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HOLOMORPHICALLY PROJECTIVE CURVATURE TENSORS

Mileva Prvanović

Mathematical Institute SANU, Knez Mihaila 35, 11001 Belgrade
P. O. Box 367, Serbia and Montenegro

Abstract. We examine the geodesic mapping of an almost Hermitian (almost para-Hermitian) manifold using the holomorphic curvature tensor and show that, if such manifold allows the geodesic mapping, then, besides the classical projective curvature tensor, there are two more invariant tensors.

1. INTRODUCTION

Let (M, g) and $(\overline{M}, \overline{g})$ be two Riemannian manifolds, $\dim M = \dim \overline{M} = m$ with the metrics g and $\overline{g}$ respectively. If we consider the mapping $f : M \mapsto \overline{M}$, both manifolds can be assigned to the coordinate system, general with respect to the mapping. This is the coordinate system in which the corresponding points $p \in M$ and $f(p) \in \overline{M}$ have the same coordinates $\{x^1, x^2, ..., x^m\}$. Thus we can write $(M, \overline{g})$ instead $(\overline{M}, \overline{g})$.

The diffeomorphism transforming all geodesics into the geodesics is said to be the geodesic mapping. It is well known that the necessary and the sufficient condition for the geodesic mapping of (M, g) onto $(M, \overline{g})$ is

$$\overline{\Gamma}^t_{ij} = \Gamma^t_{ij} + \delta^t_j \psi_i + \delta^t_i \psi_j \ , \tag{1}$$

where $\overline{\Gamma}$ and Γ are the Levi-Civita connections with respect to the metrics $\overline{g}$ and g respectively, and ψ_i is the gradient vector field. As for the curvature tensors $\overline{R}$ and R, they are related as follows

$$\overline{R}^t_{jhl} = R^t_{jhl} + \delta^t_l \psi_{jh} - \delta^t_h \psi_{jl} \tag{2}$$

where

$$\psi_{jh} = \nabla_j \psi_h - \psi_j \psi_h \ ,$$

and ∇ is the operator of the covariant derivative with respect to Γ. Also, it is well known that the projective curvature tensor

$$W^t_{ijh} = R^t_{ijh} - \frac{1}{m-1}(\delta^t_h \rho_{ij} - \delta^t_j \rho_{ih}) \tag{3}$$

where ρ_{ij} are the components of the Ricci tensor, is invariant with respect to the geodesic mapping.

If in the relation (1) ψ_i is identically zero, the geodesic mapping is said to be trivial. For example, the geodesic mapping of a Kähler manifold onto the Kähler manifold preserving the complex structure, under some additional conditions, is trivial. For such manifolds, also, the holomorphically projective mapping is defined.

A vector V at a point of a Kähler manifold and the vector $J(V)$, where J is the complex structure, are mutualy orthogonal and, consequently they are linearly independent. For a para-Kähler manifold, if V is not a null vector, V and $J(V)$ are also linearly independent. The plane element determined by V and $J(V)$ is called a holomorphic section. A curve of a Kähler (para-Kähler) manifold is a holomorphically planer curve if and only if the holomorphic sections determined by its tangent vectors are parallel along the curve itself. The diffeomorphism transforming all holomorphically planer curves into holomorphically planer curves, is said to be the holomorphically projective mapping. It is the natural generalization of the geodesic mapping. Such mappings were investigated by many authors and considered in many monographs and reviews (see, for example [5] and [9] and the references therein). Here we remind only that the necessary and the sufficient condition for the holomorphically projective mapping of a Kähler (para-Kähler) manifold, preserving the structure J, is

$$\overline{\Gamma}^t_{ij} = \Gamma^t_{ij} + \delta^t_i \theta_j + \delta^t_j \theta_i + \varepsilon \Big(J^t_i \theta_a J^a_j + J^t_j \theta_a J^a_i \Big) \ ,$$

while the curvature tensors are related as follows

$$\overline{R}^t_{jhl} = R^t_{jhl} + \delta^t_l \theta_{jh} - \delta^t_h \theta_{jl} - \varepsilon \Big(J^t_l \theta_{ja} J^a_h - J^t_h \theta_{ja} J^a_l - 2 J^t_j \theta_{ha} J^a_l \Big) \ , \tag{4}$$

where

$$\theta_{ij} = \nabla_j \theta_i - \theta_i \theta_j + \theta_a \theta_b J^a_i J^b_j \ ,$$

θ_i is a gradient and $\varepsilon = -1$ ($\varepsilon = +1$) for the Kähler (para-Kähler) manifold. The holomorphically projective curvature tensor

$$P^t_{jhl} = R^t_{jhl} - \frac{1}{2(n+1)} \Big[\delta^t_l \rho_{jh} - \delta^t_h \rho_{jl} - \varepsilon \Big(J^t_l \rho_{aj} J^a_h - J^t_h \rho_{ja} J^a_l - 2 J^t_j \rho_{ha} J^a_l \Big) \Big] \ , \tag{5}$$

where $2n = m = \dim M$, is invariant with respect to the holomorphically projective mapping. ([10], p. 265 for $\varepsilon = -1$, [6] for $\varepsilon = +1$).

In this note, we examine the geodesic mapping of an almost Hermitian (almost para-Hermitian) manifold, using the holomorphic curvature tensor. In the section 3, we obtain the corresponding invariant tensors. As a consequence, we get that for the Kähler (para-Kähler) manifolds, the tensor (5) can be obtained either as the invariant tensor with respect to the holomorphically projective mapping using the curvature tensor, or as the invariant tensor with respect to the geodesic mapping using the holomorphic curvature tensor. In the section 4, we give an example. In the section 5, we consider the holomorphically projective curvature tensors for an almost Hermitian (almost para-Hermitian) manifold satisfying the Rizza's Bianchi-type identity.

