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WALL DRIVEN STEADY FLOW AND HEAT TRANSFER IN A POROUS TUBE

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Abstract. An incompressible axisymetric wall driven steady flow of a viscous fluid and heat transfer in a uniformly porous tube of circular cross section is examined. For slow flow and low Reynolds number, the continuity equation, the Navier Stokes equations and energy equation were solved with the aid of perturbation series. An increasing or decreasing in the acceleration Reynolds number leads to increasing or decreasing in the wall shear stress and the rate of heat transfer across the wall.

1. INTRODUCTION

Flow in a pipe has been considered by many researchers due to its wide area of applications such as in the oil industries and physiological blood flow in artery and vein, gaseous diffusion in binary mixtures, natural transpiration and cooling, uniform irrigation systems etc. Several authors eg Berman [2], Makinde and Sibanda [4], Makinde [5], Makinde [6], Majdalani and Zhou [7], etc. have investigated the flow through porous media for different geometries and under various situations.

In this work, laminar flow of an incompressible viscous fluid through a uniformly porous tube of circular cross section is considered. Our objectives are to study the effect of temperature along the tube as the fluid Prandtl number and Reynolds number increases and decreases. Also to examine the wall shear stress and the rate of heat transfer in relation to the wall acceleration reynolds number. In section 2, we establish the Mathematical formulation of the problem. Graphical interpretation of the results were presented in section 3. We discuss the pertinent results in section 4.

2. MATHEMATICAL FORMULATION

Consider the laminar flow of an incompressible viscous fluid through a uniformly porous tube of circular cross section. A cylindrical polar coordinate system r, θ, z is taken where oz lies along the centre of the tube, r is the radial distance and θ is the azimulthal angle. Let u and v be the velocity components in the directions of z and r increasing respectively, a the characteristic radius, E is a constant parameter that characterises the axial wall velocity as shown in Figure 1.

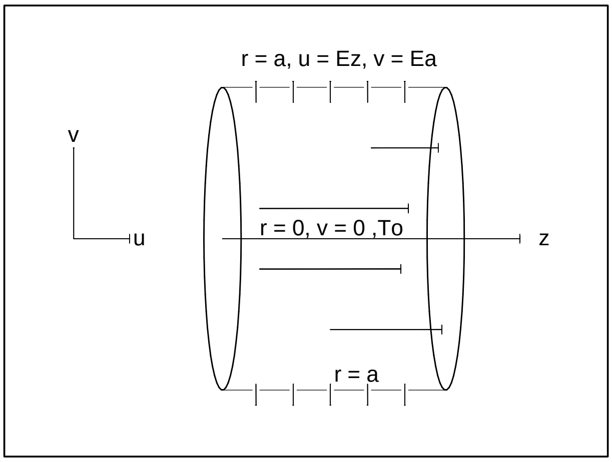

Figure 1: Wall driven steady flow in a uniformly porous tube.

For axisymetric steady incompressible viscous flow, ignoring external forces and heat source, the continuity, Navier Stokes and energy equations governing the flows are:

$$\frac{\partial rv}{\partial r} + r\frac{\partial u}{\partial z} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\nabla^2 u\right), \tag{2}$$

$$u\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 v - \frac{v}{r^2}\right), \tag{3}$$

$$\rho c_p\left(v\frac{\partial T}{\partial r} + u\frac{\partial T}{\partial z}\right) = k\nabla^2 T, \tag{4}$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$, p is the pressure, T is the temperature, ρ the density, ν the kinematic viscousity of the fliud, k the coefficient of thermometric conductivity and c_p the specific heat capacity at constant pressure.

The boundary conditions are:

Along z-axis, we have regularity of solution

$$\frac{\partial u}{\partial r} = 0, \quad v = 0, \quad \frac{\partial T}{\partial r} = 0 \quad \text{on} \quad r = 0. \tag{5}$$

The axial velocity, the normal velocity as well as temperature at the wall are prescribed as

$$u = Ez, \quad v = Ea, \quad T_w = T_0\left(1 + \frac{z}{a}\right) \quad \text{on} \quad r = a \tag{6}$$

where T_0 is the reference temperature at the center and E a contant parameter characterising the wall velocity.

