Kragujevac J. Math. 28 (2005) 215–224.

CONDITIONS OF EXISTENCE QUASIPERIODIC SOLUTIONS FOR SOME NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER AND THIRD POWER

Marija Kujumdzieva-Nikoloska[1] and Jordanka Mitevska[2]

[1]*Faculty of Electrical Engineering,*
University "St. Kiril and Methodius", Skopje, R. Macedonia
(e-mail: marekn@etf.ukim.edu.mk)

[2]*Faculty of Natural Sciences and Mathematics, Department of Mathematics*
University "St. Kiril and Methodius", Skopje, R. Macedonia
(e-mail: jordanka@iunona.pmf.ukim.edu.mk)

Abstract. Using the condition of quasiperiodicity of the solution to the nonlinear differential equation of second order (1), we reduce it to the linear differential equation of second or first order. Then we find conditions of existence quasiperiodic solutions with a constant quasiperiod and give the form of the solutions.
Used abbreviations: differential equation (DFE), quasiperiod (QP), periodic solution (PS), quasiperiodic solution (QPS), quasiperiodic coefficient (QPC)

1. PROBLEM FORMULATION

Let the differential equation

$$y''(x) + f_1(x)y'(x) + f_2(x)y(x) + f_3(x)y^3(x) + f_4(x) = 0 \quad (f_3(x) \neq 0) \tag{1}$$

be given. We want to find a quasiperiodic solution $y = y(x)$ for the equation (1), i.e. to find the solution that satisfy the relation

$$y(x+\omega) = \lambda(x)y(x), \quad x, x+\omega \in D_y \tag{2}$$

where $\omega = \omega(x)$ is a QP and $\lambda = \lambda(x)$ is a QPC of the function $y = y(x)$.

Theorem 1.1 *If the equation (1) has a QPS $y = y(x)$ with a QP $\omega = \omega(x)$ and a QPC λ, then it is reduced to the linear nonhomogeneous DFE of second order to $y(x)$*

$$\begin{aligned}
&\lambda\left(\frac{1}{t'^2} - \tfrac{f_3(t)}{f_3(x)}\lambda^2\right)y'' + \left(\tfrac{2\lambda'}{t'^2} + \left(-\tfrac{t''}{t'^3} + \tfrac{f_1(t)}{t'}\right)\lambda - \tfrac{f_3(t)f_1(x)}{f_3(x)}\lambda^3\right)y' \\
&\quad + \left(\frac{\lambda''}{t'^2} + \left(-\frac{t''}{t'^3} + \frac{f_1(t)}{t'}\right)\lambda' + f_2(t)\lambda - \frac{f_3(t)f_2(x)}{f_3(x)}\lambda^3\right)y \\
&\quad + \left(f_4(t) - \frac{f_3(t)f_4(x)}{f_3(x)}\lambda^3\right) = 0, \\
&t = x + \omega(x), \quad t' = 1 + \omega', \quad t'' = \omega''
\end{aligned} \tag{3}$$

Proof. Under the conditions of the theorem we have the following system:

$$\begin{cases}
y''(x) + f_1(x)y'(x) + f_2(x)y(x) + f_3(x)y^3(x) + f_4(x) = 0 \\
y''(t) + f_1(t)y'(t) + f_2(t)y(t) + f_3(t)y^3(t) + f_4(t)_{/t=x+\omega} = 0 \\
y(t) = \lambda(x)y(x) \\
\dfrac{d}{dx}y(t) = y'(t)t' = \lambda'(x)y(x) + \lambda(x)y'(x) \\
\dfrac{d^2}{dx^2}y(t) = y''(t)t'^2 + y'(t)t'' = \lambda''(x)y(x) + 2\lambda'(x)y'(x) + \lambda(x)y''(x)
\end{cases} \tag{4}$$

from where we are finding

$$y^3(x) = -\frac{1}{f_3(x)}\Big(f_4(x) + f_2(x)y(x) + f_1(x)y'(x) + y''(x)\Big) \tag{5}$$

$$y''(t) = \frac{1}{t'^2}\left(\Big(\lambda''(x) - \frac{t''}{t'}\lambda'(x)\Big)y(x) + \Big(2\lambda'(x) - \frac{t''}{t'}\lambda(x)\Big)y'(x) + \lambda(x)y''(x)\right) \tag{6}$$

