Kragujevac J. Math. 29 (2006) 165–173.

ON WEIGHTED NORM INTEGRAL INEQUALITY OF G. H. HARDY'S TYPE

K. Rauf and J. O. Omolehin

Department of Mathematics, University of Ilorin, Ilorin, Nigeria
(e-mails: balk_r@yahoo.com, omolehin_joseph@yahoo.com)

(Received May 25, 2005)

Abstract. In this paper, we give a necessary and sufficient condition on Hardy's integral inequality:

$$\int_X [Tf]^p w d\mu \leq C \int_X f^p v d\mu \quad \forall f \geq 0 \tag{1}$$

where w, v are non-negative measurable functions on X, a non-negative function f defined on $(0,\infty)$, $K(x,y)$ is a non-negative and measurable on $X \times X$, $(Tf)(x) = \int_0^\infty K(x,y)f(y)dy$ and C is a constant depending on K, p but independent of f. This work is a continuation of our recent result in [9].

1. INTRODUCTION

In the early twenties, G. H. Hardy proved the following result:

Theorem 1.1. *Let $1 < p < \infty$ and let f be a non-negative measurable function defined on $(0,\infty)$. Then,*

$$\int_0^\infty x^{-p} \left(\int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p dx \tag{2}$$

holds.

For a proof (see [3]).

It has wide applications in differential and integral equations, Laplace transforms and Fourier series (See [1, 2, 4, 5, 8, 10] and [11]) for related works. Recently, B. Mucheuhoupt [7] raised the question that given the weight w, under what condition will there exist a weight function v, finite $\mu-$almost everywhere on X such that the inequality (1) holds. In their attempt to simplify this problem, R. Kerman and E. Sawyer [6] provided partial solution to his question and two new open problems were generated. In [9], the proof of the following theorem, among others, on weighted norm inequalities for positive sublinear operator was established:

Theorem 1.2. *Let* $1 < p < \infty$ *and suppose* w *is a weight on* X*. Define the sublinear operator* T *by* $T(f+g)(x) = \int_X K(x,y)(f+g)(y)d\mu(y)$ *then, there exists a weight function* v*, finite* $\mu-$*almost everywhere on* X *such that* $\int_X [T(f+g)]^p w d\mu \leq C(K,p)\int_X (f^p + g^p)v d\mu$ *holds, for all* $f, g > 0$*, if and only if there is a positive function* Φ *and* θ *on* X *with* $\int_X (T\Phi)^p w d\mu < \infty$ *and* $\int_X (T\theta)^p w d\mu < \infty$ *or equivalently* $\Phi^{1-p}T^*((T\Phi)^{p-1}w) < \infty$ *and* $\Phi^{1-p}T^*((T\theta)^{p-1}w) < \infty$ *respectively,* $C(K,p) = \max[C_1(K,p), C_2(K,p)]$ *is a constant independent of* f *and* g*.*

The proof is exactly as in [9]. Thus, we omit it.

This theorem is known to have provided partial solution to the open problem number one in [6]. The aim of the present paper is to extent our recent result in [9] to the case when T is a special integral operator and to consider the case of interchanging non-negative weight functions w for which there are non trivial v *'s*. These provided partial solutions to the second open problem in [6].

Throughout this work, we let (X, ζ, μ) be a $\sigma-$finite measure space, $K(x,y)$ be a non-negative and measurable on $X \times X$. Also, set $(Tf)(x) = \int_0^\infty K(x,y)f(y)dy$ and $(T^*f)(x) = \int_0^\infty K(y,x)f(y)dy$ for non negative functions f.

Furthermore, let $p^{'}$ denotes the conjugate index of p, $p \neq 0$ and is defined by $\frac{1}{p} + \frac{1}{p^{'}} = 1$ with $p^{'} = \infty$ if $p = 1$.

2. STATEMENT OF RESULTS

In this section, we state and prove our main results by induction and show that for all $n \in \aleph$, the nth partial sum of the left hand side of the inequality (3) is less than or equal to the nth partial sum of the right hand side.