2. PRELIMINARIES

Let (M, g, J) be an almost Hermitian (almost para-Hermitian) manifold with metric g and the structure J, that is such that

$$J^2 = \varepsilon Id. \ , \qquad g(JX, JY) = -\varepsilon g(X, Y) \ , \qquad \varepsilon = \pm 1 \ ,$$

for all $X, Y \in T_p(M)$, where $T_p(M)$ is the tangent vector space of M at $p \in M$.

If $\varepsilon = -1$, J is the complex structure and (M, g, J) is an almost Hermitian manifold.

If $\varepsilon = +1$, J is the product structure and (M, g, J) is an almost para-Hermitian manifold.

In both cases, M is an even-dimensional orientable manifold. We put $\dim M = 2n$. In the case $\varepsilon = +1$, the signature of M is (n, n).

If $\nabla J = 0$, (M, g, J) is a Kähler (para-Kähler) manifold. As a consequence of $\nabla J = 0$, the Riemannian curvature tensor $R(X, Y, Z, W)$ satisfies the condition

$$R(X, Y, JZ, JW) = -\varepsilon R(X, Y, Z, W)\ . \tag{6}$$

If $\nabla J \neq 0$, (6) does not hold in general. Nevertheless, there exists for any almost Hermitian (almost para-Hermitian) manifold the algebraic curvature tensor, satisfying the condition of type (6). It is ([1], [7])

$$\begin{aligned}(HR)(X, Y, Z, W) \ &= \tfrac{1}{16}\Big\{3\Big[R(X, Y, Z, W) - \varepsilon R(X, Y, JZ, JW)\\ &\quad -\varepsilon R(JX, JY, Z, W) + R(JX, JY, JZ, JW)\Big]\\ &\quad +\varepsilon\Big[R(X, Z, JW, JY) + R(JX, JZ, W, Y)\\ &\quad +R(X, W, JY, JZ) + R(JX, JW, Y, Z)\\ &\quad -R(JX, Z, JW, Y) - R(X, JZ, W, JY)\\ &\quad -R(JX, W, Y, JZ) - R(X, JW, JY, Z)\Big]\Big\}.\end{aligned} \tag{7}$$

Namely, it is easy to see that

$$\begin{aligned}(HR)(X, Y, Z, W) \ &= -(HR)(Y, X, Z, W)\\ &= -(HR)(X, Y, W, Z)\\ &= (HR)(Z, W, X, Y),\end{aligned} \tag{8}$$

$$(HR)(X, Y, Z, W) + (HR)(Y, Z, X, W) + (HR)(Z, X, Y, W) = 0\ , \tag{9}$$

and that it also has the following properties

$$(HR)(X, Y, JZ, JW) = -\varepsilon(HR)(X, Y, Z, W)\ , \tag{10}$$

$$(HR)(X,JX,JX,X) = R(X,JX,JX,X)\ . \tag{11}$$

If (6) holds, then

$$(HR)(X,Y,Z,W) = R(X,Y,Z,W)\ . \tag{12}$$

The relation (11) shows that the holomorphic sectional curvatures with respect to R and HR are the same. This is the reason to name (7) *the holomorphic curvature tensor*.

Let $\{e_i\}$ be an orthonormal basis of $T_p(M)$. We define the Ricci tensor, associated to HR in the following way:

$$\rho(HR)(X,Y) = \sum_{i=1}^{2n}(HR)(e_i,X,Y,e_i)\ .$$

In view of (8),(9) and (10), we have

$$\rho(HR)(JX,JY) = -\varepsilon\rho(HR)(X,Y)\ , \qquad \rho(HR)(X,JY) = -\rho(HR)(JX,Y)\ .$$

Of course, in the case (12), we have

$$\rho(HR)(X,Y) = \rho(X,Y) = \sum_{i=1}^{2n} R(e_i,X,Y,e_i)\ .$$

With respect to some local coordinates, (7) can be expressed as follows

$$\begin{aligned} 16(HR)_{ijhl} \ =\ & 3\left[R_{ijhl} - \varepsilon R_{ijab}J^a_hJ^b_l - \varepsilon R_{abhl}J^a_iJ^b_j + R_{abcd}J^a_iJ^b_jJ^c_hJ^d_l\right] \\ & +\varepsilon\Big[R_{ihab}J^a_lJ^b_j + R_{ablj}J^a_iJ^b_h + R_{ilab}J^a_jJ^b_h + R_{abjh}J^a_iJ^b_l \\ & -R_{ahbj}J^a_iJ^b_l - R_{ialb}J^a_hJ^b_j - R_{aljb}J^a_iJ^b_h - R_{iabh}J^a_lJ^b_j\Big]\ , \end{aligned}$$

such that, transvecting with g^{it}, we have

$$\begin{aligned} 16(HR)^t_{jhl} \ =\ & 3\Big[R^t_{jhl} - \varepsilon R^t_{jab}J^a_hJ^b_l + \varepsilon R^a_{bhl}J^t_aJ^b_j - R^a_{bcd}J^t_aJ^b_jJ^c_hJ^d_l\Big] \\ & +\varepsilon\Big[R^t_{hab}J^a_lJ^b_j - R^a_{blj}J^t_aJ^b_h + R^t_{lab}J^a_jJ^b_h - R^a_{bjh}J^t_aJ^b_l \\ & +R^a_{hbj}J^t_aJ^b_l - R^t_{alb}J^a_hJ^b_j + R^a_{ljb}J^t_aJ^b_h - R^t_{abh}J^a_lJ^b_j\Big]. \end{aligned} \tag{13}$$

3. GEODESIC MAPPING

Let

$$f : (M, g, J) \mapsto (M, \overline{g}, J) \tag{14}$$

be the geodesic mapping. Thus, we suppose that the mapping preserves the structure.