Substituting (5) and (6) in the second equation of (4), after short transformations, we obtain (3). $\square$

Remark 1.1 In a general case, even though the equation (3) is a linear one of $y(x)$, this is not easy solvable as it is a functional differential equation of ω.

In this paper we consider the eq.(1) at some special cases for ω and λ.

2. QUASIPERIODIC SOLUTIONS WITH CONSTANT QP AND QPC

Theorem 2.1 *If the eq.(1) has a QPS $y = y(x)$ with a QP $\omega = const = c$ and QPC $\lambda = const > 0$ then it is reduced to the equation*

$$\begin{aligned}&\lambda(f_3(x) - \lambda^2 f_3(t))y'' + \lambda(f_1(t)f_3(x) - \lambda^2 f_3(t)f_1(x))y' \\ &\qquad + \lambda(f_2(t)f_3(x) - \lambda^2 f_2(x)f_3(t))y + (f_4(t)f_3(x) - \lambda^3 f_4(x)f_3(t)) = 0\end{aligned} \tag{7}$$

Proof. Substituting in (3): $\omega = c,\ \omega' = 0,\ t = x + c,\ t' = 1$, we obtain (7). □

Theorem 2.2 *If the coefficients $f_1(x), f_2(x), f_3(x), f_4(x)$ in the eq.(1) are QPF satisfying the relations*

$$f_1(t) = f_1(x), \quad f_2(t) = f_2(x), \quad f_3(t) = \frac{1}{\lambda^2} f_3(x), \quad f_4(t) = \lambda f_4(x)_{/t=x+\omega} \tag{8}$$

and (1) has a QPS, then every solution for (1) is a QPF with a QP $\omega = c$ and QPC λ.

Proof. Under the conditions of the theorem we have:

$$\begin{aligned}&y''(t) + f_1(t)y'(t) + f_2(t)y(t) + f_3(t)y^3(t) + f_4(t)_{/t=x+\omega} \\ &\qquad = \lambda y''(x) + \lambda f_1(x)y'(x) + \lambda f_2(x)y(x) + \tfrac{1}{\lambda^2} \cdot \lambda^3 f_3(x)y^3(x) + \lambda f_4(x) \\ &\qquad = \lambda(y''(x) + f_1(x)y'(x) + f_2(x)y(x) + f_3(x)y^3(x) + f_4(x)) \\ &\qquad = \lambda \cdot 0 = 0. \quad \square\end{aligned}$$

Example 2.1 Let

$$y'' - \cos x \cdot y' - \sin x \cdot y + e^{-4x} \cos x \cdot y^3 + e^{2x}(1 - 3\sin x - 4\cos x + \sin 2x - \sin^3 x \cos x) = 0.$$

All of the solutions are QPF. One particular solution is $y_1 = e^{2x} \sin x$ that is a QPF with a QP $\omega = 2\pi$ and QPC $\lambda = e^{4\pi}$.

Example 2.2 Let

$$y'' - 4y' + 5\sin x \cdot y - 5e^{-4x}y^3 + 5e^{2x}(1 - \sin x + \sin^2 x)\sin x = 0.$$

All of the solutions are QPF. One particular solution is $y_1 = e^{2x} \sin x$ that is a QPF with a QP $\omega = 2\pi$ and QPC $\lambda = e^{4\pi}$.

Using the Theorem 2.1, under the determined conditions of the coefficients in the equation (1), we have the following theorems.

