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NEW ITERATION METHODS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY SMOOTH REAL BANACH SPACES

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Abstract. In this paper, an improved modified three-step iteration method is introduced and analysed for fixed points, as well as common fixed points, of asymptotically nonepansive operators in uniformly smooth and convex Banach spaces. Our results generalize many relevant results in the literature, including those of Schu [11], Osilike and Aniagbosor [6], Xu and Noor [14], and Owojori and Imoru [8].

1. INTRODUCTION

An operator $T : K \to K$, where K is a nonempty subset of a Banach space, is called nonexpansive if for all $x, y \in K$ we have

$$\|Tx - Ty\| \leq \|x - y\|.$$

Goebel and Kirk [3] introduced the concept of asymptotically nonexpansive operators as a generalization of nonexpansive mappings. An operator T is called asymptotically nonexpansive if there exists a real sequence $\{k_n\}$ with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \qquad \forall n \in N$$

Goebel and Kirk [3] proved that every asymptotically nonexpansive selfmapping of a uniformly smooth and convex Banach space has a fixed point.

The modified Mann and Ishikawa iteration methods given by Schu [11] have been investigated by several authors for fixed points of asymptotically nonexpansive mappings. The modified Mann iteration method in the sense of Xu [17] is given for arbitrary $x_1 \in K$ by

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n, \quad n \geq 1 \tag{1.1}$$

and the modified Ishikawa iteration method in the sense of Xu [17] is also given for arbitrary $x_1 \in K$ by

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n, \\ y_n &= a_n' x_n + b_n' T^n x_n + c_n' v_n, \end{aligned}\right\} n \geq 1, \tag{1.2}$$

where K is a closed bounded convex subset of a uniformly smooth Banach space; $\{u_n\}$, $\{v_n\}$ are bounded sequences in K and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n'\}$, $\{b_n'\}$, $\{c_n'\}$ are sequences in $[0,1]$ satisfying

$$a_n + b_n + c_n = a_n' + b_n' + c_n' = 1$$

for all $n \geq 0$.

When $b_n' = c_n' = 0$ for all $n \geq 0$, then (1.2) reduces to (1.1) the modified Mann iteration scheme with errors in the sense of Xu [17].

Schu [11] established the convergence of the modified Mann iteration scheme in the sense of Liu [5] to fixed points of asymptotically nonexpansive mappings in uniformly convex Banach spaces.

Osilike and Aniagbosor [6] extended Schu's result to the modified Ishikawa iteration methods with errors given by Liu [5] and Xu [17].

Osilike and Igbokwe [7] established the convergence of the modified Ishikawa iteration scheme in the sense of Xu [17] to fixed points of asymptotically nonexpansive mappings in uniformly convex Banach spaces, when the boundedness condition is relaxed.

Also, Qihou [9] established the convergence of the modified iteration scheme in the sense of Xu [17] to the fixed point of asymptotically quasi- nonexpansive mappings in uniformly convex Banach spaces.

Xu and Noor [14] introduced and analysed the following modified three-step iteration scheme for asymptotically nonexpansive mappings in Banach spaces :

Defintion 1.1. *Let D be a nonempty subset of a normed space B and $T : D \to D$ be a mapping. For a given $x_o \in D$, compute sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ by the iteration scheme*

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T^n z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T^n x_n \end{aligned}\right\} n \geq 0 \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$.

We observe that when $\gamma_n = 0$ then (1.3) reduces to

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n \end{aligned}\right\} n \geq 0$$

which is the modified Ishikawa iteration scheme (without errors). In addition, when $\beta_n = \gamma_n = 0$, then (1.3) reduces to the modified Mann iteration scheme given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0$$

Xu and Noor [14] then established the convergence of (1.3) to the fixed point of asymptotically nonexpansive operator in uniformly smooth Banach space. Owojori and Imoru [8] also introduced a three-step modified iteration scheme with errors which include that of Xu and Noor [14] and the others as special cases. The following result was then established in [8].

Theorem 1.2. *Let B be a $p-$uniformly smooth Banach space and K a nonempty closed, bounded and convex subset of B. Suppose T is a uniformly continuous asymptotically nonexpansive selfmapping of K with real sequence $\{k_n\}$ satisfying $k_n \geq 1$ and*

$k_n^p + 1 \leq p$, $p > 1$. *For an arbitrary given* $x_1 \in K$, *define sequence* $\{x_n\}$ *generated iteratively by*

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n T^n x_n \\ y_n &= a_n^{'} x_n + b_n^{'} T^n z_n + c_n^{'} v_n \\ z_n &= a_n^{''} x_n + b_n^{''} T^n x_n + c_n^{''} \omega_n \end{aligned}\right\} n \geq 1 \tag{1.4}$$

where $\{v_n\}$, $\{\omega_n\}$ *are arbitrary sequences in* K *and* $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$ *are real sequences in* $[0,1]$ *satisfying the following conditions*
(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$
(ii) $\sum b_n = \infty$,
(iii) $\alpha_n := b_n + c_n, \beta_n := b_n{'} + c_n{'}, \gamma_n := b_n" + c_n"$,
$\alpha_n[1 + \beta_n k_n^p(1 + \gamma_n k_n^p)] \leq \frac{1}{p-1-k_n^p}$.
Then the sequence $\{x_n\}$ *converges strongly to a fixed point of* T.

