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GENERALIZED PLANE WAVE MANIFOLDS[1]

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Abstract. We show that generalized plane wave manifolds are complete, strongly geodesically convex, Osserman, Szabó, and Ivanov–Petrova. We show their holonomy groups are nilpotent and that all the local Weyl scalar invariants of these manifolds vanish. We construct isometry invariants on certain families of these manifolds which are not of Weyl type. Given k, we exhibit manifolds of this type which are k-curvature homogeneous but not locally homogeneous. We also construct a manifold which is weakly 1-curvature homogeneous but not 1-curvature homogeneous.

1. INTRODUCTION

We begin by introducing some notational conventions. Let $\mathcal{M} := (M, g)$ where g is a pseudo-Riemannian metric of signature (p, q) on smooth manifold M of dimension $m := p + q$.

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1.1. GEODESICS

We say that $\mathcal{M}$ is *complete* if all geodesics extend for infinite time and that $\mathcal{M}$ is *strongly geodesically convex* if there exists a unique geodesic between any two points of M; if $\mathcal{M}$ is complete and strongly geodesically convex, then the exponential map is a diffeomorphism from T_PM to M for any $P \in M$.

1.2. SCALAR WEYL INVARIANTS

Let $\nabla^k R$ be the k^{th} covariant derivative of the curvature operator defined by the Levi-Civita connection. Let $x := (x_1, ..., x_m)$ be local coordinates on M. Expand

$$\nabla_{\partial_{x_{j_1}}} ... \nabla_{\partial_{x_{j_l}}} R(\partial_{x_{i_1}}, \partial_{x_{i_2}})\partial_{x_{i_3}} = R_{i_1i_2i_3}{}^{i_4}{}_{;j_1...j_l}\partial_{x_{i_4}} \tag{1.a}$$

where we adopt the Einstein convention and sum over repeated indices. Scalar invariants of the metric can be formed by using the metric tensors g^{ij} and g_{ij} to fully contract all indices. For example, the scalar curvature τ, the norm of the Ricci tensor $|\rho|^2$, and the norm of the full curvature tensor $|R|^2$ are given by

$$\begin{aligned} &\tau := g^{ij}R_{kij}{}^{k}, \\ &|\rho|^2 := g^{i_1j_1}g^{i_2j_2}R_{ki_1j_1}{}^{k}R_{li_2j_2}{}^{l}, \quad \text{and} \\ &|R|^2 := g^{i_1j_1}g^{i_2j_2}g^{i_3j_3}g_{i_4j_4}R_{i_1i_2i_3}{}^{i_4}R_{j_1j_2j_3}{}^{j_4}. \end{aligned} \tag{1.b}$$

Such invariants are called *Weyl invariants*; if all possible such invariants vanish, then $\mathcal{M}$ is said to be *VSI* (vanishing scalar invariants). We refer to Pravda, Pravdová, Coley, and Milson [25] for a further discussion.

1.3. NATURAL OPERATORS DEFINED BY THE CURVATURE TENSOR

If ξ is a tangent vector, then the *Jacobi operator* $J(\xi)$ and the *Szabó operator* $\mathcal{S}(\xi)$ are the self-adjoint linear maps which are defined by:

$$J(\xi) : x \to R(x, \xi)\xi \quad \text{and} \quad \mathcal{S}(\xi) : x \to \nabla_\xi R(x, \xi)\xi\,.$$

Similarly if $\{e_1, e_2\}$ is an oriented orthonormal basis for an oriented spacelike (resp. timelike) 2-plane π, the *skew-symmetric curvature operator* $\mathcal{R}(\pi)$ is defined by:

$$\mathcal{R}(\pi) : x \to R(e_1, e_2)x\,.$$

1.4. OSSERMAN, IVANOV–PETROVA, AND SZABÓ MANIFOLDS

We say that $\mathcal{M}$ is *spacelike Osserman* (resp. *timelike Osserman*) if the eigenvalues of J are constant on the pseudo-sphere bundles of unit spacelike (resp. timelike) tangent vectors. The notions *spacelike Szabó*, *timelike Szabó*, *spacelike Ivanov–Petrova*, and *timelike Ivanov–Petrova* are defined similarly. Suppose that $p \geq 1$ and $q \geq 1$ so the conditions timelike Osserman and spacelike Osserman are both non-trivial. One can then use analytic continuation to see these two conditions are equivalent. Similarly, spacelike Szabó and timelike Szabó are equivalent notions if $p \geq 1$ and $q \geq 1$. Finally, spacelike Ivanov–Petrova and timelike Ivanov–Petrova are equivalent notions if $p \geq 2$ and $q \geq 2$. Thus we shall simply speak of *Osserman*, *Szabó*, or *Ivanov–Petrova* manifolds; see [8] for further details.

We shall refer to [6, 8] for a fuller discussion of geometry of the Riemann curvature tensor and shall content ourselves here with a very brief historical summary. Szabó [27] showed that a Riemannian manifold is Szabó if and only if it is a local symmetric space. Gilkey and Stavrov [14] showed that a Lorentzian manifold is Szabó if and only if it has constant sectional curvature.

Let $\mathcal{M}$ be a Riemannian manifold of dimension $m \neq 16$. Chi [2] and Nikolayevsky [18, 19, 20] showed that $\mathcal{M}$ is Osserman if and only if $\mathcal{M}$ either is flat or is locally isometric to a rank 1-symmetric space. This result settles in the affirmative for $m \neq 16$ a question originally posed by Osserman [24]. Work of Blažić, Bokan and Gilkey [1] and of García–Río, Kupeli and Vázquez-Abal [5] showed a Lorentzian manifold is Osserman if and only if it has constant sectional curvature.

Work of of Gilkey [7], of Gilkey, Leahy, Sadofsky [10], and of Nikolayevsky [21] showed that a Riemannian manifold is Ivanov–Petrova if and only if it either has constant sectional curvature or it is locally isometric to a warped product of an interval I with a metric of constant sectional curvature K where the warping function $f(t) = Kt^2+At+B$ is quadratic and non-vanishing for $t \in I$. This result was extended to the Lorentzian setting for $q \geq 11$ by Zhang [28]; results of Stavrov [26] provide some insight into the higher signature setting.

1.5. NILPOTENCY

The picture is very different when $p \geq 2$ and $q \geq 2$ and the classification of Osserman, Ivanov-Petrov, and Szabó manifolds is far from complete. The eigenvalue 0 plays a distinguished role. We say that $\mathcal{M}$ is *nilpotent Osserman* if 0 is the only eigenvalue of J or equivalently if $J(\xi)^m = 0$ for any tangent vector ξ; the notions *nilpotent Szabó* and *nilpotent Ivanov–Petrova* are defined similarly.

1.6. HOLONOMY

Let γ be a smooth curve in a pseudo-Riemannian manifold $\mathcal{M}$. Parallel translation along γ defines a linear isometry $P_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M$. The set of all such automorphisms where $\gamma(0) = \gamma(1)$ forms a group which is called the *holonomy group*; we shall denote this group by $\mathcal{H}_P(\mathcal{M})$.