Theorem 2.3 *Let the eq.(1) has a QPS with a QP $\omega = c$, a QPC $\lambda(\lambda > 0)$ and the coefficients $f_1(x), f_2(x), f_3(x), f_4(x)$ are QPF with the same QP $\omega = c$ and QPC $\lambda_1, \lambda_2, \lambda_3 \neq \frac{1}{\lambda^2}, \lambda_4$ respectively. Then the eq.(1) is reduced to the equation*

$$y'' + \frac{\lambda_1 - \lambda^2\lambda_3}{1 - \lambda^2\lambda_3} f_1(x)y' + \frac{\lambda_2 - \lambda^2\lambda_3}{1 - \lambda^2\lambda_3} f_2(x)y + \frac{\lambda_4 - \lambda^3\lambda_3}{\lambda(1 - \lambda^2\lambda_3)} f_4(x) = 0. \tag{9}$$

Theorem 2.4 *Let $y(x)$ be QPS to (1) with QP $\omega = c$, QPC $\lambda(\lambda > 0, \lambda \neq 1)$ and the coefficients $f_1(x), f_2(x), f_3(x), f_4(x)$ be QPF with a QP $\omega = c$ and QPC $\lambda_1, \lambda_2, \lambda_3 = \frac{1}{\lambda^2}, \lambda_4$, respectively, then the eq.(1) is transformed in*

$$\lambda(\lambda_1 - 1)f_1(x)y' + \lambda(\lambda_2 - 1)f_2(x)y + (\lambda_4 - \lambda)f_4(x) = 0 \tag{10}$$

Theorem 2.5 *Let $y(x)$ be QPS to (1) with QP $\omega = c$, QPC $\lambda(\lambda > 0, \lambda \neq 1)$ and the coefficients $f_1(x), f_2(x), f_3(x), f_4(x)$ be QPF with a QP $\omega = c$ and QPC $\lambda_1 = 1, \lambda_2, \lambda_3 = \frac{1}{\lambda^2}, \lambda_4$ respectively. Then the eq.(1) is transformed in*

$$\lambda(\lambda_2 - 1)f_2(x)y + (\lambda_4 - \lambda)f_4(x) = 0. \tag{11}$$

Theorem 2.6 *Let the eq.(1) has a QPS and the coefficients $f_1(x), f_2(x) \neq 0$, $f_3(x), f_4(x)$ be QPF with a QP $\omega = c$ and QPC $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ respectively. Then the eq.(1) has a QPS*

$$y = -\frac{f_4(x)}{f_2(x)} \tag{12}$$

with a QP $\omega = c$ and QPC $\lambda = \frac{\lambda_4}{\lambda_2}$ if the relation

$$\left(\frac{f_4(x)}{f_2(x)}\right)'' + f_1(x)\left(\frac{f_4(x)}{f_2(x)}\right)' + f_3(x)\left(\frac{f_4(x)}{f_2(x)}\right)^3 = 0 \tag{13}$$

is satisfied.

Proof. Using the conditions

$$\begin{aligned} f_1(x+c) &= \lambda_1 f_1(x), \quad f_2(x+c) = \lambda_2 f_2(x), \\ f_3(x+c) &= \lambda_3 f_3(x), \quad f_4(x+c) = \lambda_4 f_4(x) \end{aligned}$$

and the eq.(7) we have:

a) If $\lambda_3 = \dfrac{\lambda_2^2}{\lambda_4^2}, \lambda_1 = 1, \lambda_2 \neq 1$ using the Theorem 2.5 we reduce the eq.(1) to the equation

$$\lambda(\lambda_2 - 1) f_2(x) y + (\lambda_4 - \lambda) f_4(x) = 0$$

whose solution is

$$y = -\mu_1 \frac{f_4(x)}{f_2(x)} \tag{14}$$

where $\mu_1 = \dfrac{\lambda_4 - \lambda}{\lambda(\lambda_2 - 1)}$. The solution (14) is a QPF with a QP $\omega = c$ and a QPC $\lambda = \dfrac{\lambda_4}{\lambda_2}$ for which $\mu_1 = 1$. Thus, from (14) we obtain (12) and (13) since y is QPS for eq.(1).

