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GENERATING OF A CLASS OF LATTICES AND ITS APPLICATION[1]

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Abstract. In this paper we give construction of a new class of lattices from one lattice and application of this result in coding theory.

1. PRELIMINARIES

Let L be a finite complete lattice. Let denote operation infimum by $\wedge$, operation supremum by $\vee$, order relation on L by $\leq$, minimal element of L by 0 and maximal element of L by 1.

If S is nonempty set and L a complete lattice, than function $\overline{A} : S \to L$ is L-**fuzzy set** on S. $\overline{A}(x)$ is degree of belonging of the element $x \in S$ to the fuzzy set $\overline{A}$. For $\overline{A} : S \to L$ we define p-**level subset** (or p-**cutting**) of $\overline{A}$ with

$$A_p = \{x \in S : \overline{A}(x) \geq p\}.$$

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Function $\overline{A}_p : S \to \{0,1\}$ is suitable to the set A_p, such that $\overline{A}_p(x) = 1$ if and only if $\overline{A}(x) \geq p$. $\overline{A}_p$ is **characteristic function** of p-level subset.

For L-value fuzzy set on S, $\overline{A} : S \to L$, we denote **set of values** of fuzzy set $\overline{A}$ with $\overline{A}(S)$.

Element $a \in L$, $a \neq 1$ is **meet-irreducible** (or **i-irreducible)** if and only if $a = b \wedge c$ implies $a = b$ or $a = c$. Every element of the finite lattice L can be represented as infimum of meet-irreducible elements.

Theorem 1 ([7]). *Let L be a lattice of finite length, and let $\overline{A} : S \to L$ be an L-valued fuzzy set. Necessary and sufficient conditions under which all p-cuts of $\overline{A}$ are different are that the set of all meet-irreducible elements of L is a subset of $\overline{A}(S)$.*

Set of level functions of fuzzy set $\overline{A} : S \to L$ for set $S = \{1, 2, \ldots, n\}$ is **binary block-code of length** n, denoted by $V_{\overline{A}}$. For every fuzzy set $\overline{A}$ there is one correspondent code $V_{\overline{A}}$ (V for short), but for every code there can be more correspondent fuzzy sets.

For every fuzzy set $\overline{A} : S \to L$ there is a correspondent code of maximal cardinality $|L|$.

Theorem 2 ([7]). *Necessary and sufficient condition under which for L-valued fuzzy set suitable code V such that $|L| = |V|$ is the set of all meet-irreducible elements of L is a subset of $\overline{A}(S)$.*

2. RESULTS

Let L be a lattice with $|L| = m$ elements. Let $i \in N$ and $i < m$ be the number of meet-irreducible elements of lattice L. For this lattice L, fuzzy set $\overline{A} : S \to L$ can be constructed, with correspondent code V of maximal cardinality $m = |L| = |V|$. Minimal length of that code V is equal to the member of meet-irreducible elements of lattice L, so $|\overline{A}(S)| = i$ (Theorem 2).

Our goal is to construct a new lattice with minimal increment of its code length and maximal increment of its code cardinality. This can be achieved by using the following algorithms for construction of new lattices based on given lattice L. In the following, we describe these algorithms, named multiplicity and power of lattice L.

2.1. MULTIPLICITY OF LATICES

Let L, $|L| = m$, be a lattice and let $L^1, L^2, \ldots, L^i, \ldots, L^j, \ldots, L^{n-1}$ be different lattices with common zero (0) and isomorph with L. Let $\varphi^{ij} : L^i \to L^j$ be isomorphism from L^i to L^j ($i, j = 1, \ldots, n-1$, and $\varphi^{ij}(0) = 0$). Lattice L^*, $|L^*| = m$, is constructed in the following way.

Let $f^i : L^i \to L^*$ be isomorphism ($i = 1, \ldots, n-1$) defined with $f^i(0) = 0$ and $f^i(x) = X$ for all $x \in L^i$, $x \neq 0$ such that:

(1) $x < X$;

(2) for all $y \in L^i$ such that $y > x$ or $y \parallel x$ follows $y \parallel X$; [2]

(3) for all $X, Y \in L^*$ $x, y \in L^i$ exist such that $f^i(x) = X$ and $f^i(y) = Y$, it follows $X \vee Y = f^i(x \vee y)$ and $X \wedge Y = f^i(x \wedge y)$.