Theorem 2.1. *Let $1 < p < \infty$ and suppose v is a weight on X. Then there is a weight w, finite $\mu-$almost everywhere on X, such that the weighted norm inequality:*

$$\sum_{n\in\aleph}\int_{X_n}[Tf]^p w d\mu \le \sum_{n\in\aleph}\int_{X_n} f^p v d\mu \tag{3}$$

holds, for all $f > 0$, if and only if there is a positive function Φ on X with

$$\int_X [T^*\Phi]^p v^{1-p'} d\mu < \infty \tag{4}$$

or equivalently,

$$\Phi^{1-p}T^*\left[(T\Phi)^{p-1}w\right] < \infty \tag{5}$$

$\mu-$almost everywhere.

We noted that the weighted norm inequality (3) holds with v equals to the weight in (5).

Proof. Suppose that $\int_X [T^*\Phi]^p v^{1-p'} d\mu < \infty$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Since X is a $\sigma-$finite measure space. $X = \cup_{n=1}^{\infty} X_n$ with $X_n \cap X_m = \emptyset$, $n \neq m$ and $\mu(X_n) < \infty$.

We shall determine a necessary and sufficient condition on v such that the inequality (3) holds. But,

$$\int_X [Tf]^p w d\mu = \int_{\cup_{n=1}^{\infty} X_n}[Tf]^p w d\mu = \sum_{n\in\aleph}\int_{X_n}[Tf]^p w d\mu$$

and

$$\int_X f^p v d\mu = \sum_{n\in\aleph}\int_{X_n} f^p v d\mu$$

Thus we need to show that

$$\sum_{n\in\aleph}\int_{X_n}[Tf]^p w d\mu \le \sum_{n\in\aleph}\int_{X_n} f^p v d\mu$$

Let $n = 1$, we have

$$\begin{aligned}\int_{X_1}[Tf]^p w d\mu &= \int_{X_1}\left[\int K(x,y)f(y)d\mu(y)\right]^p w d\mu\\ &= \int_{X_1}\left[\int K(x,y)^{\frac{1}{p}}f(y)\Phi^{-\frac{1}{p'}}K(x,y)^{\frac{1}{p'}}\Phi^{\frac{1}{p'}}d\mu(y)\right]^p w d\mu\\ &\le \int_{X_1}\left[(\int K(x,y)f(y)^p\Phi^{-\frac{p}{p'}}d\mu(y))^{\frac{1}{p}}(\int K(x,y)\Phi d\mu(y))^{\frac{1}{p'}}\right]^p w d\mu\end{aligned}$$

By Holder's inequality

$$= \int_{X_1}\left[(Tf^p\Phi^{1-p})(T\Phi)^{p-1}w\right]d\mu$$

By a property of an inner product

$$= \int_{X_1} f^p v d\mu$$

Hence, valid for $n = 1$.

Next, assume $n = k > 1$, we have

$$\begin{aligned}\sum_{n=1}^{k}[Tf]^p w d\mu &= \int_{\cup_{n=1}^{k}X_n}[Tf]^p w d\mu\\ &= \int_{X_1}[Tf]^p w d\mu + \int_{X_2}[Tf]^p w d\mu + \ldots + \int_{X_k}[Tf]^p w d\mu\\ &\le \int_{X_1} f^p v d\mu + \int_{X_2} f^p v d\mu + \ldots + \int_{X_k} f^p v d\mu\end{aligned}$$

By Holder's inequality and using the above argument for $n = 1$

$$= \int_{\cup_{n=1}^{k}X_n} f^p v d\mu$$

Then, for $n = k + 1$, we have

$$\begin{aligned}\sum_{n=1}^{k+1}\int_{X_n}[Tf]^p w d\mu &= \textstyle\sum_{n=1}^{k}\int_{X_n}[Tf]^p w d\mu + \int_{X_{k+1}}[Tf]^p w d\mu\\ &\le \sum_{n=1}^{k}\int_{X_n} f^p v d\mu + \int_{X_{k+1}}[Tf]^p w d\mu\end{aligned}$$

by assumption when $n = k$

$$\le \sum_{n=1}^{k}\int_{X_n} f^p v d\mu + \int_{X_{k+1}} f^p v d\mu$$

since $\mu(X_{k+1}) < \infty$ and by the proof of $k = 1$. On the other hand, let us assume that (3) holds for some $v < \infty$ $\mu-$almost everywhere on X. By using the $\sigma-$finiteness of μ, we can find a positive function Φ such that $\int_X \Phi^p v d\mu < \infty$ and (4) holds.