For the manifold $(M, \overline{g}, J)$, the tensor (13) is

$$\begin{aligned} 16(H\overline{R})^t_{jhl} \;=\;& 3\left[\overline{R}^t_{jhl} - \varepsilon\overline{R}^t_{jab}J^a_hJ^b_l + \varepsilon\overline{R}^a_{bhl}J^t_aJ^b_j - \overline{R}^a_{bcd}J^t_aJ^b_jJ^c_hJ^d_l\right] \\ & +\varepsilon\Big[\overline{R}^t_{hab}J^a_lJ^b_j - \overline{R}^a_{blj}J^t_aJ^b_h + \overline{R}^t_{lab}J^a_jJ^b_h - \overline{R}^a_{bjh}J^t_aJ^b_l \\ & +\overline{R}^a_{hbj}J^t_aJ^b_l - \overline{R}^t_{alb}J^a_hJ^b_j + \overline{R}^a_{ljb}J^t_aJ^b_h - \overline{R}^t_{abh}J^a_lJ^b_j\Big] \;. \end{aligned} \tag{15}$$

Substituting (2), we get

$$8(H\overline{R})^t_{jhl} = 8(HR)^t_{jhl} + \delta^t_lQ_{jh} - \delta^t_hQ_{jl} - \varepsilon J^t_lQ_{aj}J^a_h + \varepsilon J^t_hQ_{ja}J^a_l + 2\varepsilon J^t_jQ_{ha}J^a_l, \tag{16}$$

where

$$Q_{ij} = \psi_{ij} - \varepsilon\psi_{ab}J^a_iJ^b_j \;,$$

Thus

$$Q_{aj}J^a_i = \psi_{aj}J^a_i - \psi_{ia}J^a_j \;,$$

and in view of $\psi_{ij} = \psi_{ji}$, we have

$$Q_{ij} = Q_{ji} \;, \qquad Q_{aj}J^a_i = -Q_{ai}J^a_j \;.$$

Contracting (16) with respect to t and l, we obtain

$$Q_{jh} = \frac{4}{n+1}\Big(\rho(H\overline{R})_{jh} - \rho(HR)_{jh}\Big) \;. \tag{17}$$

Substituting (17) into (16), we find

$$(H\underset{1}{\overline{W}})^t_{jhl} = (H\underset{1}{W})^t_{jhl}$$

where

$$\begin{aligned} (H\underset{1}{W})^t_{jhl} \;=\;& (HR)^t_{jhl} - \tfrac{1}{2(n+1)}\Big[\delta^t_l\rho(HR)_{jh} - \delta^t_h\rho(HR)_{jl} \\ & -\varepsilon\Big(J^t_l\rho(HR)_{ja}J^a_h - J^t_h\rho(HR)_{ja}J^a_l - 2J^t_j\rho(HR)_{ha}J^a_l\Big)\Big] \;, \end{aligned} \tag{18}$$

and the tensor $H\underset{1}{\overline{W}}$ is constructed in the same way, but with respect to the tensor $H\overline{R}$.

With respect to the geodesic mapping, $\overline{W}^t_{jhl} = W^t_{jhl}$, i.e.

$$\overline{R}^t_{jhl} - R^t_{jhl} = \frac{1}{2n-1}\Big(\delta^t_l E_{jh} - \delta^t_h E_{jl}\Big) \tag{19}$$

where

$$E_{jh} = \overline{\rho}_{jh} - \rho_{jh} \ , \tag{20}$$

Substituting (19) into the expression for

$$16(H\overline{R})^t_{jhl} - 16(HR)^t_{jhl} \ ,$$

we get

$$\begin{aligned} 8(2n-1) & \Big((H\overline{R})^t_{jhl} - (HR)^t_{jhl}\Big) \\ & = \delta^t_l\Big(E_{jh} - \varepsilon E_{ab}J^a_j J^b_h\Big) - \delta^t_h\Big(E_{jl} - \varepsilon E_{ab}J^a_j J^b_l\Big) \\ & -\varepsilon J^t_l\Big(E_{ja}J^a_h - E_{ha}J^a_j\Big) + \varepsilon J^t_h\Big(E_{ja}J^a_l - E_{la}J^a_j\Big) + 2\varepsilon J^t_j\Big(E_{ha}J^a_l - E_{la}J^a_h\Big) \ . \end{aligned}$$

This relation, in view of (20), can be rewritten in the form

$$\begin{aligned} (H\overline{R})^t_{jhl} & - \tfrac{1}{8(2n-1)}\Big[\delta^t_l\Big(\overline{\rho}_{jh} - \varepsilon\overline{\rho}_{ab}J^a_j J^b_h\Big) - \delta^t_h\Big(\overline{\rho}_{jl} - \varepsilon\overline{\rho}_{ab}J^a_j J^b_l\Big) \\ & -\varepsilon J^t_l\Big(\overline{\rho}_{ja}J^a_h - \overline{\rho}_{ha}J^a_j\Big) + \varepsilon J^t_h\Big(\overline{\rho}_{ja}J^a_l - \overline{\rho}_{la}J^a_j\Big) + 2\varepsilon J^t_j\Big(\overline{\rho}_{ha}J^a_l - \overline{\rho}_{la}J^a_h\Big)\Big] \\ = (HR)^t_{jhl} & - \tfrac{1}{8(2n-1)}\Big[\delta^t_l\Big(\rho_{jh} - \varepsilon\rho_{ab}J^a_j J^b_h\Big) - \delta^t_h\Big(\rho_{jl} - \varepsilon\rho_{ab}J^a_j J^b_l\Big) \\ & -\varepsilon J^t_l\Big(\rho_{ja}J^a_h - \rho_{ha}J^a_j\Big) + \varepsilon J^t_h\Big(\rho_{ja}J^a_l - \rho_{la}J^a_j\Big) + 2\varepsilon J^t_j\Big(\rho_{ha}J^a_l - \rho_{la}J^a_h\Big)\Big] \ . \end{aligned}$$