From relations

$$f^j(\varphi^{ij}(x)) = (\varphi^{ij} \circ f^j)(x) = f^i(x)$$

follows that

$$f^i(x) = X \in L^*$$

for all $x \in L^i$ ($i = 1, \ldots, n-1$).

Proposition 1. *If L is a lattice and $L_n = L^1 \cup L^2 \cup \ldots \cup L^{n-1} \cup L^*$, then L_n is a lattice.*

Proof. By procedure of lattice L^* construction described above, it follows that set L_n is partially ordered set. We shall prove that ordering on L_n is lattice ordering.

If $x, y \in L_n$ and there is $i \leq n$ such that $x, y \in L^*$ or $x, y \in L^i$ then $x \wedge y$ is infimum and $x \vee y$ supremum in lattice L^* or L^i.

[2] Here we use symbol $\parallel$ to denote that elements x and y are incomparable.

If $x, y \in L_n$ and $x \in L^i$, $y \in L^j$, $i \neq j$ then there is $z \in L^i$ such that $\varphi^{ij}(z) = y$ and

$$\begin{aligned} x \wedge y &= 0, \\ x \vee y &= x \vee \varphi^{ij}(z) = f^i(x) \vee f^j(\varphi^{ij}(z)) = f^i(x) \vee f^i(z) = f^i(x \vee \varphi^{ji}(y)). \end{aligned}$$

If $x, y \in L_n$, $x \in L^i$, $y \in L^*$ we have

$$\begin{aligned} x \vee y &= f^i(x) \vee y \quad \text{and} \\ x \wedge y &= \begin{cases} 0, & f^i(x) \parallel y; \\ x, & f^i(x) \leq y. \end{cases} \end{aligned}$$ □

Lattice $L_n \equiv [n \cdot L]$, $n \in N$ will be called **n-th product of lattice L**.

From construction of lattice L_n it is obvious that it has common 0 and 1 with lattice L^*, thus L^* is sublattice of L_n. Also, from procedure described above it follows that every lattice from set $\{L^1, L^2, \ldots, L^{n-1}, L^*\}$ has m elements, but 0 is common element (other elements are different) and $|L_n| = n \cdot m - (n-1) = n \cdot (m-1) + 1$.

Example 1.

(i) Let L be a lattice with seven elements, given in the Fig. 1.

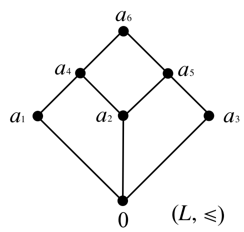


Fig. 1.

(ii) $L_2 = [2 \cdot L]$ is given in the Fig. 2.

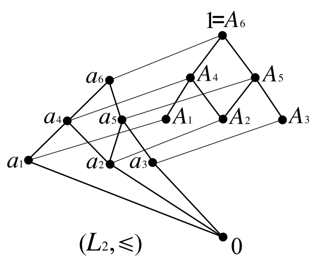


Fig. 2.

(iii) $L_3 = [3 \cdot L]$ is given in the Fig. 3.

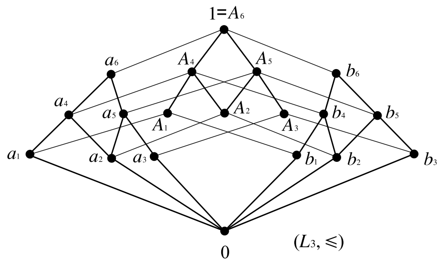


Fig. 3.

(iv) $L_5 = [5 \cdot L]$ is given in the Fig. 4.

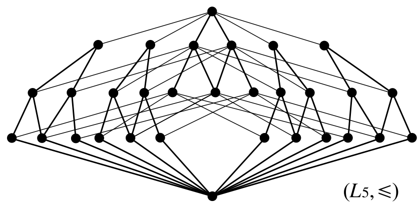


Fig. 4.