Finally, suppose (4) holds and let v denotes the weight in (5) then

$$\begin{aligned}
\sum_{n=1}^{k+1}\int_{X_n}\Phi^p v d\mu &= \int_{X_1}\Phi^p v d\mu+\int_{X_2}\Phi^p v d\mu+\ldots+\int_{X_{k+1}}\Phi^p v d\mu\\
&= \int_{X_1}\Phi^p\left[(\Phi^{1-p}T^*(T\Phi)^{p-1}w)\right]d\mu+\int_{X_2}\Phi^p\left[(\Phi^{1-p}T^*(T\Phi)^{p-1}w)\right]d\mu\\
&\quad+\ldots+\int_{X_{k+1}}\Phi^p\left[(\Phi^{1-p}T^*(T\Phi)^{p-1}w)\right]d\mu\\
&= \int_{X_1}\left[\Phi T^*\left[(T\Phi)^{p-1}w\right]\right]d\mu+\int_{X_2}\left[\Phi T^*\left[(T\Phi)^{p-1}w\right]\right]d\mu\\
&\quad+\ldots+\int_{X_{k+1}}\left[\Phi T^*\left[(T\Phi)^{p-1}w\right]\right]d\mu\\
&= \int_{X_1}\left[(T\Phi)(T\Phi)^{p-1}w\right]d\mu+\int_{X_2}\left[(T\Phi)(T\Phi)^{p-1}w\right]d\mu\\
&\quad+\ldots+\int_{X_{k+1}}\left[(T\Phi)(T\Phi)^{p-1}w\right]d\mu\\
&= \int_{X_1}[(T\Phi)^p w]\,d\mu+\int_{X_2}[(T\Phi)^p w]\,d\mu+\ldots+\int_{X_{k+1}}[(T\Phi)^p w]\,d\mu
\end{aligned}$$

$$\Rightarrow \int_{X_1+X_2+\ldots+X_{k+1}}[(Tf)^p w]\,d\mu=\sum_{n=1}^{k+1}\int_{X_n}[(T\Phi)^p w]\,d\mu<\infty$$

by (5).

We conclude that the result in (3) is valid for all $n=1,2,\ldots$ and the proof is complete. □

Remark. If we set $n=1$ with $X_1=X$ in (3), then our result yields Theorem 1 obtained by Kerman and Sawyer [6].

Theorem 2.2. *For $X\in\Re$ and $1<p<\infty$. Let $f(t)\geq 0$ and $g(t)\geq 0$ and also let*

$$h(t)=\begin{cases} f(t) & \text{if } t>0\\ g(t) & \text{if } t<0\end{cases}$$

and zero otherwise, also suppose T is a completely arbitrary integral operator, then

$$\int_{\Re}[Th](t)^p d\mu\leq\int_{\Re}f^p v_1 d\mu+\int_{\Re}g^p v_2 d\mu \tag{6}$$

If and only if there exists a positive function Φ on t with

$$v_1=\int_X[T\Phi]^p d\mu<\infty \tag{7}$$

and

$$v_2 = \int_X [T^*\Phi]^p d\mu < \infty \tag{8}$$

or equivalently,

$$v_1 = \Phi^{1-p} T^*(T\Phi)^{p-1} < \infty \tag{9}$$

and

$$v_2 = \Phi^{1-p} T(T^*\Phi)^{p-1} < \infty \tag{10}$$

Proof. Suppose $Tf = \int_0^\infty f(t)dt$ and T^* is the dual of T.

Let

$$I = Th(t)$$

Then,

$$\begin{aligned} I &= \int_\Re h(t)dt \\ &= \int_\Re [f(t) + g(t)]\, dt \\ &= \textstyle\int_0^x f(t)dt + \int_x^\infty g(t)dt \end{aligned}$$

Therefore,

$$\begin{aligned} \int_\Re [Th](t)^p d\mu &= \int_\Re \left[\int_0^x f(t)dt + \int_x^\infty g(t)dt\right]^p d\mu \\ &\leq \int_\Re \left[\int_0^x f(t)^p dt + \int_x^\infty g(t)^p dt\right] d\mu \end{aligned}$$

By Minkowski's inequality

$$\begin{aligned} &= \int_\Re (Tf)^p d\mu + \int_\Re (T^*g)^p d\mu \\ &\leq \int_\Re \left[(Tf^p\Phi^{1-p})(T\Phi)^{p-1}\right] d\mu + \int_\Re \left[(T^*g^p\Phi^{1-p})(T^*\Phi)^{p-1}\right] d\mu \\ &= \int_\Re \left[f^p\Phi^{1-p}T^*(T\Phi)^{p-1}\right] d\mu + \int_\Re \left[g^p\Phi^{1-p}T^{**}(T^*\Phi)^{p-1}\right] d\mu \end{aligned}$$