But this means that the tensor

$$\begin{aligned} (H\underset{2}{W})^t_{jhl} = (HR)^t_{jhl} & - \tfrac{1}{8(2n-1)}\Big[\delta^t_l\Big(\rho_{jh} - \varepsilon\rho_{ab}J^a_j J^b_h\Big) \\ & -\delta^t_h\Big(\rho_{jl} - \varepsilon\rho_{ab}J^a_j J^b_l\Big) - \varepsilon J^t_l\Big(\rho_{ja}J^a_h - \rho_{ha}J^a_j\Big) \\ & +\varepsilon J^t_h\Big(\rho_{ja}J^a_l - \rho_{la}J^a_j\Big) + 2\varepsilon J^t_j\Big(\rho_{ha}J^a_l - \rho_{la}J^a_h\Big)\Big] \end{aligned} \tag{21}$$

is invariant with respect to the geodesic mapping (14). Thus, we can state:

Theorem. *If an almost Hermitian (almost para-Hermitian) manifold (M, g, J) allows the geodesic mapping (14), then there are, besides (3), two more invariant tensors: (18) and (21).*

The tensors (18) and (21) are constructed using the holomorphic curvature tensor (7). Thus, it is reasonable to name them *the first and the second holomorphically projective curvature tensors*, respectively.

For the Kähler (para-Kähler) manifold, $HR = R$, $\rho(HR) = \rho$ because of which (18) reduces to (5). So we have

Proposition. *For the Kähler (para-Kähler) manifold, the holomorphically projective curvature tensor (5) can be obtained either as the invariant with respect to the holomorphically projective mapping, or as the invariant with respect to the geodesic mapping, but using the holomorphic curvature tensor (7).*

Remark. For the Kähler (para-Kähler) manifold, the tensor (5) is also invariant with respect to the holomorphically projective mapping using the holomorphic curvature tensor. Namely, substituting (4) into (15) and proceeding as before, we get (5).

For the Kähler (para-Kähler) manifold, we shall write

$$(H\underset{1}{W})^t_{jhl} = P^t_{jhl} = \underset{1}{P}{}^t_{jhl} \ , \qquad (H\underset{2}{W})^t_{jhl} = \underset{2}{P}{}^t_{jhl} \ .$$

Because of

$$-\varepsilon\rho_{ab}J^a_jJ^b_h = \rho_{jh} \ , \qquad \rho_{ja}J^a_h = -\rho_{ha}J^a_j \ ,$$

(21) becomes

$$\underset{2}{P}{}^t_{jhl} = R^t_{jhl} \quad -\tfrac{1}{4(2n-1)}\Big[\delta^t_l\rho_{jh} - \delta^t_h\rho_{jl} - \varepsilon\Big(J^t_l\rho_{ja}J^a_h - J^t_h\rho_{ja}J^a_l - 2J^t_j\rho_{ha}J^a_l\Big)\Big] \ . \qquad (22)$$

4. EXAMPLE

Let us consider the manifold M endowed with the metric

$$ds^2 = (dx^1)^2 + (x^1)^2\widetilde{g}_{\alpha\beta}dx^\alpha dx^\beta \ , \qquad (23)$$

where $\widetilde{g}_{\alpha\beta}dx^\alpha dx^\beta$, $\frac{\partial\widetilde{g}_{\alpha\beta}}{\partial x^1} = 0$, $\alpha, \beta, \gamma, \delta = 2, ..., 2n$ is the metric of a Sasakian manifold $\widetilde{M}$. From now on we shall use the tilde ($\sim$) to mark the objects of $\widetilde{M}$. The Levi-Civita connection of the metric (23) has the components

$$\Gamma^1_{11} = \Gamma^1_{\alpha 1} = \Gamma^\alpha_{11} = 0 \ ,$$

$$\Gamma^1_{\alpha\beta} = -x^1 \widetilde{g}_{\alpha\beta} \ , \qquad \Gamma^\alpha_{\beta 1} = \frac{1}{x^1}\delta^\alpha_\beta \ , \qquad \Gamma^\alpha_{\beta\gamma} = \widetilde{\Gamma}^\alpha_{\beta\gamma} \ .$$

Let $(\varphi,\ \xi,\ \eta,\ \widetilde{g})$ be the Sasakian structure on $\widetilde{M}$ and let us put

$$J^1_1 = 0 \ , \qquad J^\alpha_1 = \frac{1}{x^1}\xi^\alpha \ , \qquad J^1_\alpha = -x^1\eta_\alpha \ , \qquad J^\alpha_\beta = \varphi^\alpha_\beta \ .$$

Then $J^i_j J^j_k = -\delta^i_k$ and $\nabla J = 0$, that is (M, g, J), where g is the metric (23), is a Kähler manifold. Mikeš ([2], [3], [4]) proved that this Kähler manifold allows the geodesic mapping onto the conformally Kähler manifold $(M, \overline{g}, J)$, where $\overline{g}$ is the metric

$$d\overline{s}^2 = L[(dx^1)^2 + (x^1)^2 \widetilde{g}_{\alpha\beta} dx^\alpha dx^\beta] \tag{24}$$

and

$$L = \frac{1}{[\frac{A}{2}(x^1)^2 + B]^2} \ , \quad A, B = const \ .$$

We shall show that

$$W^t_{jhl} = \overline{W}^t_{jhl} \ , \quad \underset{1}{P}{}^t_{jhl} = (H\underset{1}{\overline{W}})^t_{jhl} \ , \qquad \underset{2}{P}{}^t_{jhl} = (H\underset{2}{\overline{W}})^t_{jhl} \ , \tag{25}$$

where the bar $(-)$ denotes the objects with respect to the metric (24). The objects with respect to the metric (23) will be without any mark.

