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THE CONNECTION BETWEEN A (3,2)-SEMIGROUP AUTOMATON AND A SEMIGROUP AUTOMATON

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Abstract. The aim of the talk is to give a connection between a (3,2)-semigroup automaton and the semigroup automaton which is constructed from himself, using the recognizable languages for themselves.

1. INTRODUCTION

The notion of semigroup automaton is introduced in [1]. Here we recall the necessary definitions and known results on (3,2)-semigroup automata.

From now on, let B be a nonempty set and let $(B, \cdot)$ be a semigroup, where $\cdot$ is a binary operation. The clasical notion of semigroup automata is the following.

A semigroup automaton is a triple $(S, (B, \cdot), f)$, where S is a set, $(B, \cdot)$ is a semigroup, and $f : S \times B \to S$ is a map satisfying

$$f(f(s, x), y) = f(s, x \cdot y) \text{ for every } s \in S, \;\; x, y \in B. \tag{1}$$

The set S is called the set of **states** of $(S,(B,\cdot),f)$ and f is called the **transition function** of $(S,(B,\cdot),f)$.

A nonempty set B with the (3,2)-operation $\{\,\}:B^3\to B^2$ is called **a (3,2)-semigroup** iff the following equality

$$\{\{xyz\}t\}=\{x\{yzt\}\} \tag{2}$$

is an identity for every $x,y,z,t\in B$. It is denoted with the pair $(B,\{\ \})$.

Example 1.1. Let $B=\{a,b\}$. Then the (3,2)-semigroup $(B,\{\ \})$ is given by Table 1.

{ }	
a a a	(b,a)
a a b	(a,a)
a b a	(a,a)
a b b	(b,a)
b a a	(a,a)
b a b	(b,a)
b b a	(b,a)
b b b	(a,at)

Table 1

A (3,2)-semigroup automaton is a triple $(S,(B,\{\ \}),f)$, where S is a set, $(B,\{\ \})$is a (3,2)-semigroup, and $f:S\times B^2\to S\times B$ is a map satisfying

$$f(f(s,x,y),z)=f(s,\{xyz\})\text{ for every }s\in S,\ x,y,z\in B. \tag{3}$$

The set S is called the set of **states** of $(S,(B,\{\ \}),f)$ and f is called the **transition function** of $(S,(B,\{\ \}),f)$.

The transition function f of a (3,2)-semigroup automaton $(S,(B,\{\ \}),f)$ can be given by a table or by a graph. When we examine a graph of a (3,2)-semigroup automaton, then the nodes of graph are the states, and the arrows of the graph are the pairs of letters.

1.1⁰ Let $(S,(B,\cdot),\varphi)$ be a semigroup automaton. Then $(S,(B,\{\ \}),f)$ is a (3,2)-semigroup automaton with (3,2)-operation $\{\ \}:B^3\to B^2$ defined by $\{xyz\}=(x\cdot y,z)$ and the transition function $f:S\times B^2\to S\times B$ defined by $f(s,x,y)=(\varphi(s,x),y)$.

1.2^0 If $(S,(B,\{\ \}),f)$ is a (3,2)-semigroup automaton, then:

i) $(B^2,*)$ is a semigroup, where the operation $*$ is defined by $(x,y)*(u,v)=\{xyuv\}$ for every $(x,y),(u,v)\in B^2$;

(ii) $(S\times B,(B^2,*),\psi)$ is a semigroup automaton, where the transition function $\psi : S\times B\times B^2\to S\times B$ is defined by $\psi((s,x),(y,z))=f(s,\{xyz\})$.

Example 1.2. Let $(B,\{\ \})$ be a (3,2)-semigroup given by Table 1 from Example 1.1 and $S=\{s_0,s_1,s_2\}$. A (3,2)-semigroup automaton $(S,(B,\{\ \}),f)$ is given by the Table 2 and the analogy graph by Figure 1.

f	(a,a)	(a,b)	(b,a)	(b,b)
s_0	(s_1,b)	(s_2,b)	(s_2,b)	(s_1,b)
s_1	(s_1,b)	(s_0,a)	(s_2,b)	(s_1,b)
s_2	(s_2,b)	(s_0,b)	(s_1,b)	(s_2,b)

