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CONDITIONS OF EXISTENCE QUASIPERIODIC SOLUTIONS FOR SOME ORDINARY DIFFERENTIAL EQUATIONS OF FIRST AND SECOND ORDER

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Abstract. In this paper we give some conditions of existence quasiperiodic solutions to the ordinary homogeneous linear differential equation of first and second order. We note that the considered problem in this paper is examined with a method different than the methods in [1] and [2].

1. INTRODUCTION

Definition 1.1 *We say that $y = \varphi(x)$, $x \in I \subseteq D_\varphi \subset \mathbf{R}$ is a quasi-periodic function (QPF) if there are: a function $\omega(x)$ and a coefficient $\lambda = \lambda(\omega(x))$ such that the relation*

$$\varphi(x + \omega(x)) = \lambda\varphi(x), \quad x, x + \omega(x) \in I \tag{1}$$

is satisfied. The function $\omega(x)$ is called a quasi-period (QP) and λ is said to be a quasi-periodic coefficient (QPC) of the function $\varphi(x)$.

Example 1.1 $\varphi(x) = e^{-2x}\sin x$ is QPF with QP $\omega = 2\pi$ and QPC $\lambda = e^{-4\pi}$, since:

$$\forall x \in \mathbf{R}, \varphi(x+2\pi) = e^{-2(x+2\pi)}\sin(x+2\pi) = e^{-4\pi}e^{-2x}\sin x = e^{-4\pi}\varphi(x).$$

Example 1.2 $\psi(x) = e^{x^2}\sin x^2$ is QPF with QP $\omega = -x+\sqrt{x^2+2\pi}$ and QPC $\lambda = e^{2\pi}$, since:

$$\forall x \in \mathbf{R}, \psi(x+\omega) = e^{(x+\omega)^2}\sin(x+\omega)^2 = e^{2\pi}e^{x^2}\sin x^2 = e^{2\pi}\psi(x).$$

Remark 1.1

1. In the general case, when $\lambda = \lambda(x, \omega(x))$, the existence of the relation (1) is very complex problem.
2. If $\omega(x) = \omega^* = const$ and $\lambda = 1$ for $x \in I$, then (1) is a definition for a periodic function in a classical sense.
3. If $\omega = \omega(x) \neq const$ and $\lambda = 1$ for $x \in I$, then (1) is a generalization of the definition for a periodic function and in this case $\omega = \omega(x)$ is a function of "repeating values" of $y = \varphi(x)$.

2. PROBLEM FORMULATION

Suppose that the function $y(x)$ is given implicitly with the linear differential equation

$$F(x, y, y', y'', ..., y^{(n)}, f(x), g(x), ..., h(x)) = 0 \tag{2}$$

where $f(x), g(x), h(x)$ are continuous and $(n-1)-$ times differentiable functions on $I \subseteq D_f \cap D_g \ldots \cap D_h \cap D_y$. Let $y(x)$ be a QPS to (2), with QP $\omega = \omega(x)$ and QPC λ, i.e.:

$$y(x+\omega(x)) = \lambda y(x), \tag{3}$$

where $\omega = \omega(x) \in C_I^n, \lambda > 0,\ \lambda \neq 1,\ x,\ x+\omega(x) \in I$.

We describe the problem of existence QPS to (2) by the system

$$\begin{cases} F(x, y, y', y'', ..., y^{(n)}, f(x), g(x), ..., h(x)) = 0 \\ y(x+\omega) = \lambda y(x) \\ y(x+\omega)^{(k)} = \lambda y^{(k)}(x), \quad k = 1, 2, ..., n \end{cases} \tag{4}$$

Using the above system (4), we transform the equation (2) to a differential equation that generally is a linear differential equation of $(n-1)$ – order to y , but it is nonlinear to $\omega(x)$.

In this paper we consider the problem of existence QPS for homogeneous linear differential equations of first and second order.

3. QUASIPERIODICITY TO A HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION OF FIRST ORDER

Let (2) be a linear differential equation of first order in the form

$$y' + a(x)y = 0 \tag{5}$$

where $a(x)$ is a continuous and differentiable function on $I \subseteq D_a \cap D_y$.

Lemma 3.1 *Let $y(x)$ be QPS to the differential equation of first order (5), with QP $\omega = \omega(x)$ and QPC λ such that $\lambda > 0, \lambda \neq 1$. If $1 + \omega' \neq 0$ then (5) is reduced to the equation*

$$y[a(x+\omega)(1+\omega') - a(x)] = 0 \tag{6}$$

Proof. For (5) the system (4) has a form

$$\begin{cases} y' + a(x)y = 0 \\ (y(x+\omega))' + a(x+\omega)y(x+\omega) = 0 \\ y(x+\omega) = \lambda y(x) \\ y'(x+\omega)(1+\omega') = \lambda y' \end{cases} \tag{7}$$

Under the condition $1 + \omega' \neq 0$ we can eliminate $y'(x), y(x+\omega), y'(x+\omega)$ from (7). Thus we obtain (6). $\square$

