completes the proof of the Theorem. □

Remark 2. Theorem 1 reduces to Theorem 2A of Olatinwo [8] if $m = 1, \delta(t) = 0, and x_k(t) = t$. According to Adeagbo-Sheikh[1], $S = \{S_1, S_2, ..., S_m\}$ is the set of the subsystems or elements of self-organizing system. The set $\{A_1, A_2, ..., A_m\}$ is the corresponding set of activities, and $y(t) = (y_1(t), y_2(t), ..., y_m(t))$ is the vector whose components measure the level or aggregate effects of respective activities from time $t_0 \geq 0$ to time t.

In this paper, we shall assume that the components of the vector $y(t)$ are implicit functions of time t, that is,

$$v(t) = y_k(\sum_{j=1}^{N}\beta_j x_j(t)), k = 1, 2, ..., m, \sum_{j=1}^{N}\beta_j = 1, \beta_j \in [0, 1]. \tag{**}$$

We are interested in finding the level of contribution or efficiency of each subsystem S_j from time t_0 to time t_n during self-organization process. We then employ it to find the probability for the overall level of the self-organization process. This idea is summarized in the next two results.

Theorem 2. *Let X_k be the event that the subsystems S_k have aggregate effects defined by (**) in the time interval$[t_0, t_n]$ during self-organization process satisfying (*). If $A_k, k = 1, 2, ..., m$ are the corresponding activities over the same time interval, then*

$$Prob\{X_k\} = \frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\sum_{j=1}^{N}\beta_j||\frac{dy_k}{dx_j}||||\frac{dx_j}{du}|| - ||\rho(u)||)du}{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\sum_{j=1}^{m}\alpha_j||\frac{dg}{dx_j}||||\frac{dx_j}{du}|| - ||\delta(u)||)du}, \tag{10}$$

such that $\rho(t)$, $\delta(t)$ *are both continuous on* $[t_0, t_n]$, $0 \leq \rho(t) \leq ||\rho(t)|| \leq \sum_{j=1}^{N}\beta_j||\frac{dy_k}{dx_j}||||\frac{dx_j}{dt}||$ *and* $0 \leq \delta(t) \leq ||\delta(t)|| < \sum_{j=1}^{m}\alpha_j||\frac{dg}{dx_j}||||\frac{dx_j}{dt}||$.

Proof. Let

$$v(t) = y_k(\sum_{j=1}^{N}\beta_j x_j(t)), k = 1, 2, ..., m.$$

Then, by Jensen's inequality, we obtain

$$v(t) = y_k(\sum_{j=1}^{N}\beta_j x_j(t)) \leq \sum_{j=1}^{N}\beta_j y_k(x_j(t)). \tag{11}$$

Differentiating both sides of (11) with respect to t yields

$$v'(t) \leq \sum_{j=1}^{N}\beta_j\frac{dy_k}{dx_j}\frac{dx_j}{dt}.$$

We get by a similar argument used to obtain (7) that

$$||v'(t)|| \leq \sum_{j=1}^{N}\beta_j||\frac{dy_k}{dx_j}||||\frac{dx_j}{dt}||. \tag{12}$$

Addition of $||\rho(t)||$ to the left-hand side of (12) leads to

$$||v'(t)|| = \sum_{j=1}^{N}\beta_j||\frac{dy_k}{dx_j}||||\frac{dx_j}{dt}|| - ||\rho(t)||. \tag{13}$$

Integrating (8) and (13) and then substituting in (4) yields

$$Prob\{X_k\} = \frac{\int_{t_0}^{t_n}(\sum_{j=1}^{N}\beta_j||\frac{dy_k}{dx_j}||||\frac{dx_j}{du}|| - ||\rho(u)||)du}{\int_{t_0}^{t_n}(\sum_{j=1}^{m}\alpha_j||\frac{dg}{dx_j}||||\frac{dx_j}{du}|| - ||\delta(u)||)du}, \tag{14}$$

where, by Lemma 2, $\int_{t_0}^{t_n}||\rho(u)||du and \int_{t_0}^{t_n}||\delta(u)||du$ are the lengths of certain curves. As in (9), application (in (14)) of the fact that finite union of intervals can be split up into disjoint ones results in (10).

This completes the proof of the Theorem. □

Remark 3. If $m = 1 = N$, $\rho(t) = \delta(t) = 0$ and $x_k(t) = t$, then, we have Theorem 2B of Olatinwo [8] as a corollary of Theorem 2.

Theorem 3. *Suppose that the subsystems $S_k, k = 1, 2, ..., m$ are independent with corresponding activities $A_k, k = 1, 2, ..., m$. Let X_k be the event that the subsystems S_k have the corresponding aggregate effects satisfying (**) in the time interval $[t_0, t_n]$, during self-organization process which satisfies (*). Then,*

$$Prob\left\{\bigcap_{k=1}^{m} X_k\right\} = \frac{\prod_{k=1}^{m}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\sum_{j=1}^{N}\beta_j||\frac{dy_k}{dx_j}||||\frac{dx_j}{du}|| - ||\rho(u)||)du}{(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\sum_{j=1}^{m}\alpha_j||\frac{dg}{dx_j}||||\frac{dx_j}{du}|| - ||\delta(u)||)du)^m}, \qquad (15)$$

such that $\rho(t) and \delta(t)$ are as in Theorem 2.

Proof. By Theorem 2, we obtain

$$(16) Prob\{X_r\} = \frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\sum_{j=1}^{N}\beta_j||\frac{dy_r}{dx_j}||||\frac{dx_j}{du}|| - ||\rho(u)||)du}{\sum_{j=1}^{n}\int_{t_{i-1}}^{t_i}(\sum_{j=1}^{m}\alpha_j||\frac{dg}{dx_j}||||\frac{dx_j}{du}|| - ||\delta(u)||)du}, r = 1, 2, ..., m.$$

Since the subsystems are independent, then $X_1, X_2, ..., X_m$ are independent events. Hence,

$$Prob\left\{\bigcap_{k=1}^{m}\right\} = Prob\{X_1\}\, Prob\{X_2\}\, ...Prob\{X_m\} = \prod_{k=1}^{m} Prob\{X_k\}. \qquad (17)$$

Substitution of (16) into (17), with $r = 1, 2, ..., m$ yields (15).

This concludes the proof of the Theorem. □

Remark 4. If $m = N = 1, \rho(t) = \delta(t) = 0$ and $x_k(t) = t$, then, we obtain Theorem 2C of Olatinwo [8] as a corollary of Theorem 3.

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