Xu [16] studied the characteristic inequalities of uniformly smooth Banach spaces with modulus of smoothness of power type $q > 1$. He established the following result.

Lemma 1.3. *Let* B *be a uniformly smooth Banach space. Then* B *has modulus of smoothness of power type* $q > 1$ *if and only if there exist* $j_q x \in J_q x$ *and a constant* $c > 0$ *such that*

$$\|x + y\|^q \leq \|x\|^q + q < y, j_q(x) > + c\|y\|^q \tag{1.5}$$

for all $x, y \in B$.

By replacing y *with* $(-y)$ *in (1.5), we obtain*

$$\left.\begin{aligned} \|x - y\|^q &\leq \|x\|^q - q < y, j(x) > + \|y\|^q \\ &\leq \|x\|^q + \|y\|^q \end{aligned}\right\} \tag{1.6}$$

for all $x, y \in B$.

Applying Lemma 1.3, Chidume and Osilike [2] estabilshed the following result in real uniformly smooth Banach spaces with modulus of smoothness of power type $q > 1$.

Lemma 1.4. (Chidume and Osilike [2]) *Let* B *be a uniformly smooth Banach space with modulus of smoothness of power type* $q > 1$. *Then for all* $x, y, z \in B$ *and*

$\lambda \in [0,1]$, *the following inequality*

$$\left.\begin{aligned}\|\lambda x+(1-\lambda)y-z\|^q \;\leq\; & [1-\lambda(q-1)]\|y-z\|^q+\lambda c\|x-z\|^q\\ & -\lambda[1-\lambda^{q-1}c]\|x-y\|^q\end{aligned}\right\} \tag{1.7}$$

holds, where c is a positive constant.

The following classical result is also required in the proofs of our results in this study.

Lemma 1.5. (Weng [13]) *Let $\{\Phi_n\}$ be a nonnegative sequence of real numbers satisfying*

$$\Phi_{n+1} \leq (1-\delta_n)\Phi_n + \sigma_n \tag{1.8}$$

where $\delta_n \in [0,1]$, $\sum \delta_n = \infty$ and $\sigma_n = o(\delta_n)$. Then $\lim\limits_{n\to\infty} \Phi_n = 0$.

It is our purpose in this paper to introduce and analyse a more acceptable modified iteration method with errors and apply it to approximate fixed point as well as common fixed point of asymptotically nonexpansive operators in uniformly smooth and convex Banach spaces.

2. PRELIMINARIES AND MAIN RESULTS

Definition 2.1. *Let K be a nonempty closed bounded convex subset of a uniformly smooth Banach space and suppose T, S are uniformly continuous asymptotically nonexpansive selfmappings of K. Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by*

$$\left.\begin{aligned}x_{n+1} &= a_n x_n + b_n T^n y_n + c_n S^n x_n\\ y_n &= a_n' x_n + b_n' S^n z_n + c_n' v_n\\ z_n &= a_n'' x_n + b_n'' T^n x_n + c_n'' \omega_n\end{aligned}\right\} n \geq 1 \tag{2.1}$$

where $\{u_n\}$, $\{v_n\}$, are arbitrary sequences in K and $\{a_n\}$, $\{a_n'\}$, $\{a_n''\}$, $\{b_n\}$, $\{b_n'\}$, $\{b_n''\}$, $\{c_n\}$, $\{c_n'\}$, $\{c_n''\}$ are real sequences in $[0,1]$ satisfying the following conditions
(i) $a_n + b_n + c_n = a_n' + b_n' + c_n' = a_n'' + b_n'' + c_n'' = 1$,

(ii) $\sum b_n = \infty$,

(iii) $\alpha_n := b_n + c_n, \beta_n := b_n' + c_n', \gamma_n := b_n" + c_n"$

We call the iteration scheme (2.1) above, 'the modified three - step iteration method with errors'.