Theorem 3.1 *The differential equation of first order (5) has a nontrivial QPS with QP ω and QPC $\lambda, (\lambda > 0, \lambda \neq 1)$ if the primitive function $A(x)$ for $a(x)$ satisfies the relation*

$$A(x+\omega) = A(x) + \ln\frac{1}{\lambda}, \quad \lambda > 0 \tag{8}$$

where $\lambda = e^{-c_1}$, $c_1 \in \mathbf{R}$

Proof. Let $y(x)$ be a nontrivial QPS with QP ω. As we supposed that $a(x)$ is a continuous function, there is the indefinite integral

$$\int a(x)dx = A(x) + C_1.$$

Then, with the substitution

$$x + \omega = z(x),$$

from Lemma 3.1 we obtain

$$A(x+\omega) = A(x) + c_1, \quad c_1 \in \mathbf{R} \tag{9}$$

At the other side, since the solution for (5) is

$$y(x) = Ce^{A(x)}$$

we obtain that $y(x)$ is a QPS for (5) if

$$e^{-(A(x+\omega)-A(x))} = \lambda, \quad \lambda > 0,$$

i.e.

$$A(x+\omega) = A(x) + \ln\frac{1}{\lambda}, \quad \lambda > 0. \qquad \square$$

Corollary 3.1 *If the equation (5) has QPS with QP $\omega = \omega(x)$ and QPC $\lambda = 1$, then the primitive function $A(x)$ to $a(x)$ is QPF with the same QP $\omega = \omega(x)$ and QPC $\mu = 1$, i.e.,*

$$A(x+\omega) = A(x). \tag{10}$$

Corollary 3.2 *If $A(x)$ is a periodic function with a period $\omega = \varpi = const$, then the solution $y(x)$ to (5) is a periodic function with a period $\omega = \varpi$ (the known result).*

Remark 3.1 If $a(x)$ is a periodic function, but its primitive function $A(x)$ is not a periodic one, then the solution $y(x)$ is a quasiperiodic function.

Example 3.1 Let

$$y' + (1 + \cos x)y = 0.$$

The coefficient $a(x) = 1 + \cos x$ is a periodic function with a base period $\omega = 2\pi$, but its primitive function $A(x) = x + \sin x$ is not a periodic one, i.e., it holds

$$A(x + 2\pi) = A(x) + 2\pi.$$

Thus the solution $y = Ce^{-(x+\sin x)}$ is QPF: $y(x + 2\pi) = e^{-2\pi}y(x)$.

Corollary 3.3 *If the equation (5) has QPS with QP ω and QPC $\lambda(\lambda > 0, \lambda \neq 1)$ and the primitive function $A(x)$ for $a(x)$ $(a(x) \neq 0)$ is a monotonous one, then QP ω is given by:*

$$\omega(x) = -x + A^{-1}\Big(A(x) + \ln\frac{1}{\lambda}\Big),$$

where A^{-1} is the inverse function of A.

Example 3.2 Let

$$y' + e^x y = 0. \tag{11}$$

The primitive functions of $a(x) = e^x$ are $A(x) = e^x + C$, that are monotonous functions. Hence, (11) has QPS $y = Ce^{-e^x}$ with QP $\omega(x) = -x + \ln(e^x - c_1)$ and QPC $\lambda = e^{c_1}C$, for $x > \ln c_1$.

Example 3.3 Let

$$y' + x\mathrm{ctg}x^2 \cdot y = 0 \tag{12}$$

The primitive function of $a(x) = x\mathrm{ctg}x^2$ is $A(x) = \dfrac{1}{2}\ln\sin x$,

$$x^2 \in (2k\pi, (2k+1)\pi), k \in \mathbf{Z},$$

that satisfies the relation (8) for

$$\omega(x) = -x \pm \sqrt{\mathrm{Arcsin}(\frac{1}{\lambda^2}\sin x^2)}, \quad \lambda = e^{c_2}, \quad c_2 \in \mathbf{R}.$$

Then $y = \dfrac{C}{\sqrt{\sin x^2}}$ are QPS for (12).

4. CONDITIONS OF EXISTENCE QPS TO HOMOGENEOUS DIFFERENTIAL EQUATIONS OF SECOND ORDER

Let (2) be a linear differential equation of second order

$$y'' + f(x)y' + g(x)y = 0 \tag{13}$$

where $f(x)$ and $g(x)$ are continuous and two times differentiable functions on $I \subseteq D_f \cap D_g \cap D_y$.

