*Kragujevac J. Math.* 29 (2006) 203–213.

# ISHIKAWA ITERATIVE SEQUENCE FOR THE GENERALIZED LIPSCHITZIAN AND Φ-STRONGLY ACCRETIVE MAPPINGS IN BANACH SPACES[1]

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*(Received Jun 15, 2005)*

**Abstract.** Let $E$ be a real uniformly smooth Banach space, $T : E \to E$ be a generalized Lipschitzian and Φ-strongly accretive mapping. It is shown that under suitable conditions the Ishikawa iterative process converges strongly to the unique solution of the equation $Tx = f$. A related result deals with approximation of the unique fixed point of a generalized Lipschitzian and Φ-strongly pseudo-contractive mapping.

## 1. INTRODUCTION

Let $E$ be real Banach space and $E^*$ be the dual space on $E$. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f \in E^* :< x, f >= ||x|| \cdot ||f|| = ||f||^2\} \tag{1}$$

[1]The author was supported by the National Science Foundation of China and Shijiazhuang Railway Institute Sciences Foundation.

for all $x \in E$, where $< \cdot, \cdot >$ denotes the generalized duality pairing. It is well known that if $E$ is an uniformly smooth Banach space, then $J$ is single-valued and such that $J(-x) = -J(x), J(tx) = tJ(x)$ for all $x \in E$ and $t \geq 0$; and $J$ is uniformly continuous on any bounded subset of $E$. In the sequel we shall denote single-valued normalized duality mapping by $j$. By means of the normalized duality mapping $J$. In the following we give some concepts.

**Definition 1.1.** *A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be strongly accretive if for any $x, y \in D(T)$, there exist a constant $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that*

$$< Tx - Ty, j(x-y) > \geq k\|x-y\|^2. \tag{2}$$

*The mapping $T$ is called $\Phi$-strongly accretive if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that the inequality*

$$< Tx - Ty, j(x-y) > \geq \Phi(\|x-y\|)\|x-y\| \tag{3}$$

*holds for all $x, y \in D(T)$. It is well known that the class of strongly accretive mappings is a proper subclass of the class of $\Phi$-strongly accretive mappings(see [1]). On the other hand, closely related to the class of accretive type mappings are those of pseudocontractive mappings.*

**Definition 1.2.** *A mapping $T : D(T) \subset E \to E$ is called strongly pseudocontractive if and only if $(I-T)$ is strongly accretive, and is called $\Phi$-strongly pseudocontractive if and only if $(I-T)$ is $\Phi$-strongly accretive, where $I$ denotes the identity mapping on $E$.*

The classes of mappings introduced above have been studied by several authors. In [2], Chidume proved that if $E = L_p$(or $l^p$), $p \geq 2$, $K$ is a nonempty closed convex and bounded subset of $E$ and $T : K \to K$ is a Lipschitz strongly pseudocontractive mapping, then Mann iteration process converges strongly to the unique fixed point of $T$. In [3], Deng extended the above result to the Ishikawa iteration process. After Tan and Xu [4] extended the results of both Chidume [2] and Deng [3] to $q$-uniformly smooth Banach spaces$(1 < q < 2)$. Recently, Osilike [1] proved that if $q > 1$, $E$ is real $q$-uniformly smooth Banach space and $T : E \to E$ is a Lipschitz $\Phi$-strongly accretive mapping and equation $Tx = f$ has a solution, then both the Mann and Ishikawa iteration sequence converges strongly to the solution. It is our purpose in this paper to generalize and extend the results of [1] from Lipschitz mapping to generalized Lipschitzian, from $q$-uniformly smooth to uniformly smooth.

