Kragujevac J. Math. 27 (2005) 11–18.

A NOTE OF A FAMILY OF QUASI-ANTIORDERS ON SEMIGROUP

Daniel Abraham Romano

Faculty of Natural Sciences and Mathematics,
78000 Banja Luka, Bosnia and Herzegovina
(e-mails: bato49@hotmail.com)

(Received February 7, 2005)

Abstract. Let $(S, =, \neq, \cdot, s)$ be an ordered semigroup under an antiorder s. If S is a subdirect product of the ordered semigroup $\{S_i : i \in I\}$, then there exists a family $\{\sigma_i : i \in I\}$ of quasi-antiorders on S which separates the elements of S. Conversely, if $\{\sigma_i : i \in I\}$ is a family of quasi-antiorders on S which separates the elements if S, then S is a subdirect product of the ordered semigroups $\{S/(\sigma_i \cup (\sigma_i)^{-1}) : i \in I\}$.

This investigation is in constructive algebra. Throughout this paper, $S = (S, =, \neq, \cdot)$ always denotes a semigroup with apartness in the sense of the books [1], [3], [8], [9] and papers [4], [5,], [6] and [7]. The apartness $\neq$ on S is a binary relation with the following properties ([1], [9]): For every elements x, y and z in S hold

$$\neg(x \neq x), x \neq y \Rightarrow y \neq x, x \neq y \wedge y = z \Rightarrow x \neq z, x \neq z \Rightarrow (\forall y \in S)(x \neq y \vee y \neq z).$$

For apartness $\neq$ we say that it is tight ([8]) if and only if $(\forall x, y \in S)(\neg(x \neq y) \Rightarrow x = y)$. It takes ([3], [5], [8]) that semigroup operation is strongly extensional in the sense

$$(\forall a, b, x, y \neq S)(ay \neq by \Rightarrow a \neq b \wedge xa \neq xb \Rightarrow a \neq b).$$

Let x be an element of S and A subset of S. We write $x\sharp A iff (\forall a \in A)(x \neq a)$, and $A' = \{x \neq S : x\sharp A\}(A'$ is the strongly compliment of $A)$. Let q be a relation on semigroup S. For q we say that it is coequality relation ([4], [6], [7]) if and only if it is consistent, symmetric and cotransitive relation on $S : q \subseteq \neq, q^{-1} = q$ and for elements x, y and z of S hold $(x, z) \neq q \Rightarrow (\forall t \in S)((x, t) \in q \bigvee (t, z) \in q), (x, y) \in q \bigwedge y = z \Rightarrow (x, z) \in q$. If the coequality relation q compatible with the semigroup operation in the next sense

$$(\forall x, y, a, b \in S)((axb, ayb) \in q \Rightarrow (x, y) \in q),$$

we say that q is anticongruence on S. Coequality relation and anticongruence studied by the author in [4], [6]and [7].

A relation α on S is *antiorder* on S if and only if

$$\begin{gathered}(\alpha \subseteq \neq),\\ (\forall x, y, z \in S)((x, z) \in \alpha \Rightarrow (x, y) \in \alpha \vee (y, z) \in \alpha)\\ (\forall x, y \in S)(x \neq y \Rightarrow (x, y) \in \alpha \vee (y, x) \in \alpha),\\ (\forall x, y, z \in S)((xz, yz) \in \alpha \Rightarrow (x, y) \in \alpha \wedge (zx, zy) \in \alpha \Rightarrow (x, y) \in \alpha).\end{gathered}$$

Let α be an antiorder relation on S. A relation s is a quasi-antiorder on S([4], [6], [7]) if and only if

$$\begin{gathered}s \subseteq \alpha\\ (\forall x, y, z \in S)((x, z) \in s \Rightarrow (x, y) \in s \vee (y, z) \in s),\\ (\forall x, y, z \in S)((xz, yz) \in s \Rightarrow (x, y) \in s \wedge (zx, zy) \in s \Rightarrow (x, y) \in s).\end{gathered}$$

In this short note we will give description of family of quasi-antiorder relations on ordered semigroup under antiorder relation.

Lemma 0. *Let s be a quasi-antiorder relation on ordered semigroup $(S, =, \neq, \cdot, \alpha)$ under antiorder relation α. Then $q = s \cup s^{-1}$ is an anticongruence on S such that S/q is an ordered semigroup under antiorder relation β defined by $(xq, yq) \in \beta \Leftrightarrow (x, y) \in s$.*

Proof. (1) It is easy to see that the relation q is an coequality relation on S. We need only to prove that q is an anticongruence on S. Let $(axb, ayb) \in q, a, b \in S$. Since $(axb, ayb) \in s \vee (ayb, axb) \in s$, we have $(x, y) \in s \vee (y, x) \in s$. Then $(x, y) \in q$.