First we note that the components of the curvature tensor R^t_{jhl} and the Ricci tensor ρ_{ij} of the metric (23) are all zero except

$$R^\delta_{\alpha\beta\gamma} = \widetilde{R}^\delta_{\alpha\beta\gamma} - (\delta^\delta_\gamma \widetilde{g}_{\alpha\beta} - \delta^\delta_\beta \widetilde{g}_{\alpha\gamma}) \ ,$$

$$\rho_{\alpha\beta} = \widetilde{\rho}_{\alpha\beta} - 2(n-1)\widetilde{g}_{\alpha\beta} \ .$$

As for the metric (24), different from zero are the components

- of the curvature tensor:

$$\overline{R}^1_{\delta\alpha 1} = P\widetilde{g}_{\delta\alpha} \ , \qquad \overline{R}^\delta_{1\alpha 1} = -\frac{P}{(x^1)^2}\delta^\delta_\alpha \ ,$$

$$\overline{R}^\tau_{\delta\alpha\beta} = \widetilde{R}^\tau_{\delta\alpha\beta} + (P-1)(\delta^\tau_\beta \widetilde{g}_{\delta\alpha} - \delta^\tau_\alpha \widetilde{g}_{\delta\beta}) \ ;$$

- of the Ricci tensor:

$$\overline{\rho}_{11} = \frac{2n-1}{(x^1)^2}P \ , \qquad \overline{\rho}_{\alpha\beta} = \widetilde{\rho}_{\alpha\beta} - 2(n-1)\widetilde{g}_{\alpha\beta} + (2n-1)P\widetilde{g}_{\alpha\beta} \ ;$$

- of the holomorphic curvature tensor (13):

$$\begin{aligned}
(H\overline{R})^1_{\delta\alpha 1} &= \tfrac{P}{4}(\widetilde{g}_{\delta\alpha} + 3\eta_\delta\eta_\alpha) \ , \\
(H\overline{R})^1_{\delta\alpha\beta} &= \tfrac{P}{4}x^1\Big[\widetilde{g}_{\delta\gamma}(\eta_\alpha\varphi^\gamma_\beta - \eta_\beta\varphi^\gamma_\alpha) + 2\eta_\delta\widetilde{g}_{\alpha\gamma}\varphi^\gamma_\beta\Big] \ , \\
(H\overline{R})^\tau_{1\alpha 1} &= -\tfrac{P}{4(x^1)^2}(\delta^\tau_\alpha + 3\xi^\tau\eta_\alpha) \ , \\
(H\overline{R})^\tau_{\delta\alpha 1} &= \tfrac{P}{4x^1}(\widetilde{g}_{\delta\gamma}\varphi^\gamma_\alpha\xi^\tau - \varphi^\tau_\alpha\eta_\delta - 2\varphi^\tau_\delta\eta_\alpha) \ , \\
(H\overline{R})^\tau_{1\alpha\beta} &= \tfrac{P}{4x^1}(\varphi^\tau_\alpha\eta_\beta - \varphi^\tau_\beta\eta_\alpha - 2\xi^\tau\widetilde{g}_{\alpha\gamma}\varphi^\gamma_\beta) \ , \\
(H\overline{R})^\tau_{\delta\alpha\beta} &= \widetilde{R}^\tau_{\delta\alpha\beta} - (\delta^\tau_\beta\widetilde{g}_{\delta\alpha} - \delta^\tau_\alpha\widetilde{g}_{\delta\beta}) + \\
&\quad + \tfrac{P}{4}\Big(\delta^\tau_\beta\widetilde{g}_{\delta\alpha} - \delta^\tau_\alpha\widetilde{g}_{\beta\delta} + \varphi^\tau_\beta\widetilde{g}_{\delta\varepsilon}\varphi^\varepsilon_\alpha - \varphi^\tau_\alpha\widetilde{g}_{\delta\varepsilon}\varphi^\varepsilon_\beta - 2\varphi^\tau_\delta\widetilde{g}_{\alpha\varepsilon}\varphi^\varepsilon_\beta\Big) \ ;
\end{aligned}$$

- of the Ricci tensor $\rho(H\overline{R})$:

$$\begin{aligned}
\rho(H\overline{R})_{11} &= \tfrac{n+1}{2}\cdot\tfrac{P}{(x^1)^2} \ , \\
\rho(H\overline{R})_{\delta\alpha} &= \widetilde{\rho}_{\delta\alpha} - 2(n-1)\widetilde{g}_{\delta\alpha} + \tfrac{n+1}{2}P\widetilde{g}_{\delta\alpha} \ ,
\end{aligned}$$

where

$$P = \frac{2A(x^1)^2}{\frac{A}{2}(x^1)^2 + B} - \frac{A^2(x^1)^4}{\Big[\frac{A}{2}(x^1)^2 + B\Big]^2} \ .$$