For this purpose, we need to introduce the following concept and some related Lemmas:

**Definition 1.3.** [8] *A mapping $T : D(T) \subset E \to E$ is called a generalized Lipschitzian if there exists a constant $C > 0$ such that*

$$\|Tx - Ty\| \leq C(1 + \|x - y\|) \tag{4}$$

*holds for all $x, y \in D(T)$. Clearly, every Lipschitz mapping is a generalized Lipschitzian, and every mapping with a bounded range too. Conversely, in general, a generalized Lipschitzian mapping neither is Lipschitzian nor the bounded range.(see, for example[8])*

**Lemma 1. 4.** [7] *Let $E$ be a real Banach space, then there exists $j(x+y) \in J(x+y)$ such that*

$$\|x + y\|^2 \leq \|x\|^2 + 2 < y, j(x + y) > \tag{5}$$

*for any $x, y \in E$.*

**Lemma 1.5.** *Let $E$ be a real Banach space, $T : E \to E$ be continuous $\Phi$-strongly pseudo-contractive mapping with $\Phi(t) \to +\infty$ as $t \to +\infty$. Then, for any given $f \in E$, the equation $x = f + Tx$ has the unique solution in $E$.*

**Proof.** We choose a positive real number sequence $\{t_n\}_{n=0}^{\infty}$ with $t_n \to 0$ as $n \to \infty$. Define an operator sequence $\{T_n\}_{n=0}^{\infty}$, where $T_n : E \to E$ by $T_n x = t_n x + x - Tx$, for any $n \geq 0$ and $x \in E$, then $T_n$ must be continuous strongly accretive mapping in $E$ for any $n \geq 0$. Thus, for any $f \in E$, the equation $T_n x = f$ has the unique solution, denote $y_n$, i.e. $t_n y_n + y_n - Ty_n = f$, where $n = 0, 1, 2, \ldots$. It yields that

$$t_n y_n - t_0 y_0 + y_n - y_0 + Ty_0 - Ty_n = 0$$

so that

$$< t_n y_n - t_0 y_0 + (I-T)y_n - (I-T)y_0, j(y_n - y_0) >= 0,$$

which implies that

$$< (I-T)y_n - (I-T)y_0, j(y_n - y_0) >= -t_n\|y_n - y_0\|^2 - < t_n y_0 - t_0 y_0, j(y_n - y_0) > .$$

Since $(I-T)$ is $\Phi$-strongly accretive mapping, then we have

$$\begin{aligned} \|y_n - y_0\|\Phi(\|y_n - y_0\|) &\leq -t_n\|y_n - y_0\|^2 - < t_n y_0 - t_0 y_0, j(y_n - y_0) > \\ &\leq - < t_n y_0 - t_0 y_0, j(y_n - y_0) > \\ &\leq |t_n - t_0| \cdot \|y_0\| \cdot \|y_n - y_0\|, \end{aligned}$$

and this implies that

$$\Phi(\|y_n - y_0\|) \leq |t_n - t_0| \cdot \|y_0\|,$$

i.e.

$$(\|y_n - y_0\|) \leq \Phi^{-1}(|t_n - t_0| \cdot \|y_0\|).$$

Since $t_n \to 0$ as $n \to \infty$, it is easily seen that $\{y_n\}_{n=0}^{\infty}$ is bounded. Therefore $y_n - Ty_n \to f$ as $n \to \infty$. Since $(I-T)$ is $\Phi$-strongly accretive mapping, we obtain that

$$\begin{aligned} \|(y_n - Ty_n) - (y_m - Ty_m)\| \cdot \|y_n - y_m\| &\geq < (I-T)y_n - (I-T)y_m, j(y_n - y_m) > \\ &\geq \Phi(\|y_n - y_m\|) \cdot \|y_n - y_m\|, \end{aligned}$$

i.e. $\|(y_n - Ty_n) - (y_m - Ty_m)\| \geq \Phi(\|y_n - y_m\|)$, then $\{y_n\}_{n=0}^{\infty}$ is a Cauchy sequence, there exists $y \in E$ such that $y_n \to y$ as $n \to \infty$. By using continuous of $T$ such that $y = f + Ty$. About uniqueness of solution, we may get it by applying definition of $\Phi$-strongly accretive mapping. The proof Lemma is completed. □

**Remark 1.6.** In Lemma 1.5, if $f = 0$, then the mapping $T$ has the unique fixed point.

**Remark 1.7.** In Lemma1.5, suppose $T : E \to E$ is a continuous $\Phi$-strongly accretive mapping, then the equation $Tx = f$ has unique solution in $E$.