(2) We define the equality, apartness and operation "$\cdot$" on S/q as follows:

$$xq = yq \Leftrightarrow (x,y)\sharp q,$$
$$xq \neq yq \Leftrightarrow (x,y) \in q,$$
$$\cdot : S/q \times S/q \ni (xq, yq) \to xyq \in S/q.$$

Since q is an anticongruence on S, the operation is well defined and $(S/q, =, \neq, \cdot)$ is a semigroup (It is known: [4], [7])

(3) We define a relation β on S/q as follows:

$$(xq, yq) \in \beta \Leftrightarrow (x,y) \in s.$$

Let (uq, vq) be an arbitrary element of β, i.e. let $(u,v) \in s$. Since $s \subseteq q$, we have $uq \neq vq$. Therefore,$\beta \subseteq \neq (inS/q)$. Let $(xq, zq) \in \beta$ and $yq \in S/q$, i.e. let $(x,z) \in s$ and $y \in S$. Since $(x,y) \in s \vee (y,z) \in s$, we have $(xq, yq) \in \beta$ or $(yq, zq) \in \beta$. Let $(xq, yq) \in \beta$ and $aq, bq \in S/q$, i.e. $(x,y) \in s$ and $a, b \in S$. Since $(axb, ayb) \in s$, we have $(aq \cdot xq \cdot bq, aq \cdot yq \cdot bq) \in \beta$. Let $xq \neq yq$, i.e. let $(x,y) \in q = s \bigcup s^{-1}$. Since $(x,y) \in s$ or $(y,x) \in s$, we have $(xq, yq) \in \beta$ or $(yq, xq) \in \beta$. So, the relation β is linear. Therefore, the relation β is an antiorder relation on S/q. $\square$

Let S be an ordered semigroup under an antiorder α,$\sum$ a family of quasi-antiorders on S. We say that $\sum$ separates the elements of S if for each $x, y \in S$ such that $(x,y) \in \alpha$ there exist $\sigma \in \sum$ such that $(x,y) \in \sigma$.

Lemma 1. *Let $(S, =, \neq, \cdot, \alpha)$ be an ordered semigroup under an anti-order α,$\sum$ a family of quasi-antiorders on S. If $\sum$ separates the elements of S, then $\alpha = \bigcup\{\sigma : \sigma \in \sum\}$. Conversely, if $\bigcup\{\sigma : \sigma \in \sum\} \supseteq \alpha$, then $\sum$ separates the elements of S.*

Proof. (1) Since $\sum$ is a family of quasi-antiorders on $S, \bigcup\{\sigma : \sigma \in \alpha\}$ is a quasi-antiorder on S. Then $\bigcup\{\sigma : \sigma \in \sum\} \subseteq \alpha$. Let $(x,y) \in \alpha$. Since $\sum$ separates the elements of S, then there exists $\rho \in \sum$ such that $(x,y) \in \rho$. Therefore $(x,y) \in \bigcup\{\sigma : \sigma \in \sum\}$.

(2) Suppose that $\bigcup\{\sigma : \sigma \in \sum\} \supseteq \alpha$. Then for each pair $x, y \in S$ there exists and an quasi-antiorder $\sigma \in \sum$ such that $(x,y) \in \sigma$. $\square$

Let $(S, =, \neq, \cdot, , s), (T, =, \neq, \cdot, t)$ be ordered semigroups under antiorders s and t respective, $f : S \to T$ a strongly extensional mapping from S onto T. f is called *isotone* if

$$(\forall x, y \in S)((x, y) \in s \Rightarrow (f(x), f(y)) \in t).$$

f is called *reverse isotone* if and only if

$$(\forall x, y \in S)((f(x), f(y)) \in t \Rightarrow (x, y) \in s).$$

f is called a homomorphism if it is isotone and satisfies

$$(\forall x, y \in S)(f(xy) = f(x)f(y)).$$

The strongly extensional and embedding mapping f is called an isomorphism if it is a homomorphism, onto, isotone and reverse isotone. S and T called isomorphic, in symbol $S\ T$, if exists an isomorphism between them.

Remarks. Every isotone mapping $f : S \to T$ satisfies the following condition:

(1) Let $x, y \in S$ and $x \neq y$. Then $(x, y) \in s$ or $(y, x) \in s$ by linearity of s and we have $(f(x), f(y)) \in t \subseteq \neq$ or $(f(y), f(x)) \in t \subseteq \neq$. So, the mapping f is an embedding.