Remark. When $S = T$ in (2.1) above, then it will reduce to (1.4). Also, since S is a continuous selfmapping of the convex set K, there exists an element $u_n \in K$ such that $S^n x_n = u_n$. Therefore, (2.1) yields

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n \\ y_n &= a_n^{'} x_n + b_n^{'} S^n z_n + c_n^{'} v_n \\ z_n &= a_n^{''} x_n + b_n^{''} T^n x_n + c_n^{''} \omega_n \end{aligned}\right\} n \geq 1 \tag{2.2}$$

Now, when $S = T$ in (2.2), we have a special case of (2.2) given by

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n \\ y_n &= a_n^{'} x_n + b_n^{'} T^n z_n + c_n^{'} v_n \\ z_n &= a_n^{''} x_n + b_n^{''} T^n x_n + c_n^{''} \omega_n \end{aligned}\right\} n \geq 1 \tag{2.3}$$

where $\{u_n\}$, $\{v_n\}$ and $\{\omega_n\}$ are bounded sequences in K and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0,1]$ satisfying

(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$,

(ii) $\sum b_n = \infty$

Observe that, if in (2.3) we set

$$c_n = c_n^{'} = c_n^{''} = 0$$

then the scheme will reduce to

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n \\ y_n &= a_n^{'} x_n + b_n^{'} T^n z_n \\ z_n &= a_n^{''} x_n + b_n^{''} T^n x_n \end{aligned}\right\} n \geq 1$$

where $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$ $\{b_n^{'}\}$, $\{b_n^{''}\}$, are real sequences in $[0,1]$ satisfying

$$a_n + b_n = a_n^{'} + b_n^{'} = a_n^{''} + b_n^{''} = 1 \quad \text{and} \quad \sum b_n = \infty$$

This is equivalent to (1.3) - the modified three-step iteration scheme of B. Xu and Noor [14], with $\alpha_n = b_n$, $\beta_n = b_n^{'}$, $\gamma_n = b_n^{''}$. Therefore, the modified three-step iteration scheme with errors given by (2.3) above is clearly an extension of the scheme (1.3) introduced by Xu and Noor [14]. Clearly, the modified iteration scheme given by (2.3) is also an extension of the existing modified Mann and Ishikawa iteration methods with (and without) errors in the sense of Liu [5] and Xu [17].

Our result is the following.

Theorem 2.2. *Let B be a uniformly smooth Banach space and K a nonempty closed bounded and convex subset of B. Suppose T is a uniformly continuous asymptotically nonexpansive selfmapping of K with real sequence $\{k_n\}$ satisfying, $k_n \geq 1, \forall n$, $\lim\limits_{n\to\infty} k_n = 1$ and $k_n^p+1 \leq p$, $p > 1$. For a given $x_1 \in K$, define sequence $\{x_n\}$ generated iteratively by*

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n \\ y_n &= a_n^{'} x_n + b_n^{'} T^n z_n + c_n^{'} v_n \\ z_n &= a_n^{''} x_n + b_n^{''} T^n x_n + c_n^{''} \omega_n \end{aligned}\right\} n \geq 1$$

where $\{u_n\}$, $\{v_n\}$, $\{\omega_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0,1]$ satisfying the following conditions

(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$,

(ii) $\sum b_n = \infty$,

(iii) $\alpha_n := b_n + c_n, \beta_n := b_n{'} + c_n{'}, \gamma_n := b_n" + c_n"$,

(iv) $(p - 1 - k_n^p) \leq \frac{1}{1+\beta_n k_n^p (1+\gamma_n k_n^p)}$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Goebel and Kirk [3], T has a fixed point in K.

Let $x^* \in K$ be a fixed point of T. Then, from our hypothesis and Lemma 1.4, we have the following estimates.

$$\begin{aligned}
\|z_n - x^*\|^p &= \|a_n^{''}x_n + b_n^{''}T^n x_n + c_n^{''}\omega_n - x^*\|^p \\
&= \|[(1-\gamma_n)(x_n - x^*) + \gamma_n(T^n x_n - x^*)] - c_n^{''}(T^n x_n - \omega_n)\|^p \\
&\leq [1-\gamma_n(p-1)]\|(x_n - x^*) - c_n^{''}(T^n x_n - \omega_n)\|^p \\
&\quad + \gamma_n c\|(T^n x_n - x^*) - c_n^{''}(T^n x_n - \omega_n)\|^p \\
&\quad - \gamma_n(1-\gamma_n^{p-1}c)\|(T^n x_n - x^*) - (x_n - x^*)\|^p
\end{aligned}$$