Table 2

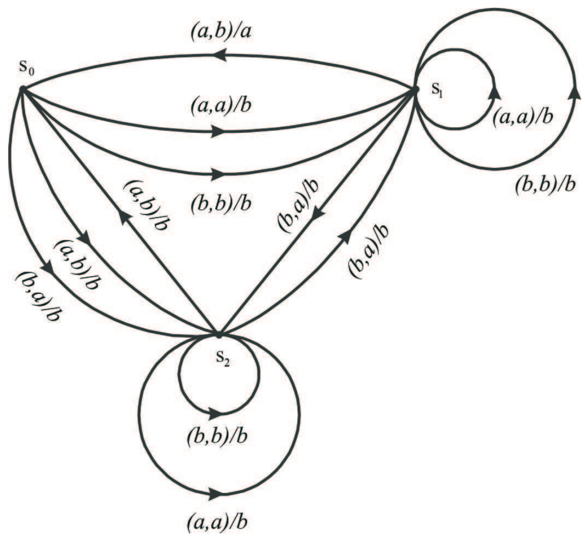

Figure 1:

2. (3,2)-LANGUAGES

Let $(Q,[\ ])$ be a free (3,2)-semigroup with a basis constructed in [2].

Any subset $L^{(3,2)}$ of the universal language $Q^* = \bigcup_{p\geq 1} Q^p$, where $(Q,[\ ])$ is a free (3,2)-semigroup with a basis B is called **a (3,2)-language** on the alphabet B.

A (3,2)-language $L^{(3,2)} \subseteq Q^*$ is called **recognizable** if there exists:

(1) a (3,2)-semigroup automaton $(S,(B,\{\ \}),f)$, where the set S is finite;

(2) an initial state $s_0 \in S$;

(3) a subset $T \subseteq S$; and

(4) a subset $C \subseteq B$,

such that

$$L^{(3,2)} = \{w \in Q^* \mid \overline{\varphi}(s_0,(w,1),(w,2)) \in T \times C\}, \tag{4}$$

where $(S,(Q,[\ ]),\bar{\varphi})$ is the (3,2)-semigroup automaton constructed in [4] for the (3,2)-semigroup automaton $(S,(B,\{\ \}),f)$.

We also say that the (3,2)-semigroup automaton $(S,(B,\{\ \}),f)$ **recognizes** $L^{(3,2)}$, or that $L^{(3,2)}$ **is recognized** by $(S,(B,\{\ \}),f)$.

Example 2.1. Let $(S,(B,\{\ \}),f)$ be a (3,2)-semigroup automaton given in Example 1.2. We construct the (3,2)-semigroup automaton $(S,(Q,[\ ]),\overline{\varphi})$ for the (3,2)-semigroup automaton $(S,(B,\{\ \}),f)$. A (3,2)-language $L^{(3,2)}$, which is recognized by the (3,2)-semigroup automaton $(S,(B,\{\ \}),f)$, with initial state s_0 and terminal state (s_2,b) is

$L^{(3,2)} = \{w \in Q^* | w = w_1 w_2 ... w_q,\ q \geq 3$, where $w_l = \begin{cases} (u_1^n, i),\ n \geq 3, u_\alpha \in Q \\ (a^* b^*)^* \end{cases}$,

$l \in \{1,2,\ldots,q\}$, and:

a) If $i = 1$, then:

a1) $(u_1^n, 1) = a$, where $\psi_{p-1}(u_1)\ \ldots\ \psi_{p-1}(u_n) = a^t b^j a^r b^h$ and

$$t + r = 2k, \quad t + j + r + h = n, \quad t,j,r,h,k \in \{0,1,2,...\}, \quad k \geq 1;$$

a2) $(u_1^n, 1) = b$, where $\psi_{p-1}(u_1)\ \ldots\ \psi_{p-1}(u_n) = a^t b^j a^r b^h$ and

$$t + r = 2k+1, \quad t + j + r + h = n, \quad t,j,r,h,k \in \{0,1,2,...\}, \quad k \geq 1;$$

a) If $i = 2$, then $(u_1^n, 2) = a$, where $\psi_{p-1}(u_1)\ \ldots\ \psi_{p-1}(u_n) = (a^*, b^*)^*$ and

$$\psi_p(w_1)\dots\psi_p(w_q) = b^*(ab^*)^{2k+1}\}.$$

For the (3,2)-semigroup automaton $(S,(B,\{\ \}),f)$, according Proposition 1.2^0 it is posible to construct semigroup automaton $(S\times B,(B^2,\ \cdot\ ),\zeta)$, where the transition function $\zeta : S\times B\times B^2 \to S\times B$ is defined with $\zeta((s,x),(y,z)) = f(s,\{xyz\})$. The appropriated table and graph of the transition function are given by Table 3 and Figure 2.