1.7. GENERALIZED PLANE WAVE MANIFOLDS

Let $x = (x_1, ..., x_m)$ be the usual coordinates on $\mathbb{R}^m$. We say $\mathcal{M} := (\mathbb{R}^m, g)$ is a *generalized plane wave manifold* if

$$\nabla_{\partial_{x_i}}\partial_{x_j} = \textstyle\sum_{k>\max(i,j)} {\Gamma_{ij}}^k(x_1, ..., x_{k-1})\partial_{x_k} \,.$$

Let $\mathcal{T}$ be the nilpotent upper triangular group of all matrices of the form:

$$T=\left(\begin{array}{cccccc}1&*&*&...&*&*\\0&1&*&...&*&*\\0&0&1&...&*&*\\...&...&...&...&...&...\\0&0&0&...&1&*\\0&0&0&...&0&1\end{array}\right).$$

Theorem 1.1 *Let $\mathcal{M}$ be a generalized plane wave manifold. Then:*

1. *$\mathcal{M}$ is complete and strongly geodesically convex.*
2. $\nabla_{\partial_{x_{j_1}}}...\nabla_{\partial_{x_{j_\nu}}}R(\partial_{x_{i_1}},\partial_{x_{i_2}})\partial_{x_{i_3}}$
   $=\sum_{k>\max(i_1,i_2,i_3,j_1,...j_\nu)}R_{i_1i_2i_3}{}^k{}_{;j_1...j_\nu}(x_1,...,x_{k-1})\partial_{x_k}.$
3. *$\mathcal{M}$ is nilpotent Osserman, nilpotent Ivanov–Petrova, and nilpotent Szabó.*
4. *$\mathcal{M}$ is Ricci flat and Einstein.*
5. *$\mathcal{M}$ is VSI.*
6. *If γ is a smooth curve in $\mathbb{R}^m$, then $P_\gamma\partial_{x_i}=\partial_{x_i}+\sum_{j>i}a^j\partial_{x_j}$.*
7. *$\mathcal{H}_P(\mathcal{M})\subset\mathcal{T}$.*

We shall establish Theorem 1.1 in §2. Since all the scalar Weyl invariants vanish, one of the central difficulties in this subject is constructing isometry invariants of such manifolds. In the remaining sections of this paper, we present several other families of examples with useful geometric properties and exhibit appropriate local invariants which are not of Weyl type.

2. GEOMETRIC PROPERTIES OF GENERALIZED PLANE WAVE MANIFOLDS

2.1. GEODESICS

We begin the proof of Theorem 1.1 by examining the geodesic structure. Consider a curve $\gamma(t)=(x_1(t),...,x_m(t))$ in $\mathbb{R}^m$; γ is a geodesic if and only

$$\begin{aligned}&\ddot{x}_1(t)=0,\quad\text{and for}\quad k>1\quad\text{we have}\\&\ddot{x}_k(t)+\textstyle\sum_{i,j<k}\dot{x}_i(t)\dot{x}_j(t)\Gamma_{ij}{}^k(x_1,...,x_{k-1})(t)=0\,.\end{aligned}$$

We solve this system of equations recursively. Let $\gamma(t;\vec{x}^{\,0},\vec{x}^{\,1})$ be defined by

$$\begin{aligned}&x_1(t):=x_1^0+x_1^1t,\quad\text{and for}\quad k>1\\&x_k(t):=x_k^0+x_k^1t-\textstyle\int_0^t\int_0^s\sum_{i,j<k}\dot{x}_i(r)\dot{x}_j(r)\Gamma_{ij}{}^k(x_1,...,x_{k-1})(r)drds\,.\end{aligned}$$

Then $\gamma(0;\vec{x}^{\,0},\vec{x}^{\,1})=\vec{x}^{\,0}$ while $\dot{\gamma}(0;\vec{x}^{\,0},\vec{x}^{\,1})=\vec{x}^{\,1}$. Thus every geodesic arises in this way so all geodesics extend for infinite time. Furthermore, given $P,Q\in\mathbb{R}^n$, there is a unique geodesic $\gamma=\gamma_{P,Q}$ so that $\gamma(0)=P$ and $\gamma(1)=Q$ where

$$\begin{aligned}&x_1^0=P_1,\quad x_1^1=Q_1-P_1,\quad\text{and for}\quad k>1\quad\text{we have}\\&x_k^0=P_k,\quad x_k^1=Q_k-P_k+\textstyle\int_0^1\int_0^s\sum_{i,j<k}\dot{x}_i(r)\dot{x}_j(r)\Gamma_{ij}{}^k(x_1,...,x_{k-1})(r)drds\,.\end{aligned}$$

This establishes Assertion (1) of Theorem 1.1.

2.2. CURVATURE

We may expand

$$\begin{aligned}R_{ijk}{}^l\;=\;&\partial_{x_i}\Gamma_{jk}{}^l(x_1,...,x_{l-1})-\partial_{x_j}\Gamma_{ik}{}^l(x_1,...,x_{l-1})\\&+\Gamma_{in}{}^l(x_1,...,x_{l-1})\Gamma_{jk}{}^n(x_1,...,x_{n-1})\\&-\Gamma_{jn}{}^l(x_1,...,x_{l-1})\Gamma_{ik}{}^n(x_1,...,x_{n-1})\,.\end{aligned}$$

As we can restrict the quadratic sums to $n<l$, $R_{ijk}{}^l=R_{ijk}{}^l(x_1,...,x_{l-1})$. Suppose $l\leq k$. Then $\Gamma_{jk}{}^l=\Gamma_{ik}{}^l=0$. Furthermore for either of the quadratic terms to be non-zero, there must exist an index n with $k<n$ and $n<l$. This is not possible if $l\leq k$. Thus $R_{ijk}{}^l=0$ if $l\leq k$. Suppose $l\leq i$. Then

$$\partial_{x_i}\Gamma_{jk}{}^l(x_1,...,x_{l-1})=0\quad\text{and}\quad\partial_{x_j}\Gamma_{ik}{}^l=\partial_{x_j}0=0\,.$$

We have $\Gamma_{in}{}^l=0$. For the other quadratic term to be non-zero, there must exist an index n so $i<n$ and $n<l$. This is not possible if $l\leq i$. This shows $R_{ijk}{}^l=0$ if $l\leq i$; similarly $R_{ijk}{}^l=0$ if $l\leq j$.

This establishes Assertion (2) of Theorem 1.1 if $\nu = 0$, i.e. for the undifferentiated curvature tensor R. To study ∇R, we expand

$$\begin{aligned} R_{ijk}{}^n{}_{;l} &= \partial_l R_{ijk}{}^n(x_1, ..., x_{n-1}) && (2.a)\\ &\quad - \textstyle\sum_r R_{rjk}{}^n(x_1, ..., x_{n-1})\Gamma_{li}{}^r(x_1, ..., x_{r-1}) && (2.b)\\ &\quad - \textstyle\sum_r R_{irk}{}^n(x_1, ..., x_{n-1})\Gamma_{lj}{}^r(x_1, ..., x_{r-1}) && (2.c)\\ &\quad - \textstyle\sum_r R_{ijr}{}^n(x_1, ..., x_{n-1})\Gamma_{lk}{}^r(x_1, ..., x_{r-1}) && (2.d)\\ &\quad - \textstyle\sum_r R_{ijk}{}^r(x_1, ..., x_{r-1})\Gamma_{lr}{}^n(x_1, ..., x_{n-1})\,. && (2.e) \end{aligned}$$

To see $R_{ijk}{}^n{}_{;l} = R_{ijk}{}^n{}_{;l}(x_1, ..., x_{n-1})$, we observe that we have:

1. $i < r < n$ in (2.b);
2. $j < r < n$ in (2.c);
3. $k < r < n$ in (2.d);
4. $r < n$ in (2.e).

To show $R_{ijk}{}^n{}_{;l} = 0$ if $n \le \max(i, j, k, l)$, we note that

1. $\partial_l R_{ijk}{}^n(x_1, ..., x_{n-1}) = 0$ if $n \le \max(i, j, k, l)$ in (2.a);
2. $n > \max(r, j, k)$ and $r > \max(i, l)$ so $n > \max(i, j, k, l)$ in (2.b);
3. $n > \max(i, r, k)$ and $r > \max(l, j)$ so $n > \max(i, j, k, l)$ in (2.c);
4. $n > \max(i, j, r)$ and $r > \max(k, l)$ so $n > \max(i, j, k, l)$ in (2.d);
5. $n > \max(l, r)$ and $r > \max(i, j, k)$ so $n > \max(i, j, k, l)$ in (2.e).