Introducing the stream function ψ and vorticity ω as follows:

$$u = \frac{1}{r}\left(\frac{\partial \psi}{\partial r}\right) \tag{7}$$

$$\omega = \frac{\partial v}{\partial z} - \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial \psi}{\partial r}. \tag{8}$$

Eliminating pressure p from (2) and (3) by using (7) and (8), we have

$$\frac{1}{r}\left(\frac{\partial \psi}{\partial r}\frac{\partial \omega}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \omega}{\partial r}\right) + \frac{\omega}{r^2}\frac{\partial \psi}{\partial z} = \nu\left(\nabla^2 \omega - \frac{\omega}{r^2}\right). \tag{9}$$

Also using (7) in (4), we obtain

$$\rho c_p \frac{1}{r}\left(\frac{\partial \psi}{\partial r}\frac{\partial T}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial T}{\partial r}\right) = k\nabla^2 T. \tag{10}$$

Introducing the following dimensionless variables

$$\overline{\omega} = \frac{\omega}{E}, \quad \overline{z} = \frac{z}{a}, \quad \overline{r} = \frac{r}{a} \quad \overline{\psi} = \frac{\psi}{Ea^3} \quad \overline{T} = \frac{T}{T_0}, \tag{11}$$

we have

$$R_e\left[\frac{1}{\overline{r}}\left(\frac{\partial \overline{\psi}}{\partial \overline{r}}\frac{\partial \overline{\omega}}{\partial \overline{z}} - \frac{\partial \overline{\psi}}{\partial \overline{z}}\frac{\partial \overline{\omega}}{\partial \overline{r}}\right) + \frac{\overline{\omega}}{\overline{r}^2}\frac{\partial \overline{\psi}}{\partial \overline{z}}\right] = \nabla^2 \overline{\omega} - \frac{\overline{\omega}}{\overline{r}^2}, \tag{12}$$

$$P_r R_e\left[\frac{1}{\overline{r}}\left(\frac{\partial \overline{\psi}}{\partial \overline{r}}\frac{\partial \overline{T}}{\partial \overline{z}} - \frac{\partial \overline{\psi}}{\partial \overline{z}}\frac{\partial \overline{T}}{\partial \overline{r}}\right)\right] = \nabla^2 \overline{T}, \tag{13}$$

where $R_e = \frac{Ea^2}{\nu}$ is the flow Reynolds number, $P_r R_e = \frac{\rho c_p E a^2}{k}$ is the product of the Prandtl number and the Reynolds number (i.e Peclet number).

We seek a similarity form of solution (Berman, 1953) that is

$$\overline{\psi} = \overline{z}F(\overline{r}), \quad \overline{\omega} = -\overline{z}G(\overline{r}), \quad \overline{T} = \theta(\overline{r}). \tag{14}$$

Equations (8), (12) and (13) become

$$G = \frac{d}{d\overline{r}}\left[\frac{1}{\overline{r}}\frac{dF}{d\overline{r}}\right], \tag{15}$$

$$\frac{d}{d\overline{r}}\left[\frac{1}{\overline{r}}\frac{d(\overline{r}G)}{d\overline{r}}\right] = R_e\left[\frac{G}{\overline{r}}\frac{dF}{d\overline{r}} - F\frac{d}{d\overline{r}}\left(\frac{G}{\overline{r}}\right)\right], \tag{16}$$

$$\frac{d}{d\overline{r}}\left[\overline{r}\frac{d\theta}{d\overline{r}}\right] = P_r R_e\left[\theta\frac{dF}{d\overline{r}} - F\frac{d\theta}{d\overline{r}}\right]. \tag{17}$$

The boundary conditions (5) and (6) become

$$F = 0, \quad \frac{d}{d\overline{r}}\left(\frac{1}{\overline{r}}\frac{dF}{d\overline{r}}\right) = 0, \quad \frac{d\theta}{d\overline{r}} = 0 \quad \text{on} \quad \overline{r} = 0, \tag{18}$$

$$F = -1, \quad \frac{dF}{d\overline{r}} = 1, \quad \theta = 1 \quad \text{on} \quad \overline{r} = 1. \tag{19}$$

For small R_e, we seek the solution of the equations (15)-(19) as a perturbation series in terms of the parameter R_e i.e.

$$\omega(\overline{r}) = \sum_{i=0}^{\infty} \omega_i R_e^i, \tag{20}$$

$$\theta\left(\overline{r}\right)=\sum_{i=0}^{\infty}\theta_i R_e^i, \tag{21}$$

where ω can be either F($\overline{r}$,R_e) or G($\overline{r}$,R_e).