ς	(a,a)	(a,b)	(b,a)	(b,b)
(s_0,a)	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,b)
(s_0,b)	(s_1,b)	(s_2,b)	(s_2,b)	(s_1,b)
(s_1,a)	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,b)
(s_1,b)	(s_1,b)	(s_2,b)	(s_2,b)	(s_1,b)
(s_2,a)	(s_1,b)	(s_2,b)	(s_2,b)	(s_1,b)
(s_2,b)	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,b)

Table 3

If we replace $(a,a) = x$, $(a,b) = y$, $(b,a) = z$, $(b,b) = t$, where $B' = B^2 = \{x,y,z,t\}$ and $(s_0,a) = s_0'$, $(s_0,b) = s_0''$, $(s_1,a) = s_1'$, $(s_1,b) = s_1''$, $(s_2,a) = s_2'$, $(s_2,b) = s_1''$, where the set of states is $S' = \{s_0',\ s_0'',\ s_1',\ s_1'',\ s_2',\ s_2''\}$, the transition function ζ for the semigroup automaton $(S',(B',\ \cdot\ ),\zeta)$ is given with Table 4, and the semigroup $(B',\cdot)$ is given with Table 5.

ς	x	y	z	t
s_0'	s_2''	s_1''	s_1''	s_2''
s_0''	s_1''	s_2''	s_2''	s_1''
s_1'	s_2''	s_1''	s_1''	s_2''
s_1''	s_1''	s_2''	s_2''	s_1''
s_2'	s_1''	s_2''	s_2''	s_1''
s_2''	s_2''	s_1''	s_1''	s_2''

Table 4

	x	y	z	t
x	x	z	z	x
y	z	x	x	z
z	z	x	x	z
t	x	z	z	x

Table 5

The language $L_{B'}^{(2,1)}$ which is recognized by the semigroup automaton $(S',(B',\ \cdot\ ),\zeta)$ in initial state s_0 and terminate state (s_2,b) is of the form

$$L_{B\prime}^{(2,1)} = \{w\in B'^* \mid w = (x\cup t)^*(((y\cup z)(x\cup t)^*)^2)^*\ \}.$$

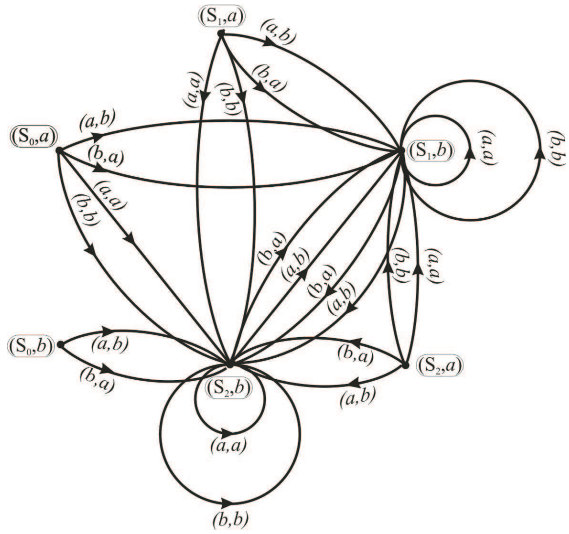

Figure 2:

If we return on the introduced replacement $x = (a, a),\ \ y = (a, b),\ \ z = (b, a),\ t = (b, b)$, we will get the (3,2)-language

$$L_{B\prime}^{(3,2)} = \{w \in B^* \mid w = (aa \cup bb)^*(((ab \cup ba)(aa \cup bb)^*)^2)^*,\ \ |w| \geq 3\ \}$$