2.2. POWER OF LATTICE

Let L be a lattice, and $|L| = m$. Let L_n be a mode lattice obtained by procedure given above. If that procedure is applied on lattice L_n k-times, then we have a new lattice

$$(L_n)_k = [k \cdot L_n] = L_{n^k}$$

called k-**th product of lattice** L_n or n^k **power of** L.

Proposition 2. *If L is a lattice, $|L| = m$, L_n is n-th product of L, and $(L_n)_k \equiv L_{n^k}$ is n^k power of L $(n, k \in N)$, then*

$$|L_{n^k}| = n^k \cdot (m-1) + 1.$$

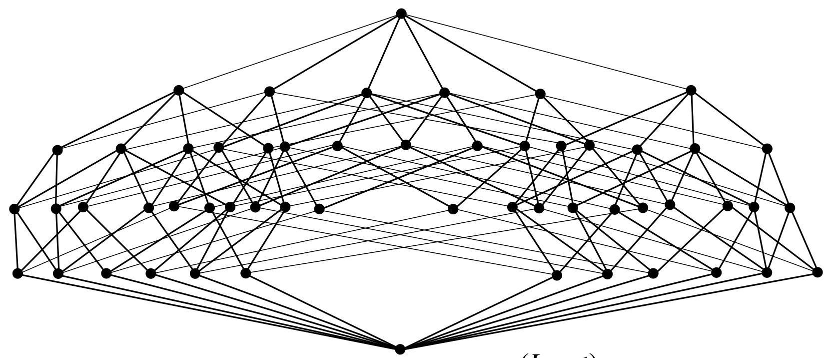
Proof. Proof is obtained by induction for $k \in N$. For $|L| = m$ and $k = 1$, lattice L_n has $n \cdot (m-1) + 1$ elements and formula $|L_{n^k}| = n^k(m-1) + 1$ holds true.

If lattice $L_{n^{k-1}}$ has $n^{k-1}(m-1)+1$ elements then from $L_{n^k} = [n \cdot L_{n^{k-1}}]$ follows

$$|L_{n^k}| = n \cdot (n^{k-1} \cdot (m-1) + 1 - 1) + 1 = n^k(m-1) + 1$$

and our formula holds true for all $k \in N$. □

Example 2. *In the Fig. 5. lattice L_{3^2} is given, where L_3 is lattice from example 1(iii).*



$(L_{3^2}, \leqslant)$

Fig. 5.

2.3. PRODUCT AND POWER OF LATTICES AND CODING

Proposition 3. *Let i, i_n and i_{n^k} $(n, k \in N)$ be numbers of meet-irreducible elements of lattices L, L_n and L_{n^k} respectively. Then the next formulas hold:*

(*a*) $i_n = i + n - 1$;

(*b*) $i_{n^k} = i + k \cdot (n-1)$.

Proof.

(*a*) From construction of n-th product of L it obviously follows that meet-irreducible elements of lattice L remain meet-irreducible in sublattice L^* and also in lattice L_n. Further, top elements of sublattices $L^1, L^2, \ldots, L^{n-1}$ are meet-irreducible elements in L_n, and our formula holds true.

(b) Straightforwardly, using (a) and induction by k. $\square$

Let $\overline{A} : S \to L$ be a fuzzy set and $\overline{A}(S)$ be a set of meet-irreducible elements of lattice L. Then code V with length $|\overline{A}(S)|$ and cardinality $|V| = |L| = m$ is correspondent to fuzzy set $\overline{A}$ (Theorem 2).

Also, from facts given above, holds:

(a) lattice L_n may have correspondent code V_n with length $i+n-1$ and cardinality

$$|V_n| = |L_n| = n \cdot (m-1) + 1;$$

(b) lattice L_{n^k} may have correspondent code V_{n^k} with length $i + k \cdot (n-1)$ and cardinality

$$|V_{n^k}| = |L_{n^k}| = n^k \cdot (m-1) + 1.$$

By products and powers of lattice L (and their combinations) described in this paper, we can construct lattices with larger desired cardinality of a given lattice L, but with slowly increasing number of meet-irreducible elements. This fact is important for coding theory, because base lattice L and described procedure can be used as the key for new codes.

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