Substituting equations (20) and (21) into equations (15)-(19), neglecting the bars for clarity, and collecting the coefficients of like powers of R_e, the resulting equations are

$$G_n=\left[\frac{1}{r}F_n^{'}\right]^{'}, \tag{22}$$

$$\left[\frac{1}{r}\left(rG_n\right)^{'}\right]^{'}=R_e\sum_{i=0}^{n-1}\left[G_i\left(\frac{F_{n-i-1}^{'}}{r}\right)-F_i\left(\frac{G_{n-i-1}}{r}\right)^{'}\right] \tag{23}$$

$$\left[r\theta_n^{'}\right]^{'}=P_rR_e\sum_{i=0}^{n-1}\left[\theta_iF_{n-i-1}^{'}-F_i\theta_{n-i-1}^{'}\right] \tag{24}$$

$$F_n=0,\quad \left[\frac{1}{r}F_n^{'}\right]^{'}=0,\quad \theta_n^{'}=0\quad \text{on}\quad r=0 \tag{25}$$

$$F_0=-1,\quad F_{n+1}=0,\quad F_0^{'}=1,\quad F_{n+1}^{'}=0\quad \theta_n=1\quad \text{on}\quad r=1, \tag{26}$$

where $n=0,1,2,...$ and the prime symbol denotes differentiation with respect to r.

Solving equations (22) and (23) using (25) and (26), we obtain the first three terms of F and G as

$$\begin{gathered}F_0=\frac{1}{2}r^2\left(3r^2-5\right),\quad F_1=\frac{1}{16}r^2\left(r^6-5r^4+7r^2-3\right),\\ F_2=\frac{1}{9600}r^2\left(r-1\right)^2\left(r+1\right)^2\left(6r^6-63r^4+218r^2-601\right)\end{gathered} \tag{27}$$

$$\begin{gathered}G_0=12r,\quad G_1=\frac{1}{2}r\left(6r^4-15r^2+7\right),\\ G_2=\frac{1}{120}r\left(9r^8-75r^6+210r^4-330r^2+142\right).\end{gathered} \tag{28}$$

Solving (24) using (25),(26) and (27), we obtain the first three terms of θ as

$$\begin{gathered}\theta_0=1,\quad \theta_1=1+\frac{P_rR_e}{8}\left(7-10r^2+3r^4\right),\\ \theta_2=1+\frac{P_rR_e}{384}\left(349-516r^2+184r^4-20r^6+3r^8\right)\\ +\frac{P_r^2R_e^2}{192}\left(157-210r^2+63r^4-10r^6\right)\end{gathered} \tag{29}$$

Substituting (29) into (21), we have

$$\begin{aligned}\theta(r) = 1 + R_e\Big[1 + \frac{P_r R_e}{8}\left(7 - 10r^2 + 3r^4\right)\Big] \\ + R_e^2\Big[1 + \frac{P_r R_e}{384}\left(347 - 516r^2 + 184r^4 - 20r^6 + 3r^8\right) \\ + \frac{P_r^2 R_e^2}{192}\left(157 - 210r^2 + 63r^4 - 10r^6\right)\Big] + O\left(R_e^3\right).\end{aligned} \tag{30}$$

Hence $T = z\theta(r)$ becomes

$$\begin{aligned}T = z\Big[1 + R_e\Big[1 + \frac{P_r R_e}{8}\left(7 - 10r^2 + 3r^4\right)\Big] \\ + R_e^2\Big[1 + \frac{P_r R_e}{384}\left(347 - 516r^2 + 184r^4 - 20r^6 + 3r^8\right) \\ + \frac{P_r^2 R_e^2}{192}\left(157 - 210r^2 + 63r^4 - 10r^6\right)\Big] + O\Big(R_e^3\Big)\Big].\end{aligned} \tag{31}$$

Equation (31) represents the temperature distribution of the fluid flow pattern in the tube.

The wall shear stress $G(1)$ is given by

$$G(1) = 12 - R_e - \frac{11}{30}R_e^2 - \frac{331}{2520}R_e^3 - \frac{1206679}{24192000}R_e^4 - ... \tag{32}$$

Also the rate of heat transfer across the wall $N_u = -\frac{d\theta}{dr}$ on $r = 1$ is given as

$$\begin{aligned}N_u = P_r R_e^2 + P_r R_e^3\left(\frac{19}{16}P_r R_e + 1\right) + P_r R_e^4\left(\frac{73}{64}P_r^2 R_e^2 + \frac{119}{96}P_r R_e + 1\right) \\ + P_r R_e^5\left(\frac{11537}{10752}P_r^3 R_e^3 + \frac{176867}{143360}P_r^2 R_e^2 + \frac{115943}{92160}P_r R_e + 1\right)\ldots\end{aligned} \tag{33}$$