2.1⁰ Let $L^{(3,2)}$ be a (3,2)-language on the set B recognized by the (3,2)-semigroup automaton $(S, (Q, [\ ]), \overline{\varphi})$. Let $(S, (Q, [\ ]), \overline{\varphi})$ be a (3,2)-semigroup automaton with an initial state s_0 and a set of terminal states $T \times C \subseteq S \times B$. Then $x\tilde{L}^{(2,1)} \subseteq L^{(3,2)}$ for each $x \in Q$ and for any language $L^{(2,1)}$, which is recognized by the semigroup automaton $(S \times Q, (Q^2, *), \psi)$ with an initial state $s_0' = (s_0, x)$ and a set of terminal states $T \times C$, where $\psi : S \times Q \times Q^2 \to S \times Q$ is a transition function defined by $\psi((s, x), (u, v)) = \bar{\varphi}(s, \{xuv\})$ and $\tilde{L}^{(2,1)} = \{\tilde{w} | w \in L^{(2,1)}\}$.

Proof. $L^{(3,2)}$ is a recognizable (3,2)-language on the set B by the (3,2)-semigroup automaton $(S, (Q, [\ ]), \overline{\varphi})$ with initial state s_0 and a set of terminal states $T \times C$, for $T \subseteq S$ and $C \subseteq B$, so

$L^{(3,2)} = \{w \in Q^* \mid \overline{\varphi}(s_0, (w, 1), (w, 2)) \in T \times C\}$.

By Proposition 1.2⁰, $(S \times Q, (Q^2, *), \psi)$ is a semigroup automaton and Q^2 is a semigroup. It recognizes a language $L^{(2,1)}$ with an initial state $s_0' = (s_0, x)$ and a set of terminal states $T \times C$, so it is of the form

$L^{(2,1)} = \{w \in (Q^2)^* \mid \psi(s'_0, w) \in T \times C\}$.

Let $w \in L^{(2,1)}$. It follows that $w \in (Q^2)^*$ and $\psi(s'_0, w) \in T \times C$. But $s'_0 = (s_0, x)$, so

$\overline{\varphi}(s_0, (x\tilde{w}, 1), (x\tilde{w}, 2)) = \overline{\varphi}(s_0, \{xw\}) = \psi((s_0, x), w) = \psi(s'_0, w) \in T \times C$.

Thus $x\tilde{w} \in L^{(3,2)}$, i.e. $x\tilde{L}^{(2,1)} \subseteq L^{(3,2)}$. □

2.2⁰ Let $L^{(3,2)}$ be a (3,2)-language on the set B recognizabled by (3,2)-semigroup automaton $(S, (Q, [\ ]), \overline{\varphi})$ with the initial state s_0 and a set of terminal states $T \times C \subseteq S \times B$, where $(B, \{\ \ \})$ is a (3,2)-semigroup. Let $x\tilde{L}^{(2,1)} \subseteq L^{(3,2)}$ where $\tilde{L}^{(2,1)} = \{\tilde{w} | w \in L^{(2,1)}\}$, for every $x \in Q$ and for the language $L^{(2,1)}$ recognizabled by the semigroup automaton $(S \times Q, (Q^2, *), \psi)$ with initial state $s'_0 = (s_0, x)$ and a set of terminal states $T \times C$, where $\psi : S \times Q \times Q^2 \to S \times Q$ be a transition function defined by

$\psi((s, x), (u, v)) = \bar{\varphi}(s, \{xuv\})$.

Then for the infinite semigroup automaton $(S \times Q, (Q^2, *), \psi)$ exists finite semigroup automaton (S', B', f') with initial state $s' = (s, b)$ for $b \in B$ and a set of terminal states $T \times C \subseteq S \times B$, which recognizes the language $L_1^{(2,1)} \subseteq (B^2)^*$ and $b\tilde{L}_1^{(2,1)} \subseteq L^{(3,2)}$.

Proof. $L^{(3,2)} = \{w \in Q^* \mid \overline{\varphi}(s_0, (w, 1), (w, 2)) \in T \times C\}$, during

$L^{(2,1)} = \{w \in (Q^2)^* \mid \ \psi(s'_0, w) \in T \times C\}$.