We also note that for the Sasakian manifold $\widetilde{M}$:

$$\begin{aligned}
rank\,(\varphi^\alpha_\beta) &= 2n-2 \ , \quad \xi^\alpha\eta_\alpha = 1 \ , \quad \varphi^\alpha_\beta\xi^\beta = 0 \ , \quad \varphi^\alpha_\beta\eta_\alpha = 0 \ , \\
\varphi^\alpha_\delta\varphi^\delta_\beta &= -\delta^\alpha_\beta + \xi^\alpha\eta_\beta \ , \quad \widetilde{g}_{\alpha\beta}\xi^\beta = \eta_\alpha \, , \\
\widetilde{g}_{\delta\gamma}\varphi^\delta_\alpha\varphi^\gamma_\beta &= \widetilde{g}_{\alpha\beta} - \eta_\alpha\eta_\beta \ , \quad \widetilde{g}_{\alpha\delta}\varphi^\delta_\beta = -\widetilde{g}_{\beta\delta}\varphi^\delta_\alpha \, , \\
\widetilde{R}_{\gamma\beta\alpha\pi}\varphi^\alpha_\sigma\varphi^\pi_\lambda &= \widetilde{R}_{\gamma\beta\sigma\lambda} + \widetilde{g}_{\gamma\sigma}\widetilde{g}_{\beta\lambda} - \widetilde{g}_{\gamma\lambda}\widetilde{g}_{\beta\sigma} - \widetilde{g}_{\gamma\varepsilon}\varphi^\varepsilon_\sigma\widetilde{g}_{\beta\mu}\varphi^\mu_\lambda + \widetilde{g}_{\gamma\varepsilon}\varphi^\varepsilon_\lambda\widetilde{g}_{\beta\mu}\varphi^\mu_\sigma \ , \\
\widetilde{\rho}_{\alpha\beta}\varphi^\alpha_\sigma\varphi^\beta_\nu &= \widetilde{\rho}_{\sigma\nu} - 2(n-1)\eta_\sigma\eta_\nu \ , \quad \widetilde{\rho}_{\alpha\sigma}\varphi^\sigma_\beta = -\widetilde{\rho}_{\beta\sigma}\varphi^\sigma_\alpha .
\end{aligned}$$

Then, after some calculation, we find the following components different from zero

- for the projective curvature tensor (3):

$$\begin{aligned}
W^1_{\beta\alpha 1} &= \overline{W}^1_{\beta\alpha 1} = -\tfrac{1}{2n-1}[\widetilde{\rho}_{\alpha\beta} - 2(n-1)\widetilde{g}_{\alpha\beta}] \ , \\
W^\delta_{\alpha\beta\gamma} &= \overline{W}^\delta_{\alpha\beta\gamma} = \widetilde{R}^\delta_{\alpha\beta\gamma} - \tfrac{1}{2n-1}\Big[\delta^\delta_\gamma(\widetilde{\rho}_{\alpha\beta} + \widetilde{g}_{\alpha\beta}) - \delta^\delta_\beta(\widetilde{\rho}_{\alpha\gamma} + \widetilde{g}_{\alpha\gamma})\Big] \ ;
\end{aligned}$$

- for the first holomorphically projective curvature tensor:

$$
\begin{aligned}
\underset{1}{P}{}^{1}_{\delta\alpha 1} &= (H\underset{1}{\overline{W}})^{1}_{\delta\alpha 1} = -\tfrac{1}{2(n+1)}[\widetilde{\rho}_{\delta\alpha} - 2(n-1)\widetilde{g}_{\delta\alpha}]\ ,\\
\underset{1}{P}{}^{1}_{\delta\alpha\beta} &= (H\underset{1}{\overline{W}})^{1}_{\delta\alpha\beta} = \tfrac{x^1}{2(n+1)}\Big[(\eta_\beta\varphi^{\varepsilon}_{\alpha} - \eta_\alpha\varphi^{\varepsilon}_{\beta})\Big(\widetilde{\rho}_{\varepsilon\delta} - 2(n-1)\widetilde{g}_{\varepsilon\delta}\Big)\\
&\qquad -2\eta_\delta\varphi^{\varepsilon}_{\beta}\Big(\widetilde{\rho}_{\varepsilon\alpha} - 2(n-1)\widetilde{g}_{\varepsilon\alpha}\Big)\Big]\ ,\\
\underset{1}{P}{}^{\tau}_{\delta\alpha 1} &= (H\underset{1}{\overline{W}})^{\tau}_{\delta\alpha 1} = -\tfrac{1}{2(n+1)x^1}\xi^\tau\varphi^{\sigma}_{\alpha}\Big(\widetilde{\rho}_{\sigma\delta} - 2(n-1)\widetilde{g}_{\sigma\delta}\Big)\ ,\\
\underset{1}{P}{}^{\tau}_{1\alpha\beta} &= (H\underset{1}{\overline{W}})^{\tau}_{1\alpha\beta} = \tfrac{1}{(n+1)x^1}\xi^\tau\varphi^{\sigma}_{\beta}\Big(\widetilde{\rho}_{\sigma\alpha} - 2(n-1)\widetilde{g}_{\sigma\alpha}\Big)\ ,\\
\underset{1}{P}{}^{\tau}_{\delta\alpha\beta} &= (H\underset{1}{\overline{W}})^{\tau}_{\delta\alpha\beta} = \widetilde{R}^{\tau}_{\delta\alpha\beta} - \tfrac{1}{2(n+1)}\Big(\delta^{\tau}_{\beta}\widetilde{\rho}_{\delta\alpha} - \delta^{\tau}_{\alpha}\widetilde{\rho}_{\delta\beta}\Big) - \tfrac{2}{n+1}\Big(\delta^{\tau}_{\beta}\widetilde{g}_{\delta\alpha} - \delta^{\tau}_{\alpha}\widetilde{g}_{\delta\beta}\Big)\\
&\qquad -\tfrac{1}{2(n+1)}\Big[\Big(\varphi^{\tau}_{\beta}\varphi^{\sigma}_{\alpha} - \varphi^{\tau}_{\alpha}\varphi^{\sigma}_{\beta}\Big)\Big(\widetilde{\rho}_{\sigma\delta} - 2(n-1)\widetilde{g}_{\sigma\delta}\Big)\\
&\qquad -2\varphi^{\tau}_{\delta}\varphi^{\sigma}_{\beta}\Big(\widetilde{\rho}_{\sigma\alpha} - 2(n-1)\widetilde{g}_{\sigma\alpha}\Big)\Big]\ ;
\end{aligned}
$$