## 2. MAIN RESULTS

Now we prove the main the results of this paper, In the sequel, we always assume that E is a uniformly smooth real Banach space.

**Theorem 2.1.** *Let $E$ be a real uniformly smooth Banach space, and $T : E \to E$ is a continuous and generalized Lipschitzian $\Phi$-strongly accretive mapping with $\Phi(t) \to +\infty$ as $t \to +\infty$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in $[0,1]$ satisfying the following conditions: (i) $\alpha_n, \beta_n \to 0$ as $n \to \infty$; (ii) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$. For any given $f \in E$, define a mapping $S : E \to E$ by $Sx = x - Tx + f$, for all $x \in E$. The Ishikawa iterative sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in E$ by (IS)*

$$\begin{cases} y_n &= (1-\beta_n)x_n + \beta_n S x_n, \quad n \geq 0, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S y_n, \quad n \geq 0. \end{cases} \tag{6}$$

*Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of the equation $Tx = f$.*

**Proof.** By Remark 1.7, we know that the equation $Tx = f$ has the unique solution in $E$, set $q$. Since $T$ is generalized Lipschitzian $\Phi$-strongly accretive mapping, then for any $x, y \in E$ such that the following inequalities hold:

$$\|Sx - Sy\| \leq L(1 + \|x - y\|), \tag{7}$$

and

$$< Sx - Sy, J(x-y) > \leq \|x - y\|^2 - \Phi(\|x - y\|)\|x - y\|. \tag{8}$$

Set $A_n = \|J(\frac{x_{n+1}-q}{1+\|x_n-q\|}) - J(\frac{y_n-q}{1+\|x_n-q\|})\|$, $B_n = \|J(\frac{y_n-q}{1+\|x_n-q\|}) - J(\frac{x_n-q}{1+\|x_n-q\|})\|$. Observe that $\frac{\|x_{n+1}-q\|}{1+\|x_n-q\|} \leq 1 + 2L + 2L^2$, $\frac{\|x_n-q\|}{1+\|x_n-q\|} \leq 1$ and $\frac{\|y_n-q\|}{1+\|x_n-q\|} \leq 1 + 2L$. It is easily obtained that, in view of the uniform continuity of $J$ on any bounded subset of $E$,

$A_n, B_n \to 0$ as $n \to \infty$. Using Lemma 1.4, (7) and (8), we computer as follows:

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&\quad = \|(1-\alpha_n)(x_n-q)+\alpha_n(Sy_n-Sq)\|^2 \\
&\quad \leq (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n < Sy_n - Sq, J(x_{n+1}-q) > \\
&\quad \leq (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n < Sy_n - Sq, J(y_n-q) > \\
&\qquad +2\alpha_n < Sy_n - Sq, J(x_{n+1}-q) - J(y_n-q) > \qquad (9)\\
&\quad \leq (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n(\|y_n-q\|^2 - \Phi(\|y_n-q\|)\|y_n-q\|) \\
&\qquad +2\alpha_n < Sy_n - Sq, J(\tfrac{x_{n+1}-q}{1+\|x_n-q\|}) - J(\tfrac{y_n-q}{1+\|x_n-q\|}\|) > (1+\|x_n-q\|) \\
&\quad \leq (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n(\|y_n-q\|^2 - \Phi(\|y_n-q\|)\|y_n-q\|) \\
&\qquad +2\alpha_n A_n L(1+\|y_n-q\|)(1+\|x_n-q\|).
\end{aligned}$$