Observe that $\gamma_n(1-\gamma_n^{p-1}c) \geq 0$. Therefore, expanding further, we obtain

$$\begin{aligned}
\|z_n - x^*\|^p &\leq [1-\gamma_n(p-1)][\|x_n - x^*\|^p + cc_n^{''}\|T^n x_n - \omega_n\|^p \\
&\quad - p < c_n^{''}(T^n x_n - \omega_n), j(x_n - x^*) >] \\
&\quad + \gamma_n c[\|T^n x_n - x^*\|^p + cc_n^{''}\|T^n x_n - \omega_n\|^p \\
&\quad - p < c_n^{''}(T^n x_n - \omega_n), j(T^n x_n - x^*) >] \\
&\leq [1-\gamma_n(p-1)][\|x_n - x^*\|^p + cc_n^{''}\|T^n x_n - \omega_n\|^p] \\
&\quad + \gamma_n c[\|T^n x_n - x^*\|^p + cc_n^{''}\|T^n x_n - \omega_n)\|^p]
\end{aligned} \tag{2.4}$$

Continuity of T and boundedness of K implies that there exists a real number $N_4 < \infty$ such that $\|T^n x_n - \omega_n\|^p \leq N_4$. Observe that $c, c_n^{''} < 1$ and T is asymptotically nonexpansive. Therefore,

$$\begin{aligned}
\|z_n - x^*\|^p &\leq [1-\gamma_n(p-1)]\|x_n - x^*\|^p + \gamma_n\|T^n x_n - x^*\|^p \\
&\quad + [1-\gamma_n(p-1)+\gamma_n]N_4 \\
&\leq [1-\gamma_n(p-{k_n}^p-1)]\|x_n - x^*\|^p + \{[1-\gamma_n(p-1)]+\gamma_n\}N_4 \\
&= [1-\gamma_n(p-{k_n}^p-1)]\|x_n - x^*\|^p + [1-\gamma_n(p-2)]N_4
\end{aligned} \tag{2.5}$$

We also have the following estimates :

$$\begin{aligned}
\|y_n - x^*\|^p &= \|a_n^{'}x_n + b_n^{'}Tz_n + c_n^{'}v_n - x^*\|^p \\
&= \|(1-\beta_n)(x_n - x^*) + \beta_n(T^n z_n - x^*) - c_n^{'}(T^n z_n - v_n)\|^p \\
&\leq [1-\beta_n(p-1)]\|(x_n - x^*) - c_n^{'}(T^n z_n - v_n)\|^p \\
&\quad + \beta_n c\|(T^n z_n - x^*) - c_n^{'}(T^n z_n - v_n)\|^p \\
&\quad - \beta_n(1-\beta_n^{p-1}c)\|(T^n z_n - x^*) - (x_n - x^*)\|^p
\end{aligned} \tag{2.6}$$

Expanding further and considering the fact that $c, {c_n}' \leq 1$ and $\beta_n(1-\beta_n^{p-1}c) \geq 0$, we have

$$\begin{aligned}\|y_n - x^*\|^p &\leq [1-\beta_n(p-1)]\{\|x_n - x^*\|^p + \|T^n z_n - v_n\|^p \\ &\quad -p < c_n(T^n z_n - v_n), j(x_n - x^*) >\} \\ &\quad +\beta_n\{\|T^n z_n - x^*\|^p + \|T^n z_n - v_n\|^p \\ &\quad -p < c_n(T^n z_n - v_n), j(T^n z_n - x^*) >\} \\ &\leq [1-\beta_n(p-1)]\{\|x_n - x^*\|^p + \|T^n z_n - v_n\|^p\} \\ &\quad +\beta_n\{\|T^n z_n - x^*\|^p + \|T^n z_n - v_n\|^p\}\end{aligned} \tag{2.7}$$

Since T is continuous on K, then there exists a real number $N_5 < \infty$ such that $\|T^n z_n - v_n\|^p \leq N_5$. Therefore

$$\begin{aligned}\|y_n - x^*\|^p &\leq [1-\beta_n(p-1)]\|x_n - x^*\|^p + \beta_n\|T^n z_n - x^*\|^p \\ &\quad +\{[1-\beta_n(p-1)] + \beta_n\}N_5 \\ &\leq [1-\beta_n(p-1)]\|x_n - x^*\|^p + \beta_n k_n^p\|z_n - x^*\|^p \\ &\quad +[1-\beta_n(p-2)]N_5\end{aligned} \tag{2.8}$$

Substituting (2.5) into (2.8) and setting $N_6 = \max\,[N_4, N_5]$, we have

$$\begin{aligned}\|y_n - x^*\|^p &\leq [1-\beta_n(p-1)]\|x_n - x^*\|^p \\ &\quad +\beta_n k_n^p[1-\gamma_n(p-k_n^p-1)](\|x_n - x^*\|^p \\ &\quad +\beta_n k_n^p[1-\gamma_n(p-2)]N_6 + [1-\beta_n(p-2)]N_6 \\ &\leq \{[1-\beta_n(p-1)] + \beta_n k_n^p[1-\gamma_n(p-k_n^p-1)]\}\|x_n - x^*\|^p \\ &\quad +\beta_n k_n^p[1-\gamma_n(p-2)]N_6 + [1-\beta_n(p-2)]N_6\end{aligned} \tag{2.9}$$