By definition of inner product

$$= \int_\Re f^p v_1 d\mu + \int_\Re g^p v_2 d\mu$$

Conversely, we can assume that (6) holds for some $v < \infty$ $\mu-$almost everywhere. By using the $\sigma-$finiteness of μ, we can obtain a positive function Φ such that $\int_X \Phi_1^p(v_1)d\mu < \infty$ and $\int_X \Phi_2^p(v_2)d\mu < \infty$, then (7) and (8) holds.

Finally, suppose (7) and (8) holds and let v denotes the weight in (9) and (10) then we can use the method of theorem 2.1 to show that

$$\int_{\Re} \Phi_1^p(v_1)d\mu + \int_{\Re} \Phi_2^p(v_2)d\mu = \int_{\Re}[Th](t)^p d\mu$$

There is a similar result for the dual operator. This completes the proof of the theorem. □

3. CONSEQUENCE OF OUR MAIN RESULT

The next thorem treats the case of a convolution operator with radially decreasing kernel on $\Re_+$.

Theorem 3.1. *Let $1 < p < \infty$ and suppose that Φ, $w(x) \geq 0$ are locally integrable with respect to Lebesque measure on $\Re_+$ and that $\Phi(x) = \Phi(|x|)$ is non-increasing as a function of $|x|$. Define the convolution operator T by*

$$Th(x) = (\Phi^* h(x) = \int_0^\infty \Phi(x-s)\varphi(h(s))d\mu(s) \tag{11}$$

Where φ is a scalar function and $h(x)$ is as defined in Theorem 2.2. Then, there exists a weight function $v(x)$ finite $\mu-$almost everywhere on X and $C \geq 0$ such that:

$$\int_X (Th)^p w d\mu(s) \leq C(K,p)\int_X h^p v d\mu(s) \tag{12}$$

if and only if for all $s \in \Re_+$

$$\int_X \Phi(x-s)^p w(x)d(x) < \infty \tag{13}$$

Proof. Equation (11) can be transformed into linear form:

$$Th(x) = \int_0^x \Phi(x-s)U(s)d\mu(s) \tag{14}$$

and a nonlinear element

$$U(t) = \varphi(h(s)) \qquad t \in \Re_+ \tag{15}$$

The proof follows readily from the proof of theorem 3.1 in [9] and Theorem 3.2, if we set $K(x,y) \equiv \Phi(x-s)$ and also theorem 2.2 of the current paper. □

References

[1] K. F. Andersen, H. G. Heinig, *Weighted norm inequalities for certain integral operators*, SIAM J. Math. Anal., **14**, No. **4** (1983), 833–844.

[2] P. R. Beesack, *Integral inequalities involving a function and its derivatives*, Amer. Math. Monthly, **78** (1971), 705–741.

[3] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge (1952).

[4] H. G. Heinig, *Weighted norm inequality for certain integral operators II*, Proc. Amer. Math. Soc., **95**, No. **3** (1985), 387–395.

[5] C. O. Imoru, *On some extensions of Hardy's inequality*, Internat. J. Math and Math. Sc., **8 (1)** (1995), 165–171.

[6] R. Kerman, E. Sawyer, *On weighted norm inequalities for positive linear operators*, American Mathematical Society, **25** (1989), 589–593.

[7] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math, **44** (1972), 31–38.

[8] J. A. Oguntuase, R. Adegoke, *Weighted norm inequality for some integral operators*, Zbornik Radova Prirodno-matematickog fakulteta u Kragujevcu, **21** (1999), 55–62.

[9] K. Rauf, C. O. Imoru, *Some generalisation of weighted norm inequalities for a certain class of integral operators*, Kragujevac Journal of Mathematics, **24** (2002), 95–105.

[10] K. Rauf, *On an improvement of Bicheng, Zhuohua and Debnath's Results*, Journal of the Mathematical Association of Nigeria, ABACUS, **31**, No. **2B** (2004), 257–261.

[11] K. Rauf, *A Characterization of Weighted Norm for the Hardy-Littlewood Maximal Function*, The Nigerian Journal of Pure and Applied Sciences, **19** (2004), 1734–1740.