b) If $\lambda_3 = \dfrac{\lambda_2^2}{\lambda_4^2}, \lambda_1 \neq 1, \lambda_2 \neq 1, \lambda_2 \neq \lambda_1$, using the Theorem 2.4 we reduce the eq.(1) to the equation

$$\lambda(\lambda_1 - 1) f_1(x) y' + \lambda(\lambda_2 - 1) f_2(x) y + (\lambda_4 - \lambda) f_4(x) = 0,$$

i.e.

$$y' + \frac{\lambda_2 - 1}{\lambda_1 - 1} \frac{f_2(x)}{f_1(x)} y + \frac{\lambda_4 - 1}{\lambda(\lambda_1 - 1)} \frac{f_4(x)}{f_1(x)} = 0, \tag{15}$$

or

$$y' + G(x) y + H(x) = 0 \tag{16}$$

where the coefficients $G(x), H(x)$ are QPF with a QP $\omega = c$ and QPC $\dfrac{\lambda_2}{\lambda_1}, \dfrac{\lambda_4}{\lambda_1}$. So, ([1], [2]), we can reduce the eq.(16) to the equation

$$\lambda(\lambda_2 - \lambda_1) G(x) y - (\lambda\lambda_1 - \lambda_4) H(x) = 0 \tag{17}$$

whose solution is

$$y = -\mu_2 \frac{f_4(x)}{f_2(x)} \tag{18}$$

where $\mu_2 = \dfrac{(\lambda_4 - \lambda)(\lambda_4 - \lambda\lambda_1)}{\lambda^2(\lambda_2 - \lambda_1)(\lambda_2 - 1)}$. The solution (18) is a QPF with a QP $\omega = c$ and QPC $\lambda = \dfrac{\lambda_4}{\lambda_2}$ from where we obtain $\mu_2 = 1$. Thus, from (18) we obtain (12), and since is QPS to (16) and to (1) we have $\left(\dfrac{f_4(x)}{f_2(x)}\right)' = 0$ and $f_3(x)\dfrac{f_4(x)}{f_2(x)} = 0$. So $f_4(x) = 0$ and $y = 0$.

c) If $\lambda_3 \neq \frac{1}{\lambda^2} = \frac{{\lambda_2}^2}{{\lambda_4}^2}$, using the Theorem 2.3 we reduce the eq.(1) to the equation (9), i.e.

$$y'' + f(x)y' + g(x)y + h(x) = 0 \tag{19}$$

where the coefficients $f(x), g(x), h(x)$ are QPF with a QP $\omega = c$ and QPC $\lambda_1, \lambda_2, \lambda_4$. Now, we reduce the eq.(19) to the equation

$$\lambda(\lambda_1 - 1)f(x)y' + \lambda(\lambda_2 - 1)g(x)y + (\lambda_4 - \lambda)h(x) = 0 \tag{20}$$

Depending on λ_1, the following two cases are possible:

1. If $\lambda_1 = 1, \lambda_2 \neq 1, \lambda_2 \neq \lambda^2\lambda_3$, then

$$y = -\mu_3 \frac{f_4(x)}{f_2(x)}, \tag{21}$$

where $\mu_3 = \frac{(\lambda_4 - \lambda)(\lambda_4 - \lambda^3\lambda_3)}{\lambda^2(\lambda_2 - \lambda^2\lambda_3)(\lambda_2 - 1)}$. The solution (21) is a QPF with a QP $\omega = c$ and QPC $\lambda = \frac{\lambda_4}{\lambda_2}$ for which $\mu_3 = 1$. Thus, from (21) we obtain (12). Since is QPS for (19) and (1) we have $\left(\frac{f_4(x)}{f_2(x)}\right)'' + f_1(x)\left(\frac{f_4(x)}{f_2(x)}\right)' = 0$ and $f_3(x)\left(\frac{f_4(x)}{f_2(x)}\right)^3 = 0$, so $f_4(x) = 0$ and $y = 0$.