Furthermore, observe that

$$\begin{aligned}
&2\alpha_n A_n L(1+\|y_n-q\|)(1+\|x_n-q\|) \\
&\qquad \leq 2\alpha_n A_n L(1+\beta_n L)(1+\|x_n-q\|)^2 \qquad (10)\\
&\qquad \leq 2\alpha_n A_n L(1+\beta_n L)(1+\|x_n-q\|^2).
\end{aligned}$$

Again using Lemma 1.4, (7) and (8), we obtain

$$\begin{aligned}
&\|y_n-q\|^2 \\
&\quad \leq (1-\beta_n)^2\|x_n-q\|^2 + 2\beta_n < Sx_n - Sq, J(y_n-q) > \\
&\quad \leq (1-\beta_n)^2\|x_n-q\|^2 + 2\beta_n < Sx_n - Sq, J(y_n-q) - J(x_n-q) > \\
&\qquad +2\beta_n < Sx_n - Sq, J(x_n-q) > \\
&\quad \leq (1-\beta_n)^2\|x_n-q\|^2 + 2\beta_n < Sx_n - Sq, J(\tfrac{y_n-q}{1+\|x_n-q\|}) - J(\tfrac{x_n-q}{1+\|x_n-q\|}\|) > \\
&\qquad \times(1+\|x_n-q\|) + 2\beta_n(\|x_n-q\|^2 - \Phi(\|x_n-q\|)\|x_n-q\|) \qquad (11)\\
&\quad \leq (1+\beta_n^2)\|x_n-q\|^2 + 2\beta_n L(1+\|x_n-q\|)B_n(1+\|x_n-q\|) \\
&\qquad -2\beta_n\Phi(\|x_n-q\|)\|x_n-q\|) \\
&\quad \leq (1+\beta_n^2)\|x_n-q\|^2 + 4\beta_n B_n L(1+\|x_n-q\|^2) \\
&\qquad -2\beta_n\Phi(\|x_n-q\|)\|x_n-q\| \\
&\quad \leq (1+\beta_n^2+4L\beta_n B_n)\|x_n-q\|^2 \\
&\qquad +4L\beta_n B_n - 2\beta_n\Phi(\|x_n-q\|)\|x_n-q\|.
\end{aligned}$$

Substituting (10) and (11) into (9) yields that

$$\begin{aligned} &\|x_{n+1}-q\|^2 \\ &\quad\le (1-\alpha_n)^2\|x_n-q\|^2+2\alpha_n((1+\beta_n^2+4L\beta_nB_n)\|x_n-q\|^2 \\ &\qquad +4L\beta_nB_n-2\beta_n\Phi(\|x_n-q\|)\|x_n-q\|-\Phi(\|y_n-q\|)\|y_n-q\|) \\ &\qquad +2\alpha_nA_nL(1+\beta_nL)(1+\|x_n-q\|^2) \qquad (12)\\ &\quad\le (1+\alpha_n^2+2\alpha_n\beta_n^2+8L\alpha_n\beta_nB_n+2\alpha_nA_nL(1+\beta_nL))\|x_n-q\|^2 \\ &\qquad +2\alpha_nA_nL(1+\beta_nL)+8L\alpha_n\beta_nB_n-2\alpha_n\Phi(\|y_n-q\|)\|y_n-q\|) \\ &\quad\le \|x_n-q\|^2+2\alpha_nC_n\|x_n-q\|^2+2\alpha_n(D_n-\Phi(\|y_n-q\|)\|y_n-q\|) \end{aligned}$$