This establishes Assertion (2) of Theorem 1.1 if $\nu = 1$ so we are dealing with ∇R. The argument is the same for higher values of ν and is therefore omitted.

2.3. THE GEOMETRY OF THE CURVATURE TENSOR

By Assertion (2) of Theorem 1.1,

$$\begin{aligned} &J(\xi)\partial_{x_i} \subset \operatorname{Span}_{k>i}\{\partial_{x_k}\}, \quad \mathcal{S}(\xi)\partial_{x_i} \subset \operatorname{Span}_{k>i}\{\partial_{x_k}\},\\ &\mathcal{R}(\pi)\partial_{x_i} \subset \operatorname{Span}_{k>i}\{\partial_{x_k}\}\,. \end{aligned}$$

Thus J, $\mathcal{R}$, and $\mathcal{S}$ are nilpotent which proves Assertion (3) of Theorem 1.1. Furthermore, because $J(\xi)$ is nilpotent, $\rho(\xi,\xi) = \mathrm{Tr}(J(\xi)) = 0$. This implies $\rho = 0$ which completes the proof of Assertion (4) of Theorem 1.1.

2.4. LOCAL SCALAR INVARIANTS

Let Θ be a Weyl monomial which is formed by contracting upper and lower indices in pairs in the variables $\{g^{ij}, g_{ij}, R_{i_1i_2i_3}{}^{i_4}{}_{;j_1\dots}\}$. The single upper index in R plays a distinguished role. We choose a representation for Θ so the number of g_{ij} variables is minimal; for example, we can eliminate the $g_{i_3i_4}$ variable in Equation (1.b) by expressing:

$$|R|^2 = g^{i_1j_1}g^{i_2j_2}R_{i_1i_2k}{}^{l}R_{j_2j_1l}{}^{k}\,.$$

Suppose there is a g_{ij} variable in this minimal representation, i.e. that

$$\Theta = g_{ij}R_{u_1u_2u_3}{}^{i}{}_{;\dots}R_{v_1v_2v_3}{}^{j}{}_{;\dots}\cdots.$$

Suppose further that $g^{u_1w_1}$ appears in Θ, i.e. that

$$\Theta = g_{ij}g^{u_1w_1}R_{u_1u_2u_3}{}^{i}{}_{;\dots}R_{v_1v_2v_3}{}^{j}{}_{;\dots}\cdots.$$

We could then raise and lower an index to express

$$\Theta = R^{w_1}{}_{u_2u_3j;\dots}R_{v_1v_2v_3}{}^{j}{}_{;\dots}\cdots = R_{ju_3u_2}{}^{w_1}{}_{;\dots}R_{v_1v_2v_3}{}^{j}{}_{;\dots}\cdots$$

which has one less $g_{..}$ variable. This contradicts the assumed minimality. Thus u_1 must be contracted against an upper index; a similar argument shows that u_2, u_3, v_1, v_2, and v_3 are contracted against an upper index as well. Consequently

$$\Theta = g_{ij}R_{u_1u_2u_3}{}^{i}{}_{;\dots}R_{v_1v_2v_3}{}^{j}{}_{;\dots}R_{w_1w_2w_3}{}^{u_1}{}_{;\dots}\cdots.$$

Suppose w_1 is not contracted against an upper index. We then have

$$\begin{aligned}\Theta &= g_{ij}g^{w_1x_1}R_{u_1u_2u_3}{}^{i}{}_{;\dots}R_{v_1v_2v_3}{}^{j}{}_{;\dots}R_{w_1w_2w_3}{}^{u_1}{}_{;\dots}\cdots\\ &= R_{u_1u_2u_3j;\dots}R_{v_1v_2v_3}{}^{j}{}_{;\dots}R^{x_1}{}_{w_2w_3}{}^{u_1}{}_{;\dots}\cdots\end{aligned}$$

$$
\begin{aligned}
&= g^{u_1y_1}R_{u_1u_2u_3j;\dots}R_{v_1v_2v_3}{}^{j}{}_{;\dots}R^{x_1}{}_{w_2w_3y_1;\dots}\cdots\\
&= R^{y_1}{}_{u_2u_3j;\dots}R_{v_1v_2v_3}{}^{j}{}_{;\dots}R^{x_1}{}_{w_2w_3y_1;\dots}\\
&= R_{ju_3u_2}{}^{y_1}{}_{;\dots}R_{v_1v_2v_3}{}^{j}{}_{;\dots}R^{x_1}{}_{w_2w_3y_1;\dots}
\end{aligned}
$$

which has one less g_{ij} variable. Thus w_1 is contracted against an upper index so

$$\Theta = g_{ij}R_{u_1u_2u_3}{}^{i}{}_{;\dots}R_{v_1v_2v_3}{}^{j}{}_{;\dots}R_{w_1w_2w_3}{}^{u_1}{}_{;\dots}R_{x_1x_2x_3}{}^{w_1}{}_{;\dots}\cdots .$$

We continue in this fashion to build a monomial of infinite length. This is not possible. Thus we can always find a representation for Θ which contains no g_{ij} variables in the summation.

We suppose the evaluation of Θ is non-zero and argue for a contradiction. To simplify the notation, group all the lower indices together. By considering the pairing of upper and lower indices, we see that we can expand Θ in cycles:

$$\Theta = R_{\dots i_r\dots}{}^{i_1}R_{\dots i_1\dots}{}^{i_2}\dots R_{\dots i_{r-1}\dots}{}^{i_r}\dots .$$

By Theorem 1.1 (2), $R_{\dots j\dots}{}^{l} = 0$ if $l \le j$. Thus the sum runs over indices where $i_r < i_1 < i_2 < \dots < i_r$. As this is the empty sum, we see that $\Theta = 0$ as desired.

2.5. HOLONOMY

Let $X = \sum_i a_i(t)\partial_{x_i}$ be a vector field which is defined along a curve $\gamma = (\gamma_1, \dots, \gamma_m)$ in $\mathbb{R}^m$. Then $\nabla_{\dot\gamma}X = 0$ if and only if

$$0 = \textstyle\sum_i \dot a_i(t)\partial_{x_i} + \sum_{i,j,k:i,j<k}\Gamma_{ij}{}^{k}(t)a_i(t)\dot\gamma_j(t)\partial_{x_k} .$$

Consequently, we can solve these equations by taking recursively

$$a_k(t) = a_k(0) - \textstyle\int_0^t \sum_{i,j<k}\Gamma_{ij}{}^{k}(a_1(s), \dots, a_{k-1}(s))a_i(s)\dot\gamma_j(s)ds .$$

If $a_i(0) = 0$ for $i < \ell$, we may conclude $a_i(t) = 0$ for all t if $i < \ell$. Assertions (6) and (7) now follow. This completes the proof of Theorem 1.1. □

3. MANIFOLDS OF SIGNATURE $(2, 2+k)$

3.1. THE MANIFOLDS $\mathcal{M}^0_{4+k,F}$

Let $(x, y, z_1, ..., z_k, \tilde{y}, \tilde{x})$ be coordinates on $\mathbb{R}^{4+k}$. Let $F(y, z_1, ..., z_k)$ be an affine function of $(z_1, ..., z_k)$, i.e.

$$F(y, z_1, ..., z_k) = f_0(y) + f_1(y)z_1 + ... + f_k(y)z_k\,.$$

Let $\mathcal{M}^0_{4+k,F} := (\mathbb{R}^{4+k}, g^0_{4+k,F})$ where:

$$\begin{aligned}
&g^0_{4+k,F}(\partial_x, \partial_{\tilde{x}}) = g^0_{4+k,F}(\partial_y, \partial_{\tilde{y}}) = g^0_{4+k,F}(\partial_{z_i}, \partial_{z_i}) = 1,\\
&g^0_{4+k,F}(\partial_x, \partial_x) = -2F(y, z_1, ..., z_k)\,.
\end{aligned}$$