2. If $\lambda_1 \neq 1, \lambda_2 \neq \lambda_1, \lambda_1 \neq \lambda\lambda_3, \lambda_2 \neq \lambda\lambda_3, \lambda_2 \neq 1, f_1(x) \neq 0$ then we reduce the eq.(1) to the equation

$$y' + \frac{(\lambda_2 - 1)(\lambda_2 - \lambda^2\lambda_3)}{(\lambda_1 - 1)(\lambda_1 - \lambda^2\lambda_3)} \cdot \frac{f_2(x)}{f_1(x)} y + \frac{(\lambda_4 - \lambda)(\lambda_4 - \lambda^3\lambda_3)}{\lambda^2(\lambda_1 - 1)(\lambda_1 - \lambda^2\lambda_3)} \cdot \frac{f_4(x)}{f_1(x)} = 0$$

i.e.

$$y' + \varphi(x)y + \psi(x) = 0. \tag{22}$$

The coefficients $\varphi(x), \psi(x)$ are QPF with a QP $\omega = c$ and QPC $\frac{\lambda_2}{\lambda_1}, \frac{\lambda_4}{\lambda_1}$ respectively. So ([1]), we reduce the last equation to the equation

$$\frac{\lambda(\lambda_2 - 1)(\lambda_2 - \lambda_1)(\lambda_2 - \lambda^2\lambda_3)}{(\lambda_1 - 1)(\lambda_1 - \lambda^2\lambda_3)} f_2(x)y + \frac{(\lambda_4 - \lambda\lambda_1)(\lambda_4 - \lambda)(\lambda_4 - \lambda^3\lambda_3)}{\lambda^2(\lambda_1 - 1)(\lambda_1 - \lambda^2\lambda_3)} f_4(x) = 0,$$

from where we find

$$y = -\mu_4 \frac{f_4(x)}{f_2(x)},$$

and $\mu_4 = \dfrac{(\lambda_4-\lambda)(\lambda_4-\lambda\lambda_1)(\lambda_4-\lambda^3\lambda_3)}{\lambda^3(\lambda_2-1)(\lambda_2-\lambda_1)(\lambda_2-\lambda^2\lambda_3)}$.

Since QPC for y is $\lambda = \dfrac{\lambda_4}{\lambda_2}$, then under the given conditions we have $\mu_4 = 1$, and the QPS for (1) is

$$y = -\frac{f_4(x)}{f_2(x)}. \quad \square$$

Using the fact that the solution (12) is also the solution to the eq.(1), we obtain that the coefficients f_2, f_3, f_4 have to satisfy the relation (13). Since y is QPS for (22),(19) and (1) we have

$$f_1(x)\left(\frac{f_4(x)}{f_2(x)}\right)' = 0, \quad \left(\frac{f_4(x)}{f_2(x)}\right)'' + \frac{{\lambda_2}^2\lambda_1 - {\lambda_4}^2\lambda_3}{{\lambda_2}^2 - {\lambda_4}^2\lambda_3} f_1(x)\left(\frac{f_4(x)}{f_2(x)}\right)' = 0 \quad \text{and}$$

$$f_3(x)\left(\frac{f_4(x)}{f_2(x)}\right)^3 = 0.$$

So we obtain $f_4(x) = 0$ and $y = 0$.