Finally,

$$\begin{aligned}\|x_{n+1} - x^*\|^p &= \|a_n x_n + b_n T^n y_n + c_n u_n - x^*\|^p \\ &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(T^n y_n - x^*) - c_n(T^n y_n - u_n)\|^p \\ &\leq [1-\alpha_n(p-1)]\|(x_n - x^*) - c_n(T^n y_n - u_n)\|^p \\ &\quad +\alpha_n c\|(T^n y_n - x^*) - c_n(T^n y_n - u_n)\|^p \\ &\quad -\alpha_n(1-\alpha_n^{(p-1)}c)\|(T^n y_n - x^*) - (x_n - x^*)\|^p\end{aligned}$$

Expanding further and observing that $\alpha_n(1-\alpha_n^{(p-1)}c) \geq 0$ for all $n > 0$, we have:

$$\begin{aligned}\|x_{n+1} - x^*\|^p &\leq [1-\alpha_n(p-1)]\{\|x_n - x^*\|^p + cc_n\|T^n y_n - u_n\|^p\} \\ &\quad +\alpha_n c\{\|T^n y_n - x^*\|^p + cc_n\|T^n y_n - u_n\|^p\} \\ &= [1-\alpha_n(p-1)]\|x_n - x^*\|^p + \alpha_n c\|T^n y_n - x^*\|^p \\ &\quad +[1-\alpha_n(p-1)c_n + \alpha_n cc_n]\|T^n y_n - u_n\|^p\end{aligned} \tag{2.10}$$

But T is asymptotically nonexpansive, $c_n \leq \alpha_n$ and $c < 1$. Therefore, simplifying (2.10), we have :

$$\begin{aligned} \|x_{n+1} - x^*\|^p \leq & [1-\alpha_n(p-1)]\|x_n - x^*\|^p + \alpha_n k_n^p\{\|y_n - x^*\|^p \\ & +[1-\alpha_n(p-2)]\|T^n y_n - u_n\|^p \end{aligned} \tag{2.11}$$

Since T is uniformly continuous on the bounded set K, then there exists a real number $N_7 < \infty$ such that $\|T^n y_n - u_n\| \leq N_7$. Substitute (2.8) into (2.10) and observe that $c_n < 1$, for all n, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^p \leq & [1-\alpha_n(p-1)]\|x_n - x^*\|^p + k_n^p\alpha_n\|y_n - x^*\|^p \\ & +[1-\alpha_n(p-2)]N_7 \\ \leq & [1-\alpha_n(p-1)]\|x_n - x^*\|^p + k_n^p\alpha_n\{[1-\beta_n(p-1)] \\ & +\beta_n k_n^p[1-\gamma_n(p-k_n^p-1)]\}\|x_n - x^*\|^p + \alpha_n\beta_n k_n^{2p}[1-\gamma_n(p-2)]N_6 \\ & +\alpha_n k_n^p[1-\beta_n(p-2)]N_6 + [1-\alpha_n(p-2)]N_7 \end{aligned}$$

Let $N_8 = \max[N_6, N_7]$, then we have

$$\begin{aligned} \|x_{n+1} - x^*\|^p \leq & \{[1-\alpha_n(p-1)] + k_n^p\alpha_n[1-\beta_n(p-1)] \\ & +\alpha_n\beta_n k_n^{2p}[1-\gamma_n(p-k_n^p-1)]\}\|x_n - x^*\|^p \\ & +\{\alpha_n\beta_n k_n^{2p}[1-\gamma_n(p-2)] + \alpha_n k_n^p[1-\beta_n(p-2)] \\ & +[1-\alpha_n(p-2)]\}N_8 \end{aligned} \tag{2.12}$$

Let

$$\rho_n = \|x_n - x^*\|^p$$

Then (2.12) becomes

$$\rho_{n+1} \leq (1-t_n)\rho_n + \sigma_n \tag{2.13}$$

where

$$\begin{aligned} t_n = & \alpha_n(p-1) - k_n^p\alpha_n[1-\beta_n(p-1)] \\ & -\alpha_n\beta_n k_n^{2p}[1-\gamma_n(p-k_n^p-1)] \end{aligned}$$

and

$$\begin{aligned} \sigma_n = & \{\alpha_n\beta_n k_n^{2p}[1-\gamma_n(p-2)] + \alpha_n k_n^p[1-\beta_n(p-2)] \\ & +[1-\alpha_n(p-2)]\}N_8 \end{aligned} \tag{2.14}$$