Theorem 4.1 *Let $y(x)$ be QPS to the equation (13) with QP $\omega = \omega(x)$ and QPC λ such that $\lambda > 0$, $\lambda \neq 1$. If $1 + \omega' \neq 0$ then (13) is reduced to the equation*

$$\begin{aligned} y'[f(x+\omega)\cdot(1+\omega')^2 - f(x)\cdot(1+\omega') - \omega''] \\ +y\cdot[g(x+\omega)\cdot(1+\omega')^3 - g(x)(1+\omega')] = 0, \end{aligned} \tag{14}$$

i.e.

$$\frac{y'}{y} = -\frac{g(x+\omega)\cdot(1+\omega')^3 - g(x)(1+\omega')}{f(x+\omega)\cdot(1+\omega')^2 - f(x)(1+\omega') - \omega''}. \tag{15}$$

Proof. For (13) the system (4) is

$$\begin{cases} y'' + f(x)y' + g(x)y = 0 \\ (y(x+\omega))'' + f(x+\omega)\cdot(y(x+\omega))' + g(x+\omega)\cdot y(x+\omega) = 0 \\ y(x+\omega) = \lambda y(x) \\ y'(x+\omega)(1+\omega') = \lambda y'(x) \\ y''(x+\omega)(1+\omega')^2 + y'(x+\omega)\omega'' = \lambda y''(x) \end{cases} \tag{16}$$

For $1 + \omega' \neq 0$ we can eliminate $y''(x), y''(x+\omega), y'(x+\omega), y(x+\omega)$ from (16). Thus we reduce the equation (13) to the equation (14), i.e. (15). $\square$

Remark 4.1 The right-hand side in (15) is a function of $f = f(x)$, $g = g(x)$ and $\omega = \omega(x)$. If we denote

$$F(x) = F(x, f, g, \omega) = \frac{g(x+\omega)\cdot(1+\omega')^3 - g(x)(1+\omega')}{f(x)(1+\omega') - f(x+\omega)\cdot(1+\omega')^2 + \omega''} \tag{17}$$

then (15) is

$$\frac{y'}{y} = F(x, f, g, \omega). \tag{18}$$

and if f, g, ω are known functions we can find the solution to (13) in the form

$$y = Ce^{\int\limits_{x_0}^{x} F(t,f,g,\omega)dt} \tag{19}$$

It follows that quasiperiodicity to the solution of (13) depends on the functions f and g. Since (14) is a functional differential equation of ω, it follows that in a general case it is not easy solvable.

Here we consider the problem of existence QPS to (13) if $\omega = \varpi = const$.

Lemma 4.1 *Let $y(x)$ be QPS to (13) with a constant QP ϖ and QPC λ such that $\lambda > 0, \lambda \neq 1$. Then it holds*

$$y'[f(x+\varpi) - f(x)] + y \cdot [g(x+\varpi) - g(x)] = 0. \tag{20}$$

Proof. Substituting $\omega = \varpi, \varpi' = \varpi'' = 0$ in (14) we obtain (20). $\square$

Theorem 4.2 *Let the coefficients $f = f(x)$ and $g = g(x)$ be QPF with a constant QP ϖ and QPC μ, ν respectively, such that $\mu \neq \nu$ and $\mu \neq 1$, $\nu \neq 1$. The equation (13) has QPS $y(x)$ with QP ϖ and QPC $\lambda(\lambda > 0, \lambda \neq 1)$, if:*

$$G^*(x+\varpi) - G^*(x) = \ln\frac{1}{\lambda}, \quad G^* = -\frac{\nu-1}{\mu-1}\int\frac{g}{f}; \quad \textit{and} \tag{21}$$

$$\Big(\frac{g}{f}\Big)' - \Big(\frac{\nu-1}{\mu-1}\Big)\cdot\Big(\frac{g}{f}\Big)^2 - \frac{\mu-\nu}{\nu-1}\cdot g = 0. \tag{22}$$

Proof. From

$$f(x+\varpi) = \mu f(x), g(x+\varpi) = \nu g(x), \mu \neq \nu, \mu \neq 1, \nu \neq 1,$$

using the Lemma 4.1., we reduce (13) to the equation

$$y' + \frac{\nu-1}{\mu-1}\cdot\frac{g}{f}\cdot y = 0. \tag{23}$$

From the Theorem 4.1. we obtain that the last equation has QPS with QP ϖ and QPC λ if the condition (21) holds. At the other hand, using the fact that the solution for (23) is also a solution for (13), both of these equations should be satisfied. Thus, after short transformation we obtain that if y is QPS to (13) then f and g have to be related by (22). $\square$

Theorem 4.3 *Let the coefficient $f(x)$ be a periodic function with a period ϖ. If the equation (13) has QPS $y(x)$ with QP ϖ and QPC λ ($\lambda > 0, \lambda \neq 1$), then the coefficient $g(x)$ is also a periodic function.*

Proof. Let $f(x+\varpi) = f(x)$. Then from Lemma 4.1. we obtain the equation

$$y \cdot [g(x+\varpi) - g(x)] = 0.$$

If $y(x)$ is a nontrivial solution for (13), the last equation is satisfied if

$$g(x+\varpi) = g(x). \quad \square$$

Example 4.1 The coefficients of the equation

$$y'' + (1 - \sin x)y' - \cos x \cdot y = 0$$

are periodic functions with a period $\varpi = 2\pi$. The solution $y = Ce^{-(x+\cos x)}$ is a QPF with QP $\varpi = 2\pi$ and QPC $\lambda = e^{-2\pi}$.

References

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