Example 2.3 The equation

$$y'' + \sin x \cdot y' + e^x y - \frac{e^{-4x}}{\sin^3 x}(3\sin x + 4\cos x + 2\sin^2 x + \sin x \cos x)y^3 - e^{3x}\sin x = 0$$

has coefficients which are QPF with a QP $\omega = 2\pi$ and QPC $\lambda_1 = 1, \lambda_2 = e^{2\pi}, \lambda_3 = e^{8\pi}, \lambda_4 = e^{6\pi}$ respectively. Thus, according to the Theorem 2.6., the QPS for the given DFE is

$$y = -\frac{f_4(x)}{f_2(x)} = e^{2x}\sin x \quad (\omega = 2\pi, \quad \lambda = \frac{\lambda_4}{\lambda_2}).$$

Example 2.4 The equation

$$y'' + e^x y + \frac{1}{\sin^2 x} y^3 - e^x \sin x = 0$$

has coefficients $f_1(x) = 0$, $f_2(x) = e^x$, $f_3(x) = \dfrac{1}{\sin^2 x}$ and $f_4(x) = e^x \sin x$ which are QPF with a QP $\omega = 2\pi$ and QPC $\lambda_1 = 1$, $\lambda_2 = e^{2\pi}$, $\lambda_3 = 1$, $\lambda_4 = e^{2\pi}$ respectively. Thus, according to the Theorem 2.6, the QPS for the given DFE is

$$y = -\frac{f_4(x)}{f_2(x)} = \sin x, \quad (\omega = 2\pi, \quad \lambda = 1 = \frac{\lambda_4}{\lambda_2}).$$

Example 2.5 The equation

$$y'' + e^{x+\cos x}\sin x \cdot y - e^{-2x}y^3 - e^{2x+\cos x}\sin x = 0$$

has coefficients $f_1(x) = 0$, $f_2(x) = e^{x+\cos x}\sin x$, $f_3(x) = -e^{-2x}$, $f_4(x) = -e^{2x+\cos x}\sin x$ which are QPF with the same QP $\omega = 2\pi$ and QPC $\lambda_1 = 1$, $\lambda_2 = e^{2\pi}$, $\lambda_3 = e^{-4\pi}$, $\lambda_4 = e^{4\pi}$, respectively. Thus, according to the Theorem 2.6., the QPS for the given DFE is

$$y = -\frac{f_4(x)}{f_2(x)} = e^x.$$

3. QPS FOR SOME SPECIAL NONLINEAR DIFFERENTIAL EQUATIONS

Theorem 3.1 *If the DFE*

$$y'' + \alpha y' + \beta y + \gamma y^3 = -f_4(x) \quad (\gamma \neq 0), \tag{23}$$

that is a DFE of nonlinear oscillations, has QPS $y = y(x)$ with a QP $\omega = c$ and a QPC $\lambda = const$ then it is reduced to

$$\lambda(1-\lambda^2)(y'' + \alpha y' + \beta y) = -f_4(x+c) + \lambda^3 f_4(x) \tag{24}$$

Proof. If $\alpha = f_1(x)$, $\beta = f_2(x)$, $\gamma = f_3(x)$, then (1) has a form (23). By the Theorem 2.1. it is reduced to the eq.(24). $\square$

Theorem 3.2 *The equation (23) has a PS with a period $\omega = c$, if and only if the coefficient $f_4(x)$ is a PF with the same period $\omega = c$.*

Proof. If $\lambda = 1$ then, from (23), we obtain $f_4(x) = f_4(x+c)$, i.e. the coefficient $f_4(x)$ has to be a PF. In reverse, if $f_4(x)$ is a PF with a period $\omega = c$, then we have $\lambda = 1$, i.e. the solution for (23) is a periodic function, since

$$\lambda(1-\lambda^2)(y'' + \alpha y' + \beta y) - (1-\lambda^3)f_4(x) = 0$$

i.e

$$(1-\lambda)(\lambda(1+\lambda)(y'' + \alpha y' + \beta y) - (1+\lambda+\lambda^2)f_4(x)) = 0 \quad \square$$