From our hypothesis, it is clear that $\sigma_n = o(t_n)$ and $\sum_{n\geq 1} t_n = \infty$. Also,

$$\begin{aligned}
t_n &= \alpha_n(p-1) - k_n^p\alpha_n[1-\beta_n(p-1)] \\
&\quad -\alpha_n\beta_n k_n^{2p}[1-\gamma_n(p-k_n^p-1)] \\
&= \alpha_n(p-1) - \alpha_n k_n^p + \alpha_n\beta_n k_n^p(p-1) - \alpha_n\beta_n k_n^{2p} \\
&\quad +\alpha_n\beta_n\gamma_n k_n^{2p}(p-1-k_n^p) \\
&= \alpha_n[p-1-k_n^p] + \alpha_n\beta_n k_n^p[p-1-k_n^p] \\
&\quad +\alpha_n\beta_n\gamma_n k_n^{2p}[p-1-k_n^p] \\
&= [p-1-k_n^p][\alpha_n + \alpha_n\beta_n k_n^p + \alpha_n\beta_n\gamma_n k_n^{2p}] \\
&\leq [p-1-k_n^p][1+\beta_n k_n^p + \beta_n\gamma_n k_n^{2p}]
\end{aligned}$$

We observe from our hypothesis that $t_n \geq 0$ and $t_n \leq 1$, i.e. $t_n \in [0,1]$. Hence, by Lemma 1.5, we have

$$\lim_{n\to\infty} \rho_n = 0$$

This implies that $\{x_n\}$ converges strongly to x^*. The proof is complete. □

Remark. Theorem 2.2 represents an extension of previous relevant results of Schu [11], Rhoades [10], Osilike and Aniagbosor [6], Osilike and Igbokwe [7] as well as Xu and Noor [14] to the more general modified Ishikawa type iteration scheme.

We now consider the modified iteration procedure given by (2.1) for fixed points of asymptotically nonexpansive operators in uniformly smooth Banach spaces. Our result is the following.

Theorem 2.3. *Let B be a uniformly smooth Banach space and K a nonempty closed bounded and convex subset of B. Suppose S, T are uniformly continuous and asymptotically nonexpansive selfmapping of K with real sequence $\{k_n\}$ satisfying $k_n \geq 1$. Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by*

$$\left.\begin{aligned}
x_{n+1} &= a_n x_n + b_n T^n y_n + c_n S^n x_n \\
y_n &= a_n^{'} x_n + b_n^{'} S^n z_n + c_n^{'} v_n \\
z_n &= a_n^{''} x_n + b_n^{''} T^n x_n + c_n^{''} v_n
\end{aligned}\right\} n \geq 1$$

where $\{u_n\}$, $\{v_n\}$, are arbitrary sequences in K and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$,$\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0,1]$ satisfying the following conditions *(i)* $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$,

(ii) $\sum b_n = \infty$,

(iii) $\alpha_n := b_n + c_n, \beta_n := b_n{}' + c_n{}', \gamma_n := b_n" + c_n"$,

$0 \leq (p - 1 - k_n^p) \leq 1,\ 1 + \beta_n k_n^p (1 + \gamma_n k_n^p) \leq \frac{1}{p-1-k_n^p}$.

If S and T have a common fixed point in K, then the sequence $\{x_n\}$ converges strongly to the common fixed point of S and T.

Proof. By Lemma Goebel and Kirk [3], S and T have fixed points in K. Let $x_o \in K$ be the common fixed point of S and T. From the above hypothesis and Lemma 1.4, we have the following estimates.

$$\begin{aligned}
\|z_n - x_o\|^p &= \|(1-\gamma_n)(x_n - x_o) + \gamma_n(T^n x_n - x_o) - c_n^{''}(T^n x_n - \omega_n\|^p \\
&\leq [1 - \gamma_n(p-1)]\|(x_n - x_o) - c_n^{''}(T^n x_n - \omega_n)\|^p \\
&\quad + \gamma_n c\|(T^n x_n - x_o) - c_n^{''}(T^n x_n - \omega_n)\|^p \\
&\quad - \gamma_n(1 - \gamma_n^{p-1} c)\|(T^n x_n - x_o) - (x_n - x_o)\|^p
\end{aligned}$$