where $C_n=\alpha_n+\beta_n^2+4L\beta_nB_n+A_nL(1+\beta_nL)$, $D_n=A_nL(1+\beta_nL)+4L\beta_nB_n$. Base on (8), we have $< x-Sx, J(x-q) >\ge \Phi(\|x-q\|)\|x-q\|, \forall x\in E$, thus $\|x-Sx\|\cdot\|x-q\|\ge \Phi(\|x-q\|)\|x-q\|$. Hence $\Phi(\|x-q\|)\le\|x-Sx\|$. At this point, we may choose any $x_0\in E$ such that $\|x_0-Sx_0\|\ne 0$, i.e. $x_0\ne q$.(If$x_0=q$, then conclusion of Theorem is obvious.) so we obtain $\|x_0-q\|\le\Phi^{-1}(\|x_0-Sx_0\|)$. Since $\alpha_n,\beta_n\to 0$ as $n\to\infty$, then there exists an integer $N$ such that $\alpha_n<\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{2(1+L+L^2)\Phi^{-1}(\|x_0-Sx_0\|)+L(1+L)}$, $\beta_n<\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{2((1+L)\Phi^{-1}(\|x_0-Sx_0\|)+L)}$, $C_n(2\Phi^{-1}(\|x_0-Sx_0\|))^2+D_n<\frac{\Phi(\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{2})\Phi^{-1}(\|x_0-Sx_0\|)}{4}$. for all $n\ge N$. Suppose $\|x_N-q\|\le 2\Phi^{-1}(\|x_0-Sx_0\|)$,by the mathe induction, we want to show $\|x_{N+1}-q\|\le 2\Phi^{-1}(\|x_0-Sx_0\|)$. If not, we assume that $\|x_{N+1}-q\|> 2\Phi^{-1}(\|x_0-Sx_0\|)$.

Using (6) and the above formule, we obtain the following inequalities

$$\begin{aligned} &\|x_N-Sy_N\| \\ &\quad= \|x_N-q+Sq-Sy_N\| \\ &\quad\le \|x_N-q\|+L(1+\|y_N-q\|) \qquad (13)\\ &\quad\le 2\Phi^{-1}(\|x_0-Sx_0\|)+L(1-\beta_N+\beta_NL)\|x_N-q\|+L(1+\beta_NL) \\ &\quad\le 2(1+L+L^2)\Phi^{-1}(\|x_0-Sx_0\|)+L(1+L), \end{aligned}$$

and get also

$$\begin{aligned} \|x_N-q\| &\ge (1-\alpha_N)\|x_N-q\| \\ &\ge \|x_{N+1}-q\|-\alpha_N\|x_N-Sy_N\| \\ &\ge 2\Phi^{-1}(\|x_0-Sx_0\|) \qquad (14)\\ &\quad -\alpha_N(2(1+L+L^2)\Phi^{-1}(\|x_0-Sx_0\|)+L(1+L)) \\ &\ge \Phi^{-1}(\|x_0-Sx_0\|), \end{aligned}$$

$$\begin{aligned}\|y_N - q\| &\geq (1-\beta_N)\|x_N - q\| - \beta_N L(1+\|x_N - q\|)\\ &= (1-\beta_N-\beta_N L)\|x_N - q\| - \beta_N L \\ &\geq \tfrac{\Phi^{-1}(\|x_0 - Tx_0\|)}{2}.\end{aligned} \tag{15}$$

Thus $\Phi(\|y_N - q\|)\|y_N - q\| \geq \Phi(\frac{\Phi^{-1}(\|x_0-Tx_0\|)}{2})\frac{\Phi^{-1}(\|x_0-Tx_0\|)}{2}$. Using (12) and above formula, we have

$$\begin{aligned}\|x_{N+1} - q\|^2 &\leq \|x_N - q\|^2 + 2\alpha_N(C_N\|x_N - q\|^2 + D_N - \Phi(\|y_N - q\|)\|y_N - q\|)\\ &\leq \|x_N - q\|^2 - \alpha_N\Phi(\frac{\Phi^{-1}(\|x_0 - Tx_0\|)}{2})\frac{\Phi^{-1}(\|x_0 - Tx_0\|)}{2}\\ &\leq \|x_N - q\|^2 \leq (2\Phi^{-1}(\|x_0 - Tx_0\|))^2\end{aligned}$$