- for the second holomorphically projective curvature tensor:

$$
\begin{aligned}
\underset{2}{P}{}^{1}_{\delta\alpha 1} &= (H\underset{2}{\overline{W}})^{1}_{\delta\alpha 1} = -\tfrac{1}{4(2n-1)}[\widetilde{\rho}_{\delta\alpha} - 2(n-1)\widetilde{g}_{\delta\alpha}]\ ,\\
\underset{2}{P}{}^{1}_{\delta\alpha\beta} &= (H\underset{2}{\overline{W}})^{1}_{\delta\alpha\beta} = \tfrac{x^1}{4(2n-1)}\Big[(\eta_\beta\varphi^{\sigma}_{\alpha} - \eta_\alpha\varphi^{\sigma}_{\beta})\Big(\widetilde{\rho}_{\sigma\delta} - 2(n-1)\widetilde{g}_{\sigma\delta}\Big)\\
&\qquad -2\eta_\delta\varphi^{\sigma}_{\beta}\Big(\widetilde{\rho}_{\sigma\alpha} - 2(n-1)\widetilde{g}_{\sigma\alpha}\Big)\Big]\ ,\\
\underset{2}{P}{}^{\tau}_{\delta\alpha 1} &= (H\underset{2}{\overline{W}})^{\tau}_{\delta\alpha 1} = -\tfrac{1}{4(2n-1)x^1}\xi^\tau\varphi^{\sigma}_{\alpha}\Big(\widetilde{\rho}_{\sigma\delta} - 2(n-1)\widetilde{g}_{\sigma\delta}\Big)\ ,\\
\underset{2}{P}{}^{\tau}_{1\alpha\beta} &= (H\underset{2}{\overline{W}})^{\tau}_{1\alpha\beta} = \tfrac{1}{2(2n-1)x^1}\xi^\tau\varphi^{\sigma}_{\beta}\Big(\widetilde{\rho}_{\sigma\alpha} - 2(n-1)\widetilde{g}_{\sigma\alpha}\Big)\ ,\\
\underset{2}{P}{}^{\tau}_{\delta\alpha\beta} &= (H\underset{2}{\overline{W}})^{\tau}_{\delta\alpha\beta} = \widetilde{R}^{\tau}_{\delta\alpha\beta} - \tfrac{3n-1}{2(2n-1)}\Big(\delta^{\tau}_{\beta}\widetilde{g}_{\delta\alpha} - \delta^{\tau}_{\alpha}\widetilde{g}_{\delta\beta}\Big) - \tfrac{1}{4(2n-1)}\Big(\delta^{\tau}_{\beta}\widetilde{\rho}_{\delta\alpha} - \delta^{\tau}_{\alpha}\widetilde{\rho}_{\delta\beta}\Big)\\
&\qquad -\tfrac{1}{4(2n-1)}\Big[\Big(\varphi^{\tau}_{\beta}\varphi^{\sigma}_{\alpha} - \varphi^{\tau}_{\alpha}\varphi^{\sigma}_{\beta}\Big)\Big(\widetilde{\rho}_{\sigma\delta} - 2(n-1)\widetilde{g}_{\sigma\delta}\Big)\\
&\qquad -2\varphi^{\tau}_{\delta}\varphi^{\sigma}_{\beta}\Big(\widetilde{\rho}_{\sigma\alpha} - 2(n-1)\widetilde{g}_{\sigma\alpha}\Big)\Big]\ .
\end{aligned}
$$

5. MANIFOLDS SATISFYING THE BIANCHI-TYPE IDENTITY

If an almost Hermitian (para-Hermitian) manifold does not satisfy the condition $HR = R$, the holomorphically projective curvature tensors (18) and (21) do not reduce to the forms (5) and (22) respectively. But in some cases, the tensors (18) and (21) can be expressed using the tensors (5) and (22). To show this, we consider (M, g, J) satisfying the condition

$$\begin{aligned} &R(X,Y,JZ,JW)+R(X,Z,JW,JY)+R(X,W,JY,JZ)\\ &\qquad +R(JX,JY,Z,W)+R(JX,JZ,W,Y)+R(JX,JW,Y,Z)=0\ . \end{aligned} \tag{26}$$

This condition was introduced by G. B. Rizza [8] and was called the Bianchi-type identity by him.

Putting into (26) $JX,\ JY$ instead of $X,\ Y$ respectively, we find

$$\begin{aligned} &R(JX,JY,JZ,JW)+\varepsilon R(JX,Z,JW,Y)+\varepsilon R(JX,W,Y,JZ)\\ &\qquad +R(X,Y,Z,W)+\varepsilon R(X,JZ,W,JY)+\varepsilon R(X,JW,JY,Z)=0\ . \end{aligned} \tag{27}$$

In view of (26) and (27), (7) becomes

$$\begin{aligned} (HR)(X,Y,Z,W)\ &=\tfrac{1}{4}\Big[R(X,Y,Z,W)-\varepsilon R(X,Y,JZ,JW)\\ &\qquad -\varepsilon R(JX,JY,Z,W)+R(JX,JY,JZ,JW)\Big]. \end{aligned} \tag{28}$$

It is easy to check that (28) satisfies the identities (8) and (10), while it satisfies the Bianchi identity (9) if and only if (26) holds.