Observe that $\gamma_n(1 - \gamma_n^{p-1}c) \geq 0$ and expand further, we have

$$\begin{aligned}
\|z_n - x_o\|^p &\leq [1 - \gamma_n(p-1)]\{\|x_n - x_o\|^p + cc_n^{''}\|T^n x_n - \omega_n\|^p \\
&\quad - p < c_n^{''}(T^n x_n - \omega_n), j(x_n - x_o) >\} \\
&\quad + \gamma_n c\{\|T^n x_n - x_o\|^p + cc_n^{''}\|T^n x_n - \omega_n\|^p \\
&\quad - p < c_n^{''}(T^n x_n - \omega_n), j(T^n x_n - x_o) >\} \\
&\leq [1 - \gamma_n(p-1)]\{\|x_n - x_o\|^p + cc_n^{''}\|T^n x_n - \omega_n\|^p\} \\
&\quad + \gamma_n c\{\|T^n x_n - x_o\|^p + cc_n^{''}\|T^n x_n - \omega_n)\|^p\}
\end{aligned} \tag{2.15}$$

By uniform continuity of T and boundedness of K, there exists real number $N_9 < \infty$ such that $\|T^n x_n - \omega_n\|^p \leq N_9$. Observe that $c, c_n^{''} < 1$. Then, from (6.20) we obtain

$$\begin{aligned}
\|z_n - x_o\|^p &\leq [1 - \gamma_n(p-1)]\|x_n - x_o\|^p + \gamma_n\|T^n x_n - x_o\|^p + [1 - \gamma_n(p-2)]N_9 \\
&\leq [1 - \gamma_n(p - k_n^p - 1)]\|x_n - x_o\|^p + [1 - \gamma_n(p-2)]N_9
\end{aligned} \tag{2.16}$$

(since T is asymptotically nonexpansive)

Furthermore, we have

$$\begin{aligned}\|y_n - x_o\|^p &\leq \|(1-\beta_n)(x_n - x_o) + \beta_n(S^n z_n - x_o) - c_n^{'}(S^n z_n - v_n)\|^p \\ &\leq [1-\beta_n(p-1)]\|(x_n - x_o) - c_n^{'}(S^n z_n - v_n)\|^p \\ &+\beta_n c\|(S^n z_n - x_o) - c_n^{'}(S^n z_n - v_n)\|^p \\ &-\beta_n(1-\beta_n^{p-1}c)\|(S^n z_n - x_o) - (x_n - x_o)\|^p\end{aligned} \tag{2.17}$$

Expanding further and considering the fact that $c, c_n{}' \leq 1$ and $\beta_n(1-\beta_n^{p-1}c) \geq 0$, we have

$$\begin{aligned}\|y_n - x_o\|^p &\leq [1-\beta_n(p-1)]\{\|x_n - x_o\|^p + \|S^n z_n - v_n\|^p \\ &\quad -p < c_n(S^n z_n - v_n), j(x_n - x_o) >\} \\ &\quad +\beta_n\{\|S^n z_n - x_o\|^p + \|S^n z_n - v_n\|^p \\ &\quad -p < c_n(S^n z_n - v_n), j(S^n z_n - x_o) >\} \\ &\leq [1-\beta_n(p-1)]\{\|x_n - x_o\|^p + \|S^n z_n - v_n\|^p\} \\ &\quad +\beta_n\{\|S^n z_n - x_o\|^p + \|S^n z_n - v_n\|^p\}\end{aligned}$$

Since S is uniformly continuous on K, there exists a real number $N_o < \infty$ such that $\|S^n z_n - v_n\|^p \leq N_o$. Observing further that S is asymptotically nonexpansive, we obtain

$$\begin{aligned}\|y_n - x_o\|^p &\leq [1-\beta_n(p-1)]\|x_n - x_o\|^p + \beta_n\|S^n z_n - x_o\|^p \\ &\quad +[1-\beta_n(p-2)]N_o \\ &\leq [1-\beta_n(p-1)]\|x_n - x^*\|^p + \beta_n k_n^p\|z_n - x_o\|^p \\ &\quad +[1-\beta_n(p-2)]N_o\end{aligned} \tag{2.18}$$

Let $Q_1 = \max[N_9, N_o]$ and substitute (2.17) into (2.18), we have

$$\begin{aligned}\|y_n - x_o\|^p &\leq [1-\beta_n(p-1)]\|x_n - x_o\|^p \\ &\quad +\beta_n k_n^p[1-\gamma_n(p-k_n^p-1)](\|x_n - x_o\|^p \\ &\quad +\beta_n k_n^p[1-\gamma_n(p-2)]Q_1 + [1-\beta_n(p-2)]Q_1 \\ &\leq \{[1-\beta_n(p-1)] + \beta_n k_n^p[1-\gamma_n(p-k_n^p-1)]\}\|x_n - x_o\|^p \\ &\quad +\{\beta_n k_n^p[1-\gamma_n(p-2)] + [1-\beta_n(p-2)]Q_1 \\ &= [1-\beta_n(p-1-k_n^p) - \beta_n\gamma_n k_n^p(p-k_n^p-1)]\|x_n - x_o\|^p \\ &\quad +[1-\beta_n(p-2) + \beta_n k_n^p(1-\gamma_n(p-2))]Q_1\end{aligned} \tag{2.19}$$