Theorem 3.1 *$\mathcal{M}^0_{4+k,F}$ is a generalized plane wave manifold of signature $(2, 2+k)$.*

Proof. The non-zero Christoffel symbols of the first kind are given by

$$\begin{aligned}
&g^0_{4+k,F}(\nabla_{\partial_x}\partial_x, \partial_y) = f_0' + \textstyle\sum_i f_i' z_i,\\
&g^0_{4+k,F}(\nabla_{\partial_y}\partial_x, \partial_x) = g^0_{4+k,F}(\nabla_{\partial_x}\partial_y, \partial_x) = -\{f_0' + \textstyle\sum_i f_i' z_i\},\\
&g^0_{4+k,F}(\nabla_{\partial_x}\partial_x, \partial_{z_i}) = f_i,\\
&g^0_{4+k,F}(\nabla_{\partial_{z_i}}\partial_x, \partial_x) = g^0_{4+k,F}(\nabla_{\partial_x}\partial_{z_i}, \partial_x) = -f_i\,.
\end{aligned}$$

Consequently the non-zero Christoffel symbols of the second kind are given by

$$\begin{aligned}
&\nabla_{\partial_x}\partial_x = \{f_0' + \textstyle\sum_i f_i' z_i\}\partial_{\tilde{y}} + \textstyle\sum_i f_i \partial_{z_i},\\
&\nabla_{\partial_y}\partial_x = \nabla_{\partial_x}\partial_y = -\{f_0' + \textstyle\sum_i f_i' z_i\}\partial_{\tilde{x}},\\
&\nabla_{\partial_{z_i}}\partial_x = \nabla_{\partial_x}\partial_{z_i} = -f_i\partial_{\tilde{x}}\,.
\end{aligned}$$

This has the required triangular form. $\square$

3.2. k-CURVATURE HOMOGENEITY

Let $\mathcal{M} := (M, g)$ be a pseudo-Riemannian manifold. If $P \in M$, let $g_P \in \otimes^2 T_P^* M$ be the restriction of g to the tangent space $T_P M$. We use the metric to lower indices and regard $\nabla^k R \in \otimes^{4+k} T^* M$; let $\nabla^k R_P$ be the restriction of $\nabla^k R$ to $T_P M$ and let

$$\mathcal{U}^k(\mathcal{M}, P) := (T_P M, g_P, R_P, ..., \nabla^k R_P)\,.$$

This is a purely algebraic object. Following Kowalski, Tricerri, and Vanhecke [16, 17], we say that $\mathcal{M}$ is *k-curvature homogeneous* if given any two points P and Q of M, there is a isomorphism $\Psi_{P,Q}$ from $\mathcal{U}^k(\mathcal{M}, P)$ to $\mathcal{U}^k(\mathcal{M}, Q)$, i.e. a linear isomorphism $\Psi_{P,Q}$ from $T_P M$ to $T_Q M$ such that

$$\Psi_{P,Q}^* g_Q = g_P \quad \text{and} \quad \Psi_{P,Q}^* \nabla^i R_Q = \nabla^i R_P \text{ for } 0 \le i \le k\,.$$

Similarly, $\mathcal{M}$ is said to be *locally homogeneous* if given any two points P and Q, there are neighborhoods U_P and U_Q of P and Q, respectively, and an isometry $\psi_{P,Q}$ from U_P to U_Q such that $\psi_{P,Q} P = Q$. Taking $\Psi_{P,Q} := (\psi_{P,Q})_*$ shows that locally homogeneous manifolds are k-curvature homogeneous for any k.

More generally, we can consider a *k-model* $\mathcal{U}^k := (V, h, A^0, ..., A^k)$ where V is an m-dimensional real vector space, where h is a non-degenerate inner product of signature (p, q) on V, and where $A^i \in \otimes^{4+i} V^*$ has the appropriate universal curvature symmetries. For example, we assume that:

$$\begin{aligned} &A^0(\xi_1, \xi_2, \xi_3, \xi_4) = A^0(\xi_3, \xi_4, \xi_1, \xi_2) = -A^0(\xi_2, \xi_1, \xi_3, \xi_4) \quad \text{and} \\ &A^0(\xi_1, \xi_2, \xi_3, \xi_4) + A^0(\xi_2, \xi_3, \xi_1, \xi_4) + A^0(\xi_3, \xi_1, \xi_2, \xi_4) = 0\,. \end{aligned} \tag{3.a}$$

We say that $\mathcal{U}^k$ *is a k-model for* $\mathcal{M}$ if given any point $P \in M$, there is an isomorphism Ψ_P from $\mathcal{U}^k(\mathcal{M}, P)$ to $\mathcal{U}^k$. Clearly $\mathcal{M}$ is k-curvature homogeneous if and only if $\mathcal{M}$ admits a k-model; one may take as the k model $\mathcal{U}^k := \mathcal{U}^k(\mathcal{M}, P)$ for any $P \in M$.

3.3. THE MANIFOLDS $\mathcal{M}^1_{6,f}$

We specialize the construction given above by taking $F = yz_1 + f(y)z_2$. One sets $\mathcal{M}^1_{6,f} := (\mathbb{R}^6, g^1_{6,f})$ where

$$\begin{aligned}&g^1_{6,f}(\partial_x,\partial_{\tilde x}) = g^1_{6,f}(\partial_y,\partial_{\tilde y}) = g^1_{6,f}(\partial_{z_1},\partial_{z_1}) = g^1_{6,f}(\partial_{z_2},\partial_{z_2}) = 1, \quad \text{and}\\ &g^1_{6,f}(\partial_x,\partial_x) = -2(yz_1 + f(y)z_2)\,.\end{aligned} \tag{3.b}$$

3.4. AN INVARIANT WHICH IS NOT OF WEYL TYPE

Set

$$\alpha^1_6(f,P) = \frac{|f'(P)|}{\sqrt{1+(f'(P))^2}}\,. \tag{3.c}$$

Theorem 3.2 *Assume that $f'' > 0$. Then*

1. $\mathcal{M}^1_{6,f}$ *is a* 0*-curvature homogeneous generalized plane wave manifold.*
2. *If* $\mathcal{U}^1(\mathcal{M}^1_{6,f_1},P_1)$ *and* $\mathcal{U}^1(\mathcal{M}^1_{6,f_2},P_2)$ *are isomorphic, then* $\alpha^1_6(f_1,P_1) = \alpha^1_6(f_2,P_2)$.
3. α^1_6 *is an isometry invariant of this family which is not of Weyl type.*
4. $\mathcal{M}^1_{6,f}$ *is not* 1*-curvature homogeneous.*

Proof. We use Theorem 3.1 to see that $\mathcal{M}^1_{6,f}$ is a generalized plane wave manifold. Furthermore, up to the usual $\mathbb{Z}_2$ symmetries, the computations performed in the proof of Theorem 3.1 show that the non-zero entries in the curvature tensor are:

$$R(\partial_x,\partial_y,\partial_y,\partial_x) = f''z_2, \quad R(\partial_x,\partial_y,\partial_{z_1},\partial_x) = 1, \quad R(\partial_x,\partial_y,\partial_{z_2},\partial_x) = f'\,.$$

We set

$$\begin{aligned}X &:= c_1\{\partial_x - \tfrac12 g^1_{6,f}(\partial_x,\partial_x)\partial_{\tilde x}\},\\ \tilde X &:= c_1^{-1}\partial_{\tilde x},\\ Y &:= c_2\{\partial_y - \varepsilon_1\partial_{z_1} - \varepsilon_2\partial_{z_2} - \tfrac12(\varepsilon_1^2+\varepsilon_2^2)\partial_{\tilde y}\},\\ \tilde Y &:= c_2^{-1}\partial_{\tilde y},\\ Z_1 &:= c_3\{\partial_{z_1} + f'\partial_{z_2} + (\varepsilon_1 + f'\varepsilon_2)\partial_{\tilde y}\},\\ Z_2 &:= c_3\{\partial_{z_2} - f'\partial_{z_1} + (\varepsilon_2 - f'\varepsilon_1)\partial_{\tilde y}\}\,.\end{aligned}$$