Let $S' = S \times B$, $B' = (B^2)^*$, and $f' = \psi|_{B^2}$. S' is a finite set because S and B are finite sets. For f' there is

$f'((s, b), (u, v)) = \psi|_{B^2}((s, b)(u, v)) = \bar{\varphi}|_{B^2}(s, \{buv\}) = f(s, \{buv\})$.

On the other hand,

$f'((s, b), \{uvtp\}) = f(s, \{buvtp\}) = f(s, \{\{buv\}tp\}) = f(f(s, \{buv\}), (t, p))$
$= f(f'(s, b), (u, v)), (t, p)) = f'(f'((s, b), (u, v)), (t, p))$.

From this it follows that f' is a transition function for the finite semigroup automaton (S', B', f'). The language which is recognized by this semigroup automaton looks like

$L_1^{(2,1)} = \{w \in L^{(2,1)} | w \in (B^2)^*\}$.

Because $x\tilde{L}^{(2,1)} \subseteq L^{(3,2)}$ for every $x \in Q$, and $L_1^{(2,1)} \subseteq L^{(2,1)}$ and $B \subseteq Q$, it follows that $b\tilde{L}_1^{(2,1)} \subseteq L^{(3,2)}$ for $b \in B$. □

2.3^0 Let $L^{(2,1)}$ be a recognizable language on the set B by a semigroup automaton $(S,(B,\|\ \|),\xi)$ with an initial state $s_0 \in S$ and a set of terminal states $T \subseteq S$, and $(S,(B,\{\ \}),f)$ be a (3,2)-semigroup automaton. Let $f : S \times B^2 \to S \times B$ is a transition function defined by $f(s,x,y) = (\xi(s,x),y)$. Then $L^{(2,1)}a \subseteq L^{(3,2)}$, for each $a \in B$, where $L^{(3,2)}$ is a recognizable (3,2)-language on the set B by the (3,2)-semigroup automaton $(S,(Q,[\ ]),\bar{\varphi})$ with an initial state $s_0 \in S$ and a set of terminal states $T \times \{a\}$.

Proof. A language $L^{(2,1)}$ is recognizable by a semigroup automaton $(S,(B,\|\ \|),\xi)$ with initial state $s_0 \in S$ and a set of terminal states $T \subseteq S$, so

$L^{(2,1)} = \{w \in B^* |\, \xi(s_0,w) \in T\}$.

By Proposition 1.1^0, $(S,(B,\{\ \}),f)$ is a (3,2)-semigroup automaton. We construct a (3,2)-semigroup automaton $(S,(Q,[\ ]),\bar{\varphi})$, where $Q = \varphi(\overline{B})$ and $\overline{\varphi} : S \times Q^2 \to S \times Q$ is a transition function defined by

$\overline{\varphi}(s,(u_1^n,i),(v_1^k,j)) = \tau_p(s,(u_1^n,i),(v_1^k,j)) = f(s,\psi_p(u_1^n,i),\psi_p(v_1^k,j))$.

It follows that a recognizable (3,2)-language $L^{(3,2)}$ on the set B by the (3,2)-semigroup automaton $(S,(Q,[]),\bar{\varphi})$, with an initial state $s_0 \in S$ and a set of terminal states $T \times \{a\}$ is of the form

$L^{(3,2)} = \{w \in Q^* \mid \overline{\varphi}(s_0,w) \in T \times \{a\}\}$.

Let $w \in L^{(2,1)}$ and $a \in B$. Then

$\overline{\varphi}(s_0,(wa,1),(wa,2)) = \overline{\varphi}(s_0,(w,a)) = (\xi(s_0,w),a) \in T \times \{a\}$.

Thus $wa \in L^{(3,2)}$, i.e $L^{(2,1)}a \subseteq L^{(3,2)}$. □

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References

[1] A. Salomaa, *Theory of automata*, Pergamon Press, Oxford (1969).

[2] D. Dimovski, *Free vector valued semigroups*, Proc. Conf. "Alegbra and Logic", Cetinje, p. 55–62.

[3] K. Trenčevski, D. Dimovski, *Complex commutative vector valued groups*, Maced. Acad. Sci. and Arts, Skopje (1992).

[4] G. Čupona, N. Celakovski, S. Markovski, D. Dimovski, *Vector valued groupoids, semigroups and groups*, Maced. Acad. Sci. and Arts, Skopje (1988), p. 3–73 .