(2) Let $x, y \in S$ and $f(x) = f(y)$. Then $\neg(f(x) \neq f(y))$ and from this we conclude $\neg((f(x), f(y)) \in t)$ and $\neg((f(y), f(x)) \in t)$. Hence $\neg((x, y) \in s)$ and $\neg((y, x) \in s)$. Therefore $\neg(x \neq y)$. If the apartness $\neq$ on semigroup S is tight, then $x = y$. So, in that case, the mapping f is an injective.

Lemma 2. *Let $(S, =, \neq, \cdot, s), (T, =, \neq, \cdot, t)$ be ordered semigroups under antiorders s and t respective, $f : S \to T$ a strongly extensional homomorphism. The relation σ on S, defined by $(x, y) \in \sigma$ if and only if $(f(x), f(y)) \in t$, is a quasi-antiorder on S.*

Proof. (1) Let $(x, y) \in \sigma$. Since $(f(x), f(y)) \in t$ and f is a reverse isotone, we have $(x, y) \in s$.

(2) Let $(x, z) \in \sigma$ and let y be an arbitrary element of S. Since $(f(x), f(z)) \in t$ we have $(f(x), f(y)) \in t$ or $(f(y), f(z)) \in t$. Then $(x, y) \in \sigma \vee (y, z) \in \sigma$.

(3) Let $x, y, a, b \in S$ and let $(axb, ayb) \in \sigma$. Since $((f(axb), f(ayb)) \in t$, f is a homomorphism of semigroups and T is ordered semigroup under the anti-order t,

we have that from $((f(a)f(x)f(b), f(a)f(y)f(b)) \in t$ follows $(f(x), f(y)) \in t$. So, $(x, y) \in \sigma$. $\square$

Let $\{(S_i, =, \neq, \cdot i, s_i) : i \in I\}$, where I is a discrete set, be an inhabited family of ordered semigroups under anti-orders. Then the inhabited Cartesian product $\prod S_i$ of $\{S_i : i \in I\}$ with the multiplication

$$\prod S_i \times \prod S_j \ni (x, y) \to z \in \prod S_k$$

and with and the antiorder α defined by

$$(\forall i \in I)(z(i) = x(i)y(i)), ((x, y) \in \alpha \Leftrightarrow (\exists i \in I)(((x(i), y(i)) \in s_i),$$

is an ordered semigroup.

Let $\{S_i : i \in I\}$ be a family of ordered semigroups under antiorders . An ordered semigroup S under an antiorder σ is a *subdirect product* of the family $\{S_i : i \in I\}$ if and only if:

(1) There exists a subsemigroup T of $\prod_{i \in \in I} S_i$ such that $T \cong S$;

(2) $(\forall i \in I)(\pi_i(T) = S_i)$.

Theorem. *Let $(S, =, \neq, \cdot, s)$ be an ordered semigroup under an antiorder s.*

If S is a subdirect product of the ordered semigroup $\{S_i : i \in I\}$, then there exists a family $\{\sigma_i : i \in I\}$ of quasi-antiorders on S which separates the elements of S.

Conversely, if $\{\sigma_i : i \in I\}$ is a family of quasi-antiorders on S which separates the elements if S, then S is a subdirect product of the ordered semigroups $\{S/(\sigma_i \bigcup (\sigma_i)^{-1}) : i \in I\}$.

Proof. (1) Let $f : S \to \prod S_i$ be a reverse isotone strongly extensional homomorphism such that $\pi_i(f(S)) = S_i (i \in I)$. For each $j \in I$, we consider the mapping $\varphi_j : S \to S_j$ by $\varphi_j(x) = \pi_j(f(x)) = f(x)(j)$.

(a) φ is a strongly extensional function because components are strongly extensional functions;

(b) Let $x, y \in S, (x, y) \in s$. Since f is isotone mapping, we have $(f(x), f(y)) \in \alpha$. Then $(\exists k \in I)(((f(x)(k), f(y)(k)) \in s_k)$. Hence $(\exists k \in I)((\pi_k(f(x)), \pi_k f(y))) \in s_k)$.

Since φ is a strongly extensional homomorphism, the relation φ_k, defined by $(x, y) \in \sigma_k$ if and only if $(\varphi_k(x), \varphi_k(y)) \in s_k$, is a quasi-antiorder for every $k \in I$, by Lemma 2. By Lemma 1, it is enough to prove the $\bigcup\{\sigma_k : k \in I\} \supseteq s$. Let $(x, y) \in s$. Then $(f(x), f(y)) \in \alpha$, because f is isotone mapping, i.e. there exists k in I such that $((f(x)(k), f(y)(k))) \in s_k$. So, $(\varphi_k(x), \varphi_k(y)) \in s_k$, i.e. $(x, y) \in \sigma_k$, which means $(x, y) \in \bigcup\{\sigma_k : k \in I\}$.