Since $R(\partial_x,\partial_y,\partial_{z_1},\partial_x)=1$ and $R(\partial_x,\partial_y,\partial_{z_2},\partial_x)\neq 0$, we may choose ε_1, ε_1, c_1, c_2, and c_3 so that

$$R(\partial_x,\partial_y,\partial_y,\partial_x)-2\varepsilon_1R(\partial_x,\partial_y,\partial_{z_1},\partial_x)-2\varepsilon_2R(\partial_x,\partial_y,\partial_{z_2},\partial_x)=0, \tag{3.d}$$

$$R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_y)-3\varepsilon_2R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_{z_2})=0, \tag{3.e}$$

$$c_3^2(1+(f')^2)=1, \tag{3.f}$$

$$c_3(1+(f')^2)c_1^2c_2=1, \tag{3.g}$$

$$c_3c_1^2c_2^2f''=1\,. \tag{3.h}$$

We show that $\mathcal{M}_{6,f}^1$ is 0-curvature homogeneous and complete the proof of Assertion (1) by noting that the possibly non-zero entries in these tensors are given by:

$$\begin{array}{ll}
g_{6,f}^1(X,\tilde X)=g_{6,f}^1(Y,\tilde Y)=1. & \\
g_{6,f}^1(Z_1,Z_1)=g_{6,f}^1(Z_2,Z_2)=1 & \text{[see equation (3.f)]},\\
R(X,Y,Y,X)=0 & \text{[see equation (3.d)]},\\
R(X,Y,Z_1,X)=1 & \text{[see equation (3.g)]},\\
R(X,Y,Z_2,X)=0\,. &
\end{array}$$

The possibly non-zero components of ∇R are:

$$\begin{array}{l}
\nabla R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_{z_2})=\nabla R(\partial_x,\partial_y,\partial_{z_2},\partial_x;\partial_y)=f''>0,\\
\nabla R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_y)=f'''z_2\,.
\end{array}$$

The possibly non-zero components of ∇R with respect to this basis are given by:

$$\begin{array}{ll}
\nabla R(X,Y,Y,X;Z_1)=\nabla R(X,Y,Z_1,X;Y)=f' & \text{[see equation (3.h)]},\\
\nabla R(X,Y,Y,X;Y)=0 & \text{[see equation (3.e)]},\\
\nabla R(X,Y,Y,X;Z_2)=\nabla R(X,Y,Z_2,X;Y)=1 & \text{[see equation (3.h)]}\,.
\end{array}$$

We shall say that a basis $\mathcal{B}=\{{}^1X,{}^1Y,{}^1Z_1,{}^1Z_2,{}^1\tilde Y,{}^1\tilde X\}$ is *normalized* if the non-zero entries in R and ∇R are

$$\begin{array}{l}
R({}^1X,{}^1Y,{}^1Z_1,{}^1X)=1,\quad \text{and}\\
\nabla R({}^1X,{}^1Y,{}^1Y,{}^1X;{}^1Z_2)=\nabla R({}^1X,{}^1Y,{}^1Z_2,{}^1X;{}^1Y)=1\,.
\end{array}$$

For example, $\mathcal{B} = \{X, Y, Z_1 - f'Z_2, Z_2, \tilde{Y}, \tilde{X}\}$ is a normalized basis. Let

$$\begin{aligned}&\ker(R) := \{\eta : R(\xi_1, \xi_2, \xi_3, \eta) = 0 \quad \forall \xi_i\},\\&\ker(\nabla R) := \{\eta : \nabla R(\xi_1, \xi_2, \xi_3, \xi_4; \eta) = 0 \text{ and } \nabla R(\xi_1, \xi_2, \xi_3, \eta; \xi_4) = 0 \quad \forall \xi_i\}\,.\end{aligned}$$

It is then immediate that

$$\ker(R) = \operatorname{Span}\{Z_2, \tilde{X}, \tilde{Y}\} \quad \text{and} \quad \ker(\nabla R) = \operatorname{Span}\{Z_1 - f'Z_2, \tilde{X}, \tilde{Y}\}\,.$$

Let $\mathcal{B} := \{{}^1X, {}^1Y, {}^1Z_1, {}^1Z_2, {}^1\tilde{Y}, {}^1\tilde{X}\}$ be any normalized basis. Since ${}^1Z_1 \in \ker(\nabla R)$ and ${}^1Z_2 \in \ker(R)$, we may expand:

$$\begin{aligned}{}^1Z_1 &= a_1(Z_1 - f'Z_2) + a_2\tilde{X} + a_3\tilde{Y},\\{}^1Z_2 &= b_1Z_2 + b_2\tilde{X} + b_3\tilde{Y}\,.\end{aligned}$$

Thus we may compute

$$\frac{|g^1_{6,f}({}^1Z_1, {}^1Z_2)|}{|{}^1Z_1| \cdot |{}^1Z_2|}(P) = \frac{|f'|}{\sqrt{1 + (f')^2}}(P) = \alpha^1_6(f, P)\,.$$

This shows $\alpha^1_6(f, P)$ is an invariant of the 1-model and establishes Assertion (2).

If $\mathcal{M}^1_{6,f}$ is curvature 1-homogeneous, then necessarily $\alpha^1_6(f)$ is constant or, equivalently, $(f')^2 = c(1 + (f')^2)$ for some constant c. Since $(f')^2 < (1 + (f')^2)$, $c < 1$. Thus we can solve for $(f')^2$ to see $(f')^2 = \frac{c}{1-c}$ is constant. This contradicts the assumption $f'' \neq 0$. $\square$

3.5. WEAK CURVATURE HOMOGENEITY

We can weaken the notion of curvature homogeneity slightly. Let $A^0 \in \otimes^4 V^*$ be an algebraic curvature tensor, i.e. A^0 has the usual symmetries of the curvature tensor given in Equation (3.a). We say that $\mathcal{M}^1$ is *weakly* 0-*curvature homogeneous* if for every point $P \in M$, there is an isomorphism $\Phi : T_PM \to V$ so that $\Phi^* A^0 = R$. There is no requirement that Φ preserve an inner product. The notion of weakly k-curvature homogeneous is similar; we consider models $(V, A^0, ..., A^k)$ where $A^i \in \otimes^{4+i}(V^*)$ has the appropriate curvature symmetries. Since we have lowered all the indices, this is a different notion from the notion of *affine k-curvature homogeneity* that will be discussed presently.

The following is an immediate consequence of the arguments given above:

Corollary 3.3 *The manifold $\mathcal{M}^1_{6,f}$ is weakly 1-curvature homogeneous but not 1-curvature homogeneous.*

3.6. AFFINE GEOMETRY

Let ∇ be a torsion free connection on TM. Since we do not have a metric, we can not raise and lower indices. Thus we must regard ∇^i as a $(i+2,1)$ tensor; instead of working with the tensor $R_{i_1i_2i_3i_4;j_1\ldots}$, we work with $R_{i_1i_2i_3}{}^{i_4}{}_{;j_1\ldots}$. We say that (M,∇) is affine *k-curvature homogeneous* if given any two points P and Q of M, there is a linear isomorphism $\phi: T_PM\rightarrow T_QM$ so that $\phi^*\nabla^iR_Q=\nabla^iR_P$ for $0\le i\le k$. Taking ∇ to be the Levi-Civita connection of a pseudo-Riemannian metric then yields that any k-curvature homogeneous manifold is necessarily affine k-curvature homogeneous by simply forgetting the requirement that ϕ be an isometry; there is no metric present in the affine setting. We refer to Opozda [22, 23] for a further discussion of the subject. The relevant models are:

$$\begin{aligned}&\mathcal{A}^k(\mathcal{M},P):=(T_PM,R_P,\nabla R_P,...,\nabla^kR_P),\quad\text{where}\\&\nabla^iR_P\in\otimes^{3+i}T_PM^*\otimes T_PM\,.\end{aligned}$$