contradicting with assumption. Hence $\|x_{N+1} - q\| \leq 2\Phi^{-1}(\|x_0 - Tx_0\|)$ holds, $\{\|x_n - q\|\}_{n=0}^{\infty}$ is bounded, so that $\{\|y_n - q\|\}_{n=0}^{\infty}$ is also bounded. Set $W = \sup\{\|x_n - q\| + \|y_n - q\|\}$, $E_n = C_n W^2 + D_n$. Then,

$$\begin{aligned}&\|x_{n+1} - q\|^2\\ &\quad\leq \|x_n - q\|^2 + 2\alpha_n(E_n - \Phi(\|y_n - q\|)\|y_n - q\|)\\ &\quad= \|x_n - q\|^2 + \alpha_n(2E_n - \Phi(\|y_n - q\|)\|y_n - q\|) - \alpha_n\Phi(\|y_n - q\|)\|y_n - q\|).\end{aligned} \tag{16}$$

Hence, $\lim_{n\to\infty}\inf\|y_n - q\| = 0$ holds. Suppose this is not true. Let $\lim_{n\to\infty}\inf\|y_n - q\| = 2\delta > 0$. Then there exists an integer $N_1$ such that $\|y_n - q\| \geq \delta$, for all $n \geq N_1$, i.e, $\Phi(\|y_n - q\|)\|y_n - q\| \geq \Phi(\delta)\delta$. Since $E_n \to 0 (n \to \infty)$, there exists an integer $N_2 > N_1$ such that $E_n \leq \Phi(\delta)\delta$ for all $n \geq N_2$, thus $E_n \leq \Phi(\|y_n - q\|)\|y_n - q\|$. Hence, for all $n \geq N_2$, we obtain that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \|x_n - q\|^2 - \alpha_n\Phi(\|y_n - q\|)\|y_n - q\|)\\ &\leq \|x_n - q\|^2 - \alpha_n\Phi(\delta)\delta,\end{aligned}$$

which implies that

$$\Phi(\delta)\delta\sum_{n=N_2}^{\infty}\alpha_n \leq \|x_{N_2} - q\|^2 < \infty$$

which is a contradiction and so $\delta = 0$. Consequently, there exists a subsequence $\left\{y_{n_j} - q\right\}_{j=0}^{\infty}$ of $\{y_n - q\}_{n=0}^{\infty}$ such that $\lim_{j\to\infty}\|y_{n_j} - q\| = 0$, and so there exists an infinite subsequence $\left\{x_{n_j} - q\right\}_{j=0}^{\infty}$ such that $\lim_{j\to\infty}\|x_{n_j} - q\| = 0$. Let $\varepsilon > 0$ be any given, $\exists j_0$

such that, for all $n_j > n_{j_0}$, $\|x_{n_j} - q\| < \varepsilon, \alpha_{n_j}(LW + L) < \frac{\varepsilon}{4}, \beta_{n_j} < \frac{\varepsilon}{4(W+LW+L)}$. Again choose an integer $N_0 \geq n_{j_0}$ such that $E_n < \Phi(\frac{\varepsilon}{2})\frac{\varepsilon}{4}$ for all $n > N_0$. By induction, we want to prove$\|x_{n_j+m} - q\| < \varepsilon$, for all $\forall m \geq 1$. We first prove that $\|x_{n_j+1} - q\| < \varepsilon$. Suppose this is not ture. Then $\exists n_{j_1} > n_{j_0}$, such that $\|x_{n_{j_1}+1} - q\| \geq \varepsilon$. Using (6), we have

