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LINEARIZABILITY OF A POLYNOMIAL SYSTEM

Diana Doličanin[1] and Valery G. Romanovski[2]

[1] *Faculty of Electronic Engineering, Department of Mathematics*
P.O. Box 73 18000 Niš, Serbia and Montenegro
(e-mail: dolicanin_d@yahoo.com)

[2] *Center for Applied Mathematics and Theoretical Physics,*
University of Maribor, Krekova 2, SI-2000 Maribor, Slovenia
(e-mail: valery.romanovsky@uni-mb.si)

Abstract. We obtain the necessary and sufficient conditions for linearizability of a polynomial system with nonlinearities of degree five.

1. INTRODUCTION

We consider a polynomial system of differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= x - \sum_{(p,q)\in S} a_{pq} x^{p+1} y^q = P(x,y), \\ -\frac{dy}{dt} &= y - \sum_{(p,q)\in S} b_{qp} x^q y^{p+1} = -Q(x,y), \end{aligned} \tag{1}$$

where $x, y, a_{\bar{\imath}}, b_{\bar{\jmath}}$ are complex variables, $S = \{\bar{\imath}_m = (p_m, q_m) | p_m + q_m \geq 1, m = \overline{1, l}\}$ is a subset of $\{-1 \cup \mathbf{N}\} \times \mathbf{N}$, and $\mathbf{N}$ is the set of non-negative integers. In recent years many studies have been devoted to the investigation of the linearizability problem for two-dimensional polynomial systems of differential equations, that is, systems of the form (1) (see, e.g. [1, 2, 8] and the references therein).

Note, that in the case when

$$x = \overline{y},\ a_{ij} = \overline{b}_{ji},\ idt = d\tau \tag{2}$$

(the bar stays for complex conjugate numbers), the system (1) is equivalent to the system

$$i\frac{dx}{d\tau} = x - \sum_{(p,q)\in S} a_{pq}x^{p+1}\bar{x}^q, \tag{3}$$

which has a center or focus at the origin in the real plane $\{(u,v) \mid x = u + iv\}$, where the system can be also written in the form

$$\dot{u} = -v + U(u,v),\ \dot{v} = u + V(u,v), \tag{4}$$

where the power series expansion of U and V at the origin starts with quadratic terms. For systems of the form (4) the notions of center and isochronicity have a simple geometric meaning. Namely, the origin of the system (4) is a center if all trajectories in its neighborhood are closed and it is an isochronous center if the period of oscillations is the same for all these trajectories. However a better understanding of integrability and isochronicity as phenomena can be obtained from considering instead of the real system (3) the complex system (1) (which is equivalent to (3) if the conditions (2) are fulfilled) in spite of the disadvantage that we lose the visible aspects of the center and isochronicity, which we have in the real case.

We recall that, as it is shown in [1] the *real* analytical system (4) has an isochronical center if and only if it can be transformed to the linear normal form

$$\dot{u} = v,\ \dot{v} = -u. \tag{5}$$

Having this fact we use the notion of isochronicity as equivalent to linearizability for *complex systems* of the form (1). Namely, we will consider the problem how to decide if a polynomial system (1) can be transformed to the linear system

$$\dot{z}_1 = z_1,\ \dot{z}_2 = -z_2 \tag{6}$$

by means of a formal change of the phase variables

$$\begin{aligned} z_1 &= x + \sum_{m+j=2}^{\infty} u^{(1)}_{m-1,j}(a^*,b^*)x^m y^j, \\ z_2 &= y + \sum_{m+j=2}^{\infty} u^{(2)}_{m,j-1}(a^*,b^*)x^m y^j. \end{aligned} \tag{7}$$

Taking derivatives with respect to t in both parts of every of equalities (7), we get

$$\dot z_1 = \dot x + \textstyle\sum_{m+j=2}^{\infty} u^{(1)}_{m-1,j}(mx^{m-1}y^j\dot x + jx^m y^{j-1}\dot y),$$

$$\dot z_2 = \dot y + \textstyle\sum_{m+j=2}^{\infty} u^{(2)}_{m,j-1}(mx^{m-1}y^j\dot x + jx^m y^{j-1}\dot y).$$

Equating coefficients of the terms $x^{q_1+1}y^{q_2}$, $x^{q_1}y^{q_2+1}$, correspondingly, we obtain the recurrence formulae

$$(q_1-q_2)u^{(1)}_{q_1q_2} = \sum_{s_1+s_2=0}^{q_1+q_2-1}[(s_1+1)u^{(1)}_{s_1s_2}a_{q_1-s_1,q_2-s_2} - s_2u^{(1)}_{s_1s_2}b_{q_1-s_1,q_2-s_2}], \tag{8}$$

$$(q_1-q_2)u^{(2)}_{q_1q_2} = \sum_{s_1+s_2=0}^{q_1+q_2-1}[s_1u^{(2)}_{s_1s_2}a_{q_1-s_1,q_2-s_2} - (s_2+1)u^{(2)}_{s_1s_2}b_{q_1-s_1,q_2-s_2}], \tag{9}$$

where $s_1, s_2 \geq -1$, $q_1, q_2 \geq -1$, $q_1+q_2 \geq 0$, $u^{(1)}_{1,-1} = u^{(1)}_{-1,1} = 0$, $u^{(2)}_{1,-1} = u^{(2)}_{-1,1} = 0$, $u^{(1)}_{00} = u^{(2)}_{00} = 1$, and we put $a_{qm} = b_{mq} = 0$, if $(q,m) \notin S$.