In fact the invariant α^1_6 is an *affine invariant.* We use note that:

$$\begin{array}{ll}
R(X,Y)Z_1=\tilde X, & R(X,Y)X=-Z_1,\\
R(X,Z_1)Y=\tilde X, & R(X,Z_1)X=-\tilde Y,\\
\nabla_{Z_1}R(X,Y)Y=f'\tilde X, & \nabla_{Z_2}R(X,Y)Y=\tilde X,\\
\nabla_{Z_1}R(X,Y)X=-f'\tilde Y, & \nabla_{Z_2}R(X,Y)X=-\tilde Y,\\
\nabla_YR(X,Y)Z_1=f'\tilde X, & \nabla_YR(X,Y)Z_2=\tilde X,\\
\nabla_YR(X,Z_1)Y=f'\tilde X, & \nabla_YR(X,Z_2)Y=\tilde X,\\
\nabla_YR(X,Z_1)X=-\tilde Y, & \nabla_YR(X,Z_2)X=-\tilde Y,\\
\nabla_YR(X,Y)X=-f'Z_1-Z_2\,. &
\end{array}$$

We define the following subspaces:

$$\begin{aligned}
W_1 : &= \mathrm{Range}(R) = \mathrm{Span}\{R(\xi_1,\xi_2)\xi_3 : \xi_i \in \mathbb{R}^6\},\\
W_2 : &= \mathrm{Range}(\nabla R) = \mathrm{Span}\{\nabla_{\xi_1} R(\xi_2,\xi_3)\xi_4 : \xi_i \in \mathbb{R}^6\},\\
W_3 : &= \mathrm{Span}\{R(\xi_1, R(\xi_2,\xi_3)\xi_4)\xi_5 : \xi_i \in \mathbb{R}^6\},\\
W_4 : &= \ker(R) = \{\eta \in \mathbb{R}^6 : R(\xi_1,\xi_2)\eta = 0 \ \forall\ \xi_i \in \mathbb{R}^6\},\\
W_5 : &= \ker(\nabla R) = \{\eta \in \mathbb{R}^6 : \nabla_{\xi_1} R(\xi_2,\xi_3)\eta = 0 \ \forall\ \xi_i \in \mathbb{R}^6\}\,.
\end{aligned}$$

Lemma 3.4 *We have*

1. $W_1 = \mathrm{Span}\{\tilde X, \tilde Y, Z_1\}$,
2. $W_2 = \mathrm{Span}\{\tilde X, \tilde Y, f' Z_1 + Z_2\}$,
3. $W_3 = \mathrm{Span}\{\tilde X, \tilde Y\}$,
4. $W_4 = \mathrm{Span}\{\tilde X, \tilde Y, Z_2\}$,
5. $W_5 = \mathrm{Span}\{\tilde X, \tilde Y, Z_1 - f' Z_2\}$.
6. *If* $\mathcal{A}^1(\mathcal{M}^6_{6,f_1}, P_1)$ *and* $\mathcal{A}^1(\mathcal{M}^6_{6,f_2}, P_2)$ *are isomorphic, then* $\alpha^1_6(f_1,P_1) = \alpha^1_6(f_2,P_2)$.

Proof. Assertions (1) and (2) are immediate. We compute

$$\begin{aligned}
&R(X, R(X,Y)X)X = R(X,-Z_1)X = \ \ \tilde Y,\\
&R(X, R(X,Y)X)Y = R(X,-Z_1)Y = -\tilde X, \qquad \text{so} \qquad \mathrm{Span}\{\tilde X, \tilde Y\} \subset W_3\,.
\end{aligned}$$

We establish Assertion (3) by establishing the reverse inclusion:

$$R(\xi_1, R(\xi_2,\xi_3)\xi_4)\xi_5 = R(\xi_1, aZ_1 + b\tilde X + c\tilde Y)\xi_5 = R(dX, aZ_1)\xi_5 \in \mathrm{Span}\{\tilde X, \tilde Y\}\,.$$

It is clear $W_4 \subset \mathrm{Span}\{\tilde X, \tilde Y, Z_2\}$. Let $\eta = aX + bY + cZ_1 + dZ_2 + e\tilde X + f\tilde Y \in W_4$. As $R(X,Y)\eta = 0$, we have $-aZ_1 + c\tilde X = 0$ so $a = 0$ and $c = 0$. As $R(X,Z_1)\eta = 0$, we have $-a\tilde Y + b\tilde X = 0$ so $b = 0$ as well. Assertion (4) now follows.

It is clear $W_5 \subset \mathrm{Span}\{\tilde X, \tilde Y, Z_1 - f' Z_2\}$. Let η be as above. As $\nabla_{Z_2} R(X,Y)\eta = 0$, $-a\tilde Y + b\tilde X = 0$ so $a = b = 0$. Since $\nabla_Y R(X,Y)\eta = 0$, $(cf' + d) = 0$ so $d = -cf'$; this establishes Assertion (5).

Suppose we have an isomorphism from $\mathcal{A}^1(\mathcal{M}^6_{6,f_1}, P_1)$ to $\mathcal{A}^1(\mathcal{M}^6_{6,f_2}, P_2)$. We ignore the X and Y variables. Then we have an isomorphism ϕ from $\mathbb{R}^6$ to itself so that $\phi(W_i(f_1, P_1)) = W_i(f_2, P_2)$ for $1 \leq i \leq 5$. We can work in the spaces W_i/W_3 to see that we must have the relations:

$$\begin{aligned}\phi(Z_1) = a_1 Z_1, &\quad \phi(f_1' Z_1 + Z_2) = a_2(f_2' Z_1 + Z_2),\\ \phi(Z_2) = a_3 Z_2, &\quad \phi(Z_1 - f' Z_2) = a_4(Z_1 - f_2' Z_2)\,.\end{aligned}$$

This yields $a_1 f_1' Z_1 + a_3 Z_2 = a_2 f_2' Z_1 + a_2 Z_2$ and $a_1 Z_1 - a_3 f_1' Z_2 = a_4 Z_1 - a_4 f_2' Z_2$. Thus $a_1 = a_4$ and $a_3 = a_2$ so $a_1 f_1' = a_2 f_2'$ and $a_2 f_1' = a_1 f_2'$. Consequently,

$$a_1 a_2 f_1' f_1' = a_2 a_1 f_2' f_2'\,.$$

Since the coefficients a_i are non-zero, the desired conclusion follows. $\square$

4. NEUTRAL SIGNATURE GENERALIZED PLANE WAVE MANIFOLDS

4.1. THE MANIFOLDS $\mathcal{M}^2_{2p,\psi}$

Let $p \geq 2$. Introduce coordinates $(x_1, ..., x_p, y_1, ..., y_p)$ on $\mathbb{R}^{2p}$. Let $\psi(x)$ be a symmetric 2-tensor field on $\mathbb{R}^p$. We define a neutral signature metric $g^2_{2p,\psi}$ on $\mathbb{R}^{2p}$ and a corresponding pseudo-Riemannian manifold $\mathcal{M}^2_{2p,\psi}$ by:

$$g^2_{2p,\psi}(\partial_{x_i}, \partial_{x_j}) = \psi_{ij}(x), \quad g^2_{2p,\psi}(\partial_{x_i}, \partial_{y_j}) = \delta_{ij}, \quad \text{and} \quad g^2_{2p,\psi}(\partial_{y_i}, \partial_{y_j}) = 0\,.$$

Theorem 4.1 *$\mathcal{M}^2_{2p,\psi}$ is a generalized plane wave manifold of signature (p, p).*