We finally have the following estimates.

$$\begin{aligned}
\|x_{n+1}-x_o\|^p &= \|a_nx_n+b_nT^ny_n+c_nS^nx_n-x_o\|^p\\
&= \|(1-\alpha_n)(x_n-x_n)+\alpha_n(T^ny_n-x_o)-c_n(T^ny_n-S^nx_n)\|^p\\
&\leq [1-\alpha_n(p-1)]\|(x_n-x_o)-c_n(T^ny_n-S^nx_n)\|^p\\
&\quad +\alpha_nc\|(T^ny_n-x_o)-c_n(T^ny_n-S^nx_n)\|^p\\
&\quad -\alpha_n(1-\alpha_n^{(p-1)}c)\|(Ty_n-x_o)-(x_n-x_o)\|^p
\end{aligned}$$

But $\alpha_n(1-\alpha_n^{(p-1)}c)\geq 0$, therefore further expansion yields

$$\begin{aligned}
\|x_{n+1}-x_o\|^p &\leq [1-\alpha_n(p-1)]\{\|x_n-x_o\|^p+cc_n\|T^ny_n-S^nx_n\|^p\\
&\quad -p<c_n(T^ny_n-S^nx_n)j(x_n-x_o)>\}\\
&\quad +\alpha_nc\{\|T^ny_n-x_o\|^p+cc_n\|T^ny_n-S^nx_n\|^p\\
&\quad -p<c_n(T^ny_n-S^nx_n)j(T^ny_n-x_o)>\}\\
&\leq [1-\alpha_n(p-1)]\|x_n-x_o\|^p\\
&\quad +[1-\alpha_n(p-2)]\|T^ny_n-S^nx_n\|^p+\alpha_nk_n^p\|y_n-x_o\|^p
\end{aligned} \tag{2.20}$$

T and S are uniformly continuous on K implies there exists a real number $Q_2<\infty$ such that $\|T^ny_n-S^nx_n\|^p\leq Q_2$. Therefore substituting (2.19) into (2.20), we have

$$\begin{aligned}
\|x_{n+1}-x_o\|^p &\leq [1-\alpha_n(p-1)]\|x_n-x_o\|^p+[1-\alpha_n(p-2)]Q_2\\
&\quad +\alpha_nk_n^p[1-\beta_n(p-1-k_n^p)(1+\gamma_nk_n^p)]\|x_n-x_o\|^p\\
&\quad +\alpha_nk_n^p[1-\beta_n(p-2-k_n^p)+\beta_nk_n^p\gamma_n(p-2)]Q_1\\
&= \{[1-\alpha_n(p-1)]\\
&\quad +\alpha_nk_n^p[1-\beta_n(p-1-k_n^p)(1+\gamma_nk_n^p)]\}\|x_n-x_o\|^p+\sigma_n\\
&= \{1-\alpha_n(p-1-k_n^p)[1+\beta_nk_n^p)(1+\gamma_nk_n^p)]\}\|x_n-x_o\|^p\\
&\quad +\sigma_n
\end{aligned} \tag{2.21}$$

where

$$\sigma_n=[1-\alpha_n(p-2)]Q_2+\alpha_nk_n^p[1-\beta_n(p-2-k_n^p)+\beta_nk_n^p\gamma_n(p-2)]Q_1$$

Let

$$t_n=\alpha_n(p-1-k_n^p)[1-\beta_nk_n^p)(1+\gamma_nk_n^p)]$$

From our hypothesis, we see that $t_n \in [0,1]$. Also $\sum t_n = \infty$ and $\sigma_n = o(t_n)$. Now, let

$$\|x_n - x_o\|^p = \Phi_n,$$

then (2.21) reduces to

$$\Phi_{n+1} \leq (1 - t_n)\Phi_n + \sigma_n$$

Hence, Lemma 1.5, $\Phi_n \to 0$ as $n \to \infty$. This implies that $\{x_n\}$ converges strongly to the common fixed point of S and T. The proof is complete. □

Remark. Theorem 2.3 generalizes relevant results on fixed points of asymptotically nonexpansive operators to the modified general Ishikawa iteration procedure in Banach spaces. When $S = T$ in Theorem 2.3, then we obtain Theorem 1.2 as a special case. Hence Theorem 2.3 is a generalization of the result of Owojori and Imoru [8], Theorem 2.2 and other relevant results in the literature.

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