Thus we see, that the coefficients $u^{(1)}_{q_1q_2}, u^{(2)}_{q_1q_2}$ of the transformation (7) can be computed step by step using the formulae (8), (9). In the case $q_1 = q_2 = q$ the coefficients $u^{(1)}_{qq}, u^{(2)}_{qq}$ can be chosen arbitrary (we set $u^{(1)}_{qq} = u^{(2)}_{qq} = 0$). The system is linearizable only if the quantities in the right-hand side of (8), (9) are equal to zero for all $q \in \mathbf{N}$. As a matter of definition, in the case $q_1 = q_2 = q$ we denote the polynomials in the right-hand side of (8) by i_{qq} and in the right-hand side of (9) by $-j_{qq}$ and call them *q-th linearizability quantities.* We see that the system (1) with the given coefficients (a^*, b^*) is linearizable if and only if $i_{kk}(a^*,b^*) = j_{kk}(a^*,b^*) = 0$ for all $k \in \mathbf{N}$.

We recall that (by definition) a Darboux linearization [3] of the system (1) is a change of variables

$$z_1 = H_1(x,y), \quad z_2 = H_2(x,y) \tag{10}$$

which transforms the system to the linear system (6), and such that at least one of the functions H_1, H_2 is of the form $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$, α_j's being complex numbers, where the $f_i(x,y)$ are invariant algebraic curves of the system (1) defined by $f_i(x,y) = 0$, that is, polynomials satisfying the equation

$$\frac{\partial f_i}{\partial x} P + \frac{\partial f_i}{\partial y} Q = K_i f_i. \tag{11}$$

The polynomial $K_i(x,y)$ is called the cofactor of the invariant curve $f_i(x,y)$.

It is easy to see that if

$$P(x,y)/x + \sum_{i=1}^{k} \alpha_i K_i = 1, \tag{12}$$

then after the substitution

$$z_1 = x f_1^{\alpha_1} \cdots f_k^{\alpha_k}, \tag{13}$$

we get

$$\dot{z}_1 = z_1,$$

and if

$$Q(x,y)/y + \sum_{i=1}^{k} \alpha_i K_i = -1, \tag{14}$$

then the second equation of the system (1) is linearized by the change

$$z_2 = y f_1^{\alpha_1} \cdots f_k^{\alpha_k}. \tag{15}$$

If the system (1) is such that only one of the conditions (12), (14) is satisfied, let say (14), but the system (1) has a Lyapunov first integral $\Psi(x,y)$ of the form

$$\Psi(x,y) = xy + \sum_{\substack{l+j=3 \\ l,j \geq 0}}^{\infty} v_{l,j} x^l y^j, \tag{16}$$

then (1) is linearizable by the change

$$z_1 = \Psi(x,y)/H_2(x,y), \quad z_2 = H_2(x,y), \tag{17}$$

and, correspondingly, if (12) holds, then the linearizing transformation is given by

$$z_1 = H_1(x,y), \quad z_2 = \Psi(x,y)/H_1(x,y)), \tag{18}$$

as can be verified by a straightforward calculation [3].

2. THE LINEARIZABILITY CONDITIONS FOR SYSTEM (19)

In the present paper we find the necessary and sufficient conditions of linearizability for the polynomial system

$$\begin{array}{rcl}\dot{x} & = & x(1-a_{20}x^2-a_{11}xy-a_{02}y^2-a_{-13}x^{-1}y^3-a_{22}x^2y^2),\\ \dot{y} & = & -y(1-b_{20}x^2-b_{11}xy-b_{02}y^2-b_{3,-1}x^3y^{-1}-b_{22}x^2y^2).\end{array} \tag{19}$$

For particular values of parameters, namely when $a_{ij}=\bar{b}_{ji}$ and a_{22} and b_{22} are real, system (19) has been studied in [5].

The given below conditions for linearizability generalize the conditions obtained in [5].

Theorem 1. *The system (19) is linearizable if and only if one of the following conditions holds:*

1) $b_{11}=b_{20}=b_{22}=a_{22}=a_{-13}=3a_{02}+b_{02}=a_{11}=a_{20}=0$,

2) $b_{11}=b_{20}=b_{22}=a_{22}=b_{3,-1}=a_{11}=a_{20}=0$,

3) $b_{11}=b_{22}=a_{22}=7a_{02}+3b_{02}=a_{11}=3a_{20}+7b_{20}=21a_{-13}b_{3,-1}+16b_{20}b_{02}=112b_{20}^3+27b_{3,-1}^2b_{02}=49a_{-13}b_{20}^2-9b_{3,-1}b_{02}^2=343a_{-13}^2b_{20}+48b_{02}^3=0$,

4) $b_{02}=b_{11}=b_{22}=a_{22}=b_{3,-1}=a_{02}=a_{11}=a_{20}+3b_{20}=0$.