For an (M,g,J), besides the Ricci tensor $\rho(X,Y)=\sum_{i=1}^{2n}R(e_i,X,Y,e_i)$, we consider also the $*$ -Ricci tensor

$$\rho^*(X,Y)=\sum_{i=1}^{2n}R(e_i,X,JY,Je_i)=\frac{1}{2}\sum_{i=1}^{2n}R(Je_i,e_i,X,JY)\ .$$

Then, putting $X=W=e_i$ into (27) and summing up, we obtain

$$\rho(Y,Z)-\varepsilon\rho(JY,JZ)=-\varepsilon\Big(\rho^*(Y,Z)+\rho^*(Z,Y)\Big)\ . \tag{29}$$

On the other hand, putting into (28) $X=W=e_i$ and summing up, we find

$$\rho(HR)(Y,Z)=\frac{1}{4}\Big(\rho(Y,Z)-\varepsilon\rho^*(Z,Y)-\varepsilon\rho^*(Y,Z)-\varepsilon\rho(JY,JZ)\Big)\ ,$$

which, in view of (29), becomes

$$\rho(HR)(Y,Z)=\frac{1}{2}\Big(\rho(Y,Z)-\varepsilon\rho(JY,JZ)\Big)\ . \tag{30}$$

From (30), it follows

$$\rho(HR)(Y,JZ)=\frac{1}{2}\Big(\rho(Y,JZ)-\rho(JY,Z)\Big)\ . \tag{31}$$

The relation (18) can be rewritten in the form

$$\begin{aligned}(H\underset{1}{W})(X,Y,Z,U) = \ & (HR)(X,Y,Z,U)\\ & -\tfrac{1}{2(n+1)}\Big[g(X,U)\rho(HR)(Y,Z) - g(X,Z)\rho(HR)(Y,U)\\ & -\varepsilon\Big(g(X,JU)\rho(HR)(Y,JZ) - g(X,JZ)\rho(HR)(Y,JU)\\ & -2g(X,JY)\rho(HR)(Z,JU)\Big)\Big]\ ,\end{aligned}$$

such that, using (28), (30) and (31), we get

$$\begin{aligned}(H\underset{1}{W})(X,Y,Z,U) = \ & \tfrac{1}{4}\Big[R(X,Y,Z,U) - \varepsilon R(X,Y,JZ,JU)\\ & -\varepsilon R(JX,JY,Z,U) + R(JX,JY,JZ,JU)\Big]\\ & -\tfrac{1}{4(n+1)}\Big\{g(X,U)\Big[\rho(Y,Z) - \varepsilon\rho(JY,JZ)\Big]\\ & -g(X,Z)\Big[\rho(Y,U) - \varepsilon\rho(JY,JU)\Big]\\ & -\varepsilon g(X,JU)\Big[\rho(Y,JZ) - \rho(JY,Z)\Big]\\ & +\varepsilon g(X,JZ)\Big[\rho(Y,JU) - \rho(JY,U)\Big]\\ & +2\varepsilon g(X,JY)\Big[\rho(Z,JU) - \rho(JZ,U)\Big]\Big\}.\end{aligned} \tag{32}$$

On the other hand, (5) can be rewritten in the form

$$\begin{aligned}\underset{1}{P}(X,Y,Z,U) = R(X,Y,Z,U) - \tfrac{1}{2(n+1)}\Big(&g(X,U)\rho(Y,Z) - g(X,Z)\rho(Y,U) -\\ & -\varepsilon g(X,JU)\rho(Y,JZ) + \varepsilon g(X,JZ)\rho(Y,JU) + 2\varepsilon g(X,JY)\rho(Z,JU)\Big),\end{aligned}$$

from which we obtain

$$\begin{aligned}\underset{1}{P}(X,Y,Z,U) - \varepsilon\underset{1}{P}(X,Y,JZ,JU) - \varepsilon\underset{1}{P}(JX,JY,Z,U) + \underset{1}{P}(JX,JY,JZ,JU)&\\ = R(X,Y,Z,U) - \varepsilon R(X,Y,JZ,JU)&\\ -\varepsilon R(JX,JY,Z,U) + R(JX,JY,JZ,JU)&\\ -\tfrac{1}{n+1}\Big[g(X,U)\Big(\rho(Y,Z) - \varepsilon\rho(JY,JZ)\Big)&\\ -g(X,Z)\Big(\rho(Y,U) - \varepsilon\rho(JY,JU)\Big)&\\ -\varepsilon g(X,JU)\Big(\rho(Y,JZ) - \rho(JY,Z)\Big)&\\ +\varepsilon g(X,JZ)\Big(\rho(Y,JU) - \rho(JY,U)\Big)&\\ +2\varepsilon g(X,JY)\Big(\rho(Z,JU) - \rho(JZ,U)\Big)\Big].&\end{aligned} \tag{33}$$

The relations (32) and (33) show that

$$\begin{aligned}(H\underset{1}{W})(X,Y,Z,U) = {} & \tfrac{1}{4}\Big(\underset{1}{P}(X,Y,Z,U) - \varepsilon \underset{1}{P}(X,Y,JZ,JU) \\ & -\varepsilon \underset{1}{P}(JX,JY,Z,U) + \underset{1}{P}(JX,JY,JZ,JU)\Big).\end{aligned} \tag{34}$$

In a similar way we can see that

$$\begin{aligned}(H\underset{2}{W})(X,Y,Z,U) = {} & \tfrac{1}{4}\Big(\underset{2}{P}(X,Y,Z,U) - \varepsilon \underset{2}{P}(X,Y,JZ,JU) \\ & -\varepsilon \underset{2}{P}(JX,JY,Z,U) + \underset{2}{P}(JX,JY,JZ,JU)\Big).\end{aligned} \tag{35}$$

Thus, we can state

Theorem. *If an almost Hermitian (almost para-Hermitian) manifold satisfies the Rizza's Bianchi-type identity (26), then (34) and (35) hold.*

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