$$\begin{aligned}
\|x_{n_{j_1}+1} - q\| &\leq (1 - \alpha_{n_{j_1}})\|x_{n_{j_1}} - q\| + \alpha_{n_{j_1}}\|Sy_{n_{j_1}} - Sq\| \\
&\leq (1 - \alpha_{n_{j_1}})\|x_{n_{j_1}} - q\| + \alpha_{n_{j_1}}(L\|y_{n_{j_1}} - q\| + L) \\
&\leq \|x_{n_{j_1}} - q\| + \alpha_{n_{j_1}}(LW + L) \\
&\leq \|x_{n_{j_1}} - q\| + \frac{\varepsilon}{4}
\end{aligned}$$

thus $\|x_{n_{j_1}} - q\| > \|x_{n_{j_1}+1} - q\| - \frac{\varepsilon}{4} \geq \frac{3\varepsilon}{4}$. By (6), we obtain

$$\begin{aligned}
\|y_{n_{j_1}} - q\| &\geq (1 - \beta_{n_{j_1}})\|x_{n_{j_1}} - q\| - \beta_{n_{j_1}}(L\|x_{n_{j_1}} - q\| + L) \\
&= \|x_{n_{j_1}} - q\| - (\beta_{n_{j_1}} + \beta_{n_{j_1}}L)\|x_{n_{j_1}} - q\| - \beta_{n_{j_1}}L \\
&> \frac{3\varepsilon}{4} - (\beta_{n_{j_1}} + \beta_{n_{j_1}}L)W - \beta_{n_{j_1}}L \\
&> \frac{\varepsilon}{2}.
\end{aligned}$$

Thus, $\Phi(\|y_{n_{j_1}} - q\|)\|y_{n_{j_1}} - q\| > \Phi(\frac{\varepsilon}{2})\frac{\varepsilon}{2}$. Applying (16) and the above form, we obtain

$$\begin{aligned}
\varepsilon^2 &\leq \|x_{n_{j_1}+1} - q\|^2 \\
&\leq \|x_{n_{j_1}} - q\|^2 + 2\alpha_{n_{j_1}}(E_{n_{j_1}} - \Phi(\|y_{n_{j_1}} - q\|)\|y_{n_{j_1}} - q\|) \\
&< \varepsilon^2 + 2\alpha_{n_{j_1}}(\Phi(\frac{\varepsilon}{2})\frac{\varepsilon}{4} - \Phi(\frac{\varepsilon}{2})\frac{\varepsilon}{2}) \\
&= \varepsilon^2 - \alpha_{n_{j_1}}\Phi(\frac{\varepsilon}{2})\frac{\varepsilon}{2} \\
&\leq \varepsilon^2
\end{aligned}$$

contradiction. Hence the conclusion holds for $m = 1$. Assume now it holds for $m$. Following the above argument, we easily proves that it holds for $m + 1$. This shows that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q$ as $n \to \infty$, completing proof of Theorem 2.1. □

**Theorem 2.2.** *Let $E$ be a real uniformly smooth Banach space, and $T : E \to E$ is a continuous and generalized Lipschitzian $\Phi$-strongly pseudo-contractive mapping* *with* $\Phi(t) \to +\infty$ *as* $t \to +\infty$. *Let* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *be two real sequences in* $[0,1]$ *satisfying the following conditions: (i)* $\alpha_n, \beta_n \to 0$ *as* $n \to \infty$*; (ii)* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *The Ishikawa iterative sequence* $\{x_n\}_{n=0}^{\infty}$ *generated from an arbitrary* $x_0 \in E$ *by (IS)*

$$\begin{cases} y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 0. \end{cases} \tag{17}$$

*Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* $T$.

**Proof.** Using Lemma 1.5, we know that the mapping $T$ has unique fixed point, let $q$ denote the fixed point. Since $T$ is a continuous and generalized Lipschitzian $\Phi$-strongly pseudo-contractive mapping, then the conclusion of Theorem 2.2 follows exactly from Theorem 2.1. This completes the proof. □

**Acknowledgements:** The author is very grateful to the referees for careful reading of the original version of this paper and for some good suggestions.

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