5) $b_{02}=b_{11}=b_{22}=a_{22}=a_{-13}=a_{02}=a_{11}=0$,

6) $b_{11}=b_{22}=a_{22}=b_{3,-1}=a_{-13}=a_{02}+b_{02}=a_{11}=a_{20}+b_{20}=0$,

7) $b_{11}=a_{22}=b_{3,-1}=a_{-13}=a_{02}=a_{11}=b_{20}b_{02}+b_{22}=0$,

8) $b_{11}=b_{20}=b_{22}=b_{3,-1}=a_{-13}=a_{11}=a_{20}a_{02}+a_{22}=0$.

Proof. In order to obtain the above conditions for linearizability we computed the first six linearizability quantities i_{kk},j_{kk}. The polynomials are too long, so we do not present them here, however one can easily compute them using Mathematica or any other computer algebra system and the formulae (8)–(9). Then, using the computer algebra program Singular [4], we find the primary decomposition of the ideal $\langle i_{11},j_{11},...,i_{66},j_{66}\rangle$ and obtain the necessary conditions for linearizability (condition 1)-8) of Theorem 1). To prove that the obtained conditions are also the sufficient conditions for linearizability we look for a Darboux linearization. Actually, we have to study only the cases 7) and 8), because it has been shown in [3] that the conditions 1)-6) yield the linearizability of the corresponding systems.

Consider first case 7). In this case the system has the form

$$\begin{aligned} \dot{x} &= x(1-a_{20}x^2), \\ \dot{y} &= -y(1-b_{20}x^2)(1-b_{02}y^2). \end{aligned} \tag{20}$$

System (20) has the invariant curves

$$f_1 = x,\ f_2 = y,\ f_3 = 1-a_{20}x^2,\ f_4 = 1-b_{02}y^2,$$

with the corresponding cofactors

$$k_1 = 1-a_{20}x^2,\ k_2 = -(1-b_{20}x^2)(1-b_{02}y^2),\ k_3 = -2a_{20}x^2,\ k_4 = 2b_{02}(1-b_{20}x^2)y^2.$$

In order to find linearizing transformation for the first equation of system (20) we use (12). Namely, the equation

$$1-a_{20}x^2 + ak_3 + bk_4 = 1$$

has a solution $a = -1/2, b = 0$. Therefore the first equation of (20) is linearizable by the substitution

$$z_1 = \frac{x}{\sqrt{1-a_{20}x^2}}.$$

Similarly, using the equation (14) we see that if $a_{20} \neq 0$ then the second equation of system (20) is linearizable by the substitution

$$z_2 = \frac{y(1-a_{20}x^2)^{\frac{b_{20}}{2a_{20}}}}{\sqrt{1-b_{02}y^2}}.$$

If $a_{20} = 0$ then the first equation of (20) is already linear. According to [6] in this case the system (20) is time reversible. Therefore, it has a Lyapunov first integral of the form (16). Hence by (18) the second equation of the system is linearizable by the substitution

$$z_2 = \Psi(x,y)/x.$$

Consider now case 8). In this case the system has the form

$$\begin{aligned} \dot{x} &= x(1-a_{20}x^2)(1-a_{02}y^2), \\ \dot{y} &= -y(1-b_{02}y^2). \end{aligned} \tag{21}$$

and the invariant lines are the same as for system (20). Thus, similarly as above, we see that if $b_{02} \neq 0$ then the first equation of (21) is linearizable by the substitution

$$z_1 = \frac{x\,(1-b_{02}y^2)^{\frac{a_{02}}{2b_{02}}}}{\sqrt{1-a_{20}x^2}}$$

and the linearization of the second equation is given by

$$z_2 = \frac{y}{\sqrt{1-b_{02}y^2}}.$$

When $b_{02} = 0$ the second equation of (21) is linear and the first one is linearizable by the transformation

$$z_1 = \Psi(x,y)/y,$$

where $\Psi(x,y)$ is a Lyapunov first integral of the form (16). $\square$

Remark. We note that when $a_{20} \neq 0$ ($b_{02} \neq 0$) the systems (20) and (21) have the first integrals

$$\Psi(x,y) = \frac{xy\,(1-a_{20}\,x^2)^{\frac{b_{20}-a_{20}}{2\,a_{20}}}}{\sqrt{1-b_{02}\,y^2}}$$

and

$$\Psi(x,y) = \frac{xy\,(1-b_{02}y^2)^{\frac{a_{02}-b_{02}}{2b_{02}}}}{\sqrt{1-a_{20}\,x^2}},$$

respectively.

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