Proof. The non-zero Christoffel symbols of the first kind are given by:

$$\Gamma^x_{ijk} := g^2_{2p,\psi}(\nabla_{\partial_{x_i}} \partial_{x_j}, \partial_{x_k}) = \tfrac{1}{2}\{\partial_{x_j}\psi_{ik} + \partial_{x_i}\psi_{jk} - \partial_{x_k}\psi_{ij}\}\,.$$

From this, it is immediate that:

$$\nabla_{\partial_{x_i}} \partial_{x_j} = \textstyle\sum_k \Gamma^x_{ij}{}^k(x) \partial_{y_k}\,.$$

We set $x_{p+i} = y_i$ to see $\mathcal{M}^2_{2p,\psi}$ is a generalized plane wave manifold. $\square$

4.2. HOLONOMY

The manifolds $\mathcal{M}^2_{2p,\psi}$ present a special case. Let $\mathfrak{o}(p)$ be the Lie algebra of the orthogonal group; this is the additive group of all skew-symmetric $p \times p$ real matrices. If A_p is such a matrix, let $\mathcal{G}_{2p}$ be the set of all matrices of the form

$$G(A_p) = \left(\begin{array}{cc} I_p & A_p \\ 0 & I_p \end{array} \right) .$$

The map $A_p \to G(A_p)$ identifies $\mathfrak{o}(p)$ with a subgroup of the upper triangular matrices.

Lemma 4.2 *$\mathcal{H}_P(\mathcal{M}^2_{2p,\psi}) \subset \mathfrak{o}(p)$.*

Proof. Let γ be a closed loop in $\mathbb{R}^{2p}$. Let $H_\gamma\partial_{x_i} = X_i$ and $H_\gamma\partial_{y_i} = Y_i$. Since $\nabla\partial_{y_i} = 0$, $Y_i = \partial_{y_i}$. Expand $X_i = \sum_j(a_{ij}\partial_{x_j} + b_{ij}\partial_{y_j})$. Since H_γ is an isometry,

$$g^2_{2p,\psi}(X_i, X_j) = \psi_{ij}, \quad g^2_{2p,\psi}(X_i, Y_j) = \delta_{ij}, \quad \text{and} \quad g^2_{2p,\psi}(Y_i, Y_j) = 0 .$$

The relation $g^2_{2p,\psi}(X_i, Y_j) = \delta_{ij}$ and the observation that $Y_i = \partial_{y_i}$ shows that $a_{ij} = \delta_{ij}$. Thus

$$g^2_{2p,\psi}(X_i, X_j) = \psi_{ij} + b_{ij} + b_{ji} = \psi_{ij} .$$

This shows $b \in \mathfrak{o}(p)$. $\square$

4.3. JORDAN NORMAL FORM

The eigenvalue structure does not determine the Jordan normal form of a self-adjoint or of a skew-adjoint endomorphism if the metric is indefinite. We say that $\mathcal{M}$ is *spacelike* (resp. *timelike*) *Jordan Osserman* if the Jordan normal form of the Jacobi operator J is constant on the pseudo-sphere bundles of spacelike (resp. timelike) unit vectors. These two notions are not equivalent. The notions *spacelike Jordan Ivanov–Petrova*, *timelike Jordan Ivanov–Petrova*, *spacelike Jordan Szabó*, and *timelike Jordan Szabó* are defined similarly. There are no known examples of spacelike or timelike Jordan Szabó manifolds which are not locally symmetric; $\mathcal{S}(\cdot)$ vanishes identically if and only if $\nabla R = 0$.

4.4. THE MANIFOLDS $\mathcal{M}^3_{2p,f}$

Let $f(x_1,...,x_p)$ be a smooth function on $\mathbb{R}^p$ and let $\mathcal{M}^3_{2p,f} := (\mathbb{R}^{2p}, g^3_{2p,f})$ where $g^3_{2p,f}$ is defined by $\psi_{ij} := \partial_{x_i} f \cdot \partial_{x_j} f$, i.e.

$$\begin{aligned} &g^3_{2p,f}(\partial_{x_i},\partial_{y_j}) = \delta_{ij}, \quad g^3_{2p,f}(\partial_{y_i},\partial_{y_j}) = 0, \quad \text{and} \\ &g^3_{2p,f}(\partial_{x_i},\partial_{x_j}) = \partial_{x_i}(f)\cdot\partial_{x_j}(f)\,. \end{aligned}$$

Let $H_{f,ij} := \partial_{x_i}\partial_{x_j} f$ be the Hessian. We use Theorem 4.1 and results of Gilkey, Ivanova, and Zhang [9] to see that:

Theorem 4.3 *Assume that H_f is non-degenerate. Then*

1. *$\mathcal{M}^3_{2p,f}$ is a generalized plane wave manifold which is isometric to a hypersurface in a flat space of signature $(p,p+1)$.*
2. *$\mathcal{M}^3_{2p,f}$ is spacelike and timelike Jordan Ivanov–Petrova.*
3. *If $p=2$, then $\mathcal{M}^3_{2p,f}$ is spacelike and timelike Jordan Osserman.*
4. *If $p\geq 3$ and if H_f is definite, $\mathcal{M}^3_{2p,f}$ is spacelike and timelike Jordan Osserman.*
5. *If $p\geq 3$ and if H_f is indefinite, $\mathcal{M}^3_{2p,f}$ is neither spacelike nor timelike Jordan Osserman.*
6. *The following conditions are equivalent:*
    (a) f is quadratic.
    (b) $\nabla R = 0$.
    (c) $\mathcal{M}^3_{2p,f}$ is either spacelike or timelike Jordan Szabó.

4.5. AN INVARIANT WHICH IS NOT OF WEYL TYPE

If H_f is definite, set

$$\alpha^3_{2p}(f,P) := \{H_f^{i_1j_1}H_f^{i_2j_2}H_f^{i_3j_3}H_f^{i_4j_4}H_f^{i_5j_5}R(i_1i_2i_3i_4;i_5)R(j_1j_2j_3j_4;j_5)\}(P) \tag{4.a}$$

where H_f^{ij} denotes the inverse matrix and where we sum over repeated indices. One has the following result of Dunn and Gilkey [3]:

Theorem 4.4 *Let* $p \geq 3$*. Assume that the Hessian* H_f *is definite. Then:*

1. $\mathcal{M}^3_{2p,f}$ *is* 0*-curvature homogeneous.*
2. *If* $\mathcal{U}(\mathcal{M}^3_{2p,f_1}, P_1)$ *is isomorphic to* $\mathcal{U}(\mathcal{M}^3_{2p,f_2}, P_2)$*, then* $\alpha^3_{2p}(f_1, P_1) = \alpha^3_{2p}(f_2, P_2)$*.*
3. $\mathcal{M}^3_{2p,f}$ *is not locally homogeneous for generic* f*.*

4.6. THE MANIFOLDS $\mathcal{M}^4_{4,f}$

Let (x_1, x_2, y_1, y_2) be coordinates on $\mathbb{R}^4$. We consider another subfamily of the examples considered in Theorem 4.1. Let $f = f(x_2)$. Let

$$g^4_{4,f}(\partial_{x_1}, \partial_{x_1}) = -2f(x_2), \quad g^4_{4,f}(\partial_{x_1}, \partial_{y_1}) = g^4_{4,f}(\partial_{x_2}, \partial_{y_2}) = 1$$

define $\mathcal{M}^4_{4,f}$. Results of Dunn, Gilkey, and Nikčević [4] show:

Theorem 4.5 *Assume that* $f^{(2)}$ *and* $f^{(3)}$ *are never vanishing. The manifold* $\mathcal{M}^4_{4,f}$ *is a generalized plane wave manifold of neutral signature* $(2, 2)$ *which is* 1*-curvature homogeneous but not symmetric. The following assertions are equivalent:*