(2) Converse statement: Let $\{\sigma_i : i \in I\}$ be a family of quasi-antiorders relation on $(S, =, \neq, \cdot, s)$ which separates the elements of S. We can construct the anticongruence $q_k = \sigma_k \bigcup(\sigma_k)^{-1}$, and semigroup $S_k = S/q_k$ for every $k \in I$. By lemma 0, there exists the quasi-antiorder β_k on S/q_k, defined by $(xq_k, yq_k) \in \beta_k$ if and only if $(x, y) \in \sigma_k$, such that the semigroup S/q_k is ordered semigroup under antiorder β_k. Now, we can construct the Cartesian product $\prod_{k\in I}(S/q_k, =, \neq, \cdot, \beta_k)$ with

$$\begin{aligned} a = b &\Leftrightarrow (\forall k \in I)(a(k), b(k) \in S/q_k \wedge a(k) = b(k)) \\ a \neq b &\Leftrightarrow (\exists k \in I)(a(k), b(k) \in S/q_k \wedge a(k) \neq b(k)) \\ &(\forall k \in I)((a \cdot b)(k) = a(k) \cdot b(k)) \\ &(\forall k \in I)(\pi_k(a) = a(k) \in S/q_k). \end{aligned}$$

and with antiorder α , defined by

$$(a, b) \in \alpha \Leftrightarrow (\exists k \in I)((a(k), b(k)) \in \beta_k).$$

We consider the mapping $f : S \to \prod S/q_k$ by $f(x)(k) = xq_k, (k \in I)$.

(a) f is correct defined.

If $x \in S$, then $xq_k \in S/q_k(\forall k \in I)$. So, we have $f(x) \in \prod S/q_k$.

Let $x, y \in S, x = y$. Then $(x, y) \in (q_i)$' for every $i \in I$. So, for every $i \in I$ we have $xq_i = yq_i$, i.e. $(\forall i \in I)(f(x)(i) = f(y)(i))$, i.e. $f(x) = f(y)$. So, f is a function.

Let $x, y \in S$, and let $f(x) \neq f(y)$ in $\prod S/q_k$. Then $(\exists_j \in I)((f(x)(j) \neq f(y)(j))$, i.e. $(\exists_j \in I)(xq_j \neq yq_j)$. Hence $(x, y) \in q_j = \sigma_j \bigcup(\sigma_j)^{-1}$, we conclude $(x, y) \in \sigma_j \vee (y, x) \in \sigma_j$. Therefore, $x \neq y$. So, the mapping f is strongly extensional function from S into $\prod S/q_k$.

(b) f is a homomorphism:

If $x, y \in S$, then $(\forall k \in I)(f(xy)(k) = xyq_k = xq_kyq_k = f(x)(k) \cdot f(y)(k))$. So, $f(xy) = f(x) \cdot f(y)$.

(c) f is an isotone function:

Let $x, y \in S, (x, y) \in s = \bigcup\{\sigma_k : k \in I\}$ because the family $\{\sigma_k : k \in I\}$ separates the elements of S. Then $(\exists k \in I)((x, y) \in \sigma_k)$. i.e. $(\exists k \in I)((xq_k, yq_k) \in \beta_k)$. Therefore, $(f(x), f(y)) \in \alpha$.

(d) f is reverse isotone function:

Let $x, y \in S, (f(x), f(y)) \in \alpha$. Since $(\exists k \in I)((f(x), f(y)) \in \beta_k)$, i.e. $(\exists k \in I)((xq_k, yq_k) \in \beta_k)$, we have $(x, y) \in \sigma_k$. Since $\beta_k \subseteq s$, we have $(x, y) \in s$.

(e) $(\forall k \in I)(\pi_k(f(S)) = S/q_k)$:

Let $x \in S$, then $f(x) \in \prod S/q_k$, thus $(\forall k \in I)(\pi_k(f(x)) = xq_k \in S/q_k)$. This means $(\forall k \in I)(\pi_k(f(S)) \subseteq S/q_k)$. Let $y \in \prod S/q_k$, i.e. let $(\forall k \in I)(y(k) \in S/q_k)$. Then there exists the element x in S such that $f(x) = y$, defined by $(\forall\forall k \in I)(y(k) = f(x)(k))$. So, $(\forall k \in I)(\pi_k(f(x)) = y(k) \in S/q_k)$. Therefore, $(\forall k \in I)(\pi_k(f(S)) \supseteq S/q_k)$. $\square$

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