3. GRAPHICAL RESULTS

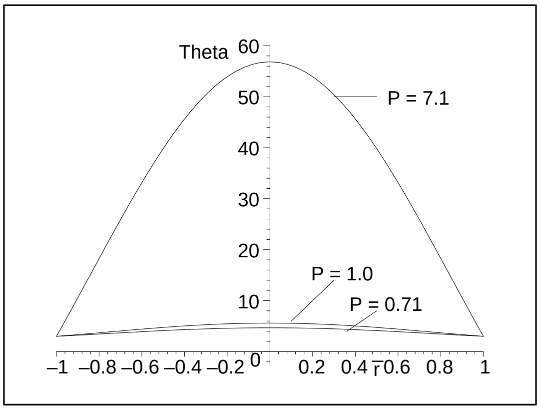

Figure 2: Temperature distribution profile (R = 1)

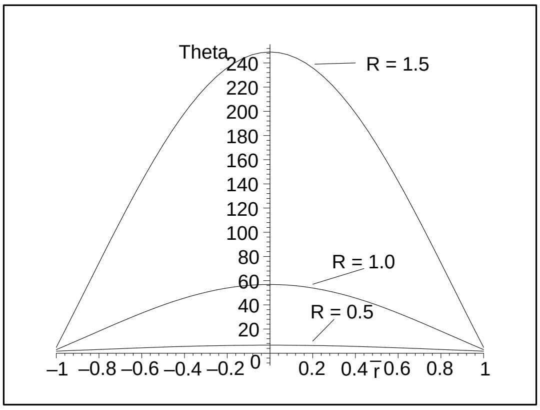

Figure 3: Temperature distribution profile (P = 7.1)

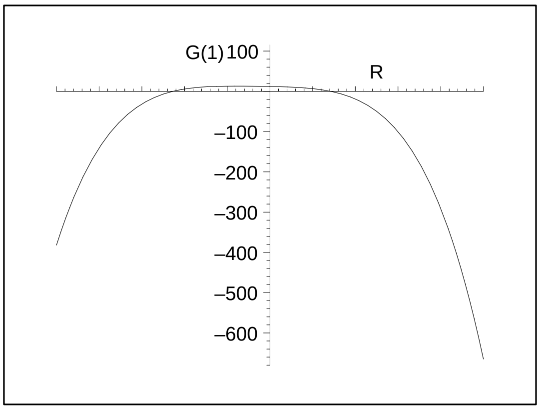

Figure 4: The wall shear stress

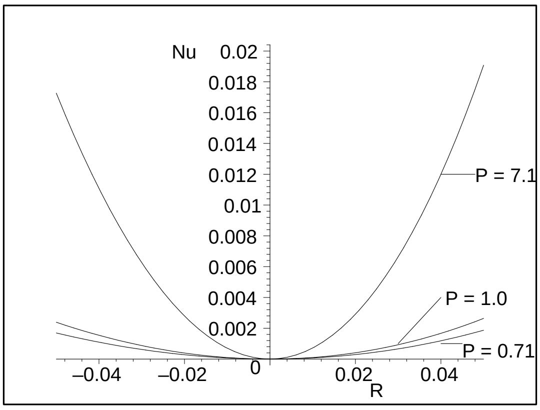

Figure 5: Heat transfer across the wall

4. DISCUSSION AND CONCLUSION

We have made use of various Prandtl number such as air ($P_r = 0.71$), water ($P_r = 7.1$) etc. in our numerical calculation. Figure (2) show that the fluid temperature increases with an increase in fluid Prandtl number with maximum value at the pipe centreline. The fluid temperature decreases transversly with minimum value at the wall. In figure (3), we fixed the Prandtl number of fluid like water ($P_r = 7.1$) and varying the wall acceleration Reynolds number i.e. varying Reynolds number. An increase in Reynolds number causes an increase in the fluid temperature with maximum magnitude at the centre of pipe and minimum at the wall. Figure (4) shows that the wall shear stress is parabolic in nature with maximum shear stress of $G(1) = 12$ corresponding to $R_e = 0$. Also, figure (5) shows the rate of heat transfer across the wall. The rate of heat transfer increases monotonically as the fluid Reynolds number increases and decreases monotonically as the fluid Reynolds number decreases. Also, using different Prandtl numbers such as $P_r = 0.71, 1.0, 7.1$ etc, the rate of heat transfer is more when the Prandtl number is increased.

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