1. $f^{(2)} = ae^{\lambda y}$ *for some* $a, \lambda \in \mathbb{R} - \{0\}$*.*
2. $\mathcal{M}^4_{4,f}$ *is homogeneous.*
3. $\mathcal{M}^4_{4,f}$ *is* 2*-curvature homogeneous.*

4.7. AN INVARIANT WHICH IS NOT OF WEYL TYPE

If $f^{(3)}$ is never vanishing, we set

$$\alpha^4_{4,p}(f, P) := \frac{f^{(p+2)}\{f^{(2)}\}^{p-1}}{\{f^{(3)}\}^{-p}}(P) \quad \text{for} \quad p = 2, 3, \dots. \tag{4.b}$$

In the real analytic context, these form a complete family of isometry invariants that are not of Weyl type. Again, we refer to Dunn, Gilkey, and Nikčević [4] for:

Theorem 4.6 *Assume that f_i are real analytic functions on $\mathbb{R}$ and that $f_i^{(2)}$ and $f_i^{(3)}$ are positive for $i = 1, 2$. The following assertions are equivalent:*

1. *There exists an isometry $\phi : (\mathcal{M}^4_{f_1}, P_1) \to (\mathcal{M}^4_{f_2}, P_2)$.*
2. *We have $\alpha^4_{4,p}(f_1)(P_1) = \alpha^4_{4,p}(f_2)(P_2)$ for $p \geq 2$.*

4.8. THE MANIFOLDS $\mathcal{M}^5_{2p+6,f}$

We consider yet another subfamily of the examples considered in Theorem 4.1. Introduce coordinates on $\mathbb{R}^{2p+6}$ of the form $(x, y, z_0, ..., z_p, \bar{x}, \bar{y}, \bar{z}_0, ..., \bar{z}_p)$. Let $\mathcal{M}^5_{2p+6,f} := (\mathbb{R}^{2p+6}, g^5_{2p+6,f})$ be the pseudo-Riemannian manifold of signature $(p + 3, p + 3)$ where:

$$\begin{aligned}
&g^5_{2p+6,f}(\partial_{z_i}, \partial_{\bar{z}_j}) = \delta_{ij}, \quad g^5_{2p+6,f}(\partial_x, \partial_{\bar{x}}) = 1, \quad g^5_{2p+6,f}(\partial_y, \partial_{\bar{y}}) = 1, \\
&g^5_{2p+6,f}(\partial_x, \partial_x) = -2(f(y) + yz_0 + y^2 z_1 + ... + y^{p+1} z_p) .
\end{aligned}$$

4.9. AN INVARIANT WHICH IS NOT OF WEYL TYPE

If $f^{(p+4)} > 0$, set

$$\alpha^5_{2p+6,k}(f, P) := \frac{f^{(k+p+3)}\{f^{(p+3)}\}^{k-1}}{\{f^{(p+4)}\}^k}(P) \quad \text{for} \quad k \geq 2 . \tag{4.c}$$

The following result follows from work of Gilkey and Nikčević [12, 13].

Theorem 4.7 *Assume that $f^{(p+3)} > 0$ and that $f^{(p+4)} > 0$. Then:*

1. *$\mathcal{M}^5_{2p+6,f}$ is a generalized plane wave manifold of signature $(p + 3, p + 3)$.*
2. *$\mathcal{M}^5_{2p+6,f}$ is $p + 2$-curvature homogeneous.*
3. *If $k \geq 2$ and if $\mathcal{A}^{k+p+1}(M^5_{2p+6,f_1}, P_1)$ and $\mathcal{A}^{k+p+1}(\mathcal{M}^5_{2p+6,f_2}, P_2)$ are isomorphic, then $\alpha^5_{2p+6,k}(f_1, P_1) = \alpha^5_{2p+6,k}(f_2, P_2)$.*
4. *$\alpha^5_{2p+6,k}$ is preserved by any affine diffeomorphism and by any isometry.*
5. *If f_i are real analytic, if $f_i^{(p+3)} > 0$, if $f_i^{(p+4)} > 0$, and if for all $k \geq 2$ we have that $\alpha^5_{2p+6,k}(f_1, P_1) = \alpha^5_{2p+6,k}(f_2, P_2)$, then there exists an isometry ϕ from $\mathcal{M}^5_{2p+6,f_1}$ to $\mathcal{M}^5_{2p+6,f_2}$ with $f(P_1) = P_2$.*

6. *The following assertions are equivalent:*
   - (a) $\mathcal{M}^5_{2p+6,f}$ *is affine* $p+3$*-curvature homogeneous.*
   - (b) $\alpha^5_{2,p}(f)$ *is constant.*
   - (c) $f^{(p+3)} = ae^{\lambda y}$ *for* $a \neq 0$ *and* $\lambda \neq 0$.
   - (d) $\mathcal{M}^5_{2p+6,f}$ *is homogeneous.*

5. GENERALIZED PLANE WAVE MANIFOLDS OF SIGNATURE $(2s, s)$

5.1. THE MANIFOLDS $\mathcal{M}^6_{3s,F}$

Let $s \geq 2$. Introduce coordinates $(\vec{u}, \vec{t}, \vec{v})$ on $\mathbb{R}^{3s}$ for

$$\vec{u} := (u_1, ..., u_s), \quad \vec{t} := (t_1, ..., t_s), \quad \text{and} \quad \vec{v} := (v_1, ..., v_s)\,.$$

Let $F = (f_1, ..., f_s)$ be a collection of smooth real valued functions of one variable. Let $\mathcal{M}^6_{3s,F} = (\mathbb{R}^{3s}, g^6_{3s,F})$ be the pseudo-Riemannian manifold of signature $(2s, s)$:

$$g^6_{3s,F}(\partial_{u_i}, \partial_{u_i}) = -2\{f_1(u_1) + ... + f_s(u_s) - u_1t_1 - ... - u_st_s\},$$
$$g^6_{3s,F}(\partial_{u_i}, \partial_{v_i}) = g^6_{3s,F}(\partial_{v_i}, \partial_{u_i}) = 1, \quad \text{and} \quad g^6_{3s,F}(\partial_{t_i}, \partial_{t_i}) = -1\,.$$

5.2. AN INVARIANT WHICH IS NOT OF WEYL TYPE

Define

$$\alpha^6_{3s}(F, P) := \textstyle\sum_{1 \leq i \leq s}\{f_i'''(u_i) + 4u_i\}^2(P)\,. \tag{5.a}$$

We refer to Gilkey-Nikčević [11] for the proof of the following result:

Theorem 5.1 *Let* $s \geq 3$. *Then*

1. $\mathcal{M}^6_{3s,F}$ *is a generalized plane wave manifold of signature* $(2s, s)$.
2. $\mathcal{M}^6_{3s,F}$ *is* 0*-curvature homogeneous.*
3. $\mathcal{M}^6_{3s,F}$ *is spacelike Jordan Osserman.*
4. $\mathcal{M}^6_{3s,F}$ *is spacelike Jordan Ivanov–Petrova of rank* 4.

5. *$\mathcal{M}^6_{3s,F}$ is not timelike Jordan Osserman.*
6. *$\mathcal{M}^6_{3s,F}$ is not timelike Jordan Ivanov–Petrova.*
7. *If $\mathcal{U}^1(\mathcal{M}^6_{3s,F_1},P_1)$ and $\mathcal{U}^1(\mathcal{M}^6_{3s,F_2},P_2)$ are isomorphic, then $\alpha^6_{3s}(F_1,P_1)=\alpha^6_{3s}(F_2,P_2)$.*
8. *α^6_{3s} is an isometry invariant.*
9. *The following assertions are equivalent:*
   - *(a) $f_i^{(3)}(u_i)+4u_i=0$ for $1\le i\le s$.*
   - *(b) $\mathcal{M}^6_{3s,F}$ is a symmetric space.*
   - *(c) $\mathcal{M}^6_{3s,F}$ is 1-curvature homogeneous.*

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