Theorem 3.3 *Let $f_4(x)$ is a QPF with a QP $\omega = c$ and a QPC $\lambda_4 = \lambda^3$, $\lambda \neq 1$. Then QPS for (23) is*

$$y^3 = -\frac{f_4(x)}{\gamma} \tag{25}$$

if

$$3f_4''(x) \cdot f_4(x) - 2{f_4'}^2(x) + 3\alpha f_4'(x) \cdot f_4(x) + 9\beta {f_4}^2(x) = 0 \tag{26}$$

Proof. From (24) we have

$$\lambda(1-\lambda)(y'' + \alpha y' + \beta y) = (\lambda^3 - \lambda_4)f_4(x).$$

Since $\lambda \neq 1$, it follows

$$y'' + \alpha y' + \beta y = 0 \tag{27}$$

Then from (27) and (23) we obtain

$$\gamma y^3 + f_4(x) = 0$$

from where

$$y = -\sqrt[3]{\frac{f_4(x)}{\gamma}}$$

that is QPS for (23) if the relation (26) is satisfied. $\square$

Remark 3.1 From (27) we have that $y = e^{rx}$ where r is a solution for the characteristic equation

$$r^2 + \alpha r + \beta = 0$$

i.e.

$$r_{1/2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$$

Thus, we can consider the following cases:

1. If $D^2 = \alpha^2 - 4\beta > 0$ then $f_4(x) = ke^{\frac{3}{2}(-\alpha-D)x}$ or $f_4(x) = ke^{\frac{3}{2}(-\alpha+D)x}$ and the QPS is $y = Ce^{\frac{1}{2}(-\alpha-D)x}$ with a QPC $\lambda = e^{\frac{1}{2}(-\alpha-D)c}$, or $y = Ce^{\frac{1}{2}(-\alpha+D)x}$ with a QPC $\lambda = e^{\frac{1}{2}(-\alpha+D)c}$.
2. If $D^2 = \alpha^2 - 4\beta = 0$ then $f_4(x) = ke^{-\frac{3}{2}\alpha x}$ and the QPS is $y = Ce^{-\frac{\alpha}{2}x}$ with a QPC $\lambda = e^{-\frac{\alpha}{2}c}$.

3. If $-D^2 = \alpha^2 - 4\beta < 0$ then $f_4(x) = e^{-\frac{3}{2}\alpha x}\left(A\cos\dfrac{3}{2}Dx + B\sin\dfrac{3}{2}Dx\right)$ and the QPS is $y = e^{-\frac{\alpha}{2}x}(a\cos\frac{D}{2}x + b\sin\frac{D}{2}x)$ with a QPC $\lambda = e^{-\frac{\alpha}{2}c}$.

Example 3.1 The equation

$$y'' + \alpha y' + \frac{\alpha^2}{4}y + y^3 - e^{-\frac{3}{2}\alpha x} = 0$$

has QPS $y = e^{-\frac{\alpha}{2}x}$.

Example 3.2 The equation

$$y'' - 4y' + 5y - y^3 + e^{6x}\cos^3 x = 0$$

has a QPS $y = e^{2x}\cos x$.

Example 3.3 The equation

$$y'' + y - y^3 + (1-\cos 2x)\sqrt{2}\sin x = 0$$

has periodic solutions $y = \sqrt{2}\sin x$.

References

[1] J. Митевска, М. Кујумџиева Николоска, *Квазипериодичност на решенијата на линеарна диференцијална равенка од* I *ред*, Мат. билт. **28** (2004).

[2] J. Mitevska, M. Kujumdzieva Nikoloska, D. Dimitrovski, *Quaziperiodicity of the solutions for linear nonhomogeneous differential equations of second order*, Math. Maced. **2** (2004), 83-88.

[3] Э. Камке, *Справочник по обыкновенным дифференциальным уравнениям*, Москва (1971).