Kragujevac J. Math. 27 (2005) 19–30.

A GOOD λ ESTIMATE FOR MULTILINEAR SINGULAR INTEGRAL OPERATORS WITH VARIABLE CALDERÓN-ZYGMUND KERNEL

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(Received November 14, 2004)

Abstract. In this paper, a good λ estimate for the multilinear operators associated to the singular integral operators with variable Calderón-Zygmund kernel is obtained. Under this result, we get the (L^p, L^q)-boundedness of the multilinear operators.

1. INTRODUCTION

Let T be a singular integral operator. Cohen and Gosselin studied [5, 6, 7] the $L^p(p > 1)$ boundedness of the multilinear operator T^A associated to T, which is defined by

$$T^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x-y|^m} K(x, y) f(y) dy,$$

where $D^\alpha A \in BMO$ for $|\alpha| = m$. The boundedness of the multilinear operator on the Triebel-Lizorkin and Lebesgue spaces, where $D^\alpha A \in Lip\beta$ for $|\alpha| = m$ is found [3]. Calderón and Zygmund introduce some singular integral operators with variable kernel and discuss their boundedness [1]. It is obtained the boundedness for the commutators generated by the singular integral operators with variable kernel and BMO functions [4, 10]. The boundedness for the multilinear oscillatory singular integral operators generated by the operators and BMO functions is also found [13]. The purpose of this paper is to find the good λ estimate for the multilinear operators associated to the singular integral operators with variable Calderón-Zygmund kernel where $D^\alpha A \in Lip\beta$ for $|\alpha| = m$ and to find (L^p, L^q)-boundedness for the multilinear operators.

2. PRELIMINARIES AND THEOREMS

Let us introduce some notations [11, 14, 15]. Throughout this paper, Q will denote a cube of R^n with side parallel to the axes. Let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^\#(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. For $1 \le p < \infty$ and $0 \le \delta < n$, let

$$M_{\delta,p}(f)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|^{1-p\delta/n}}\int_Q |f(y)|^p dy\right)^{1/p}.$$

If $\delta = 0$, $M_{\delta,r}(f) = M_r(f)$ it is the Hardy-Littlewood maximal function when $r = 1$. For $\beta > 0$ and $p > 1$, the Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$||f||_{\dot{\Lambda}_\beta} = \sup_{\substack{x,h\in R^n\\ h\neq 0}} \left|\Delta_h^{[\beta]+1} f(x)\right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator [14].

In this paper, we will study a class of multilinear operators associated to the singular integral operators with variable kernel.

Definition 1. *Let* $k(x) = \Omega(x)/|x|^n : R^n \setminus \{0\} \longrightarrow R$. *$k$ is a Calderón-Zygmund kernel if*

(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;

(b) Ω *is homogeneous of degree zero;*

(c) $\int_\Sigma \Omega(x)x^\alpha d\sigma(x) = 0$ *for all multi-indices* $\alpha \in (N\bigcup\{0\})^n$ *with* $|\alpha| = N$, *where* $\Sigma = \{x \in R^n : |x| = 1\}$ *is the unit sphere of* R^n.

Definition 2. *Let* $k(x,y) = \Omega(x,y)/|y|^n : R^n \times (R^n \setminus \{0\}) \longrightarrow R$. *$k$ is a variable Calderón-Zygmund kernel if*

(d) $k(x,\cdot)$ *is a Calderón-Zygmund kernel for a.e.* $x \in R^n$*;*

(e) $\max_{|\gamma|\le 2n} \left\| \frac{\partial^{|\gamma|}}{\partial^\gamma y}\Omega(x,y) \right\|_{L^\infty(R^n\times\Sigma)} = M < \infty$.

Let m_i *be the positive integers* ($i = 1,\cdots,l$), $m_1 + \cdots + m_l = m$ *and* A_i *be the functions on* R^n ($i = 1,\cdots,l$). *Let*

$$R_{m_i+1}(A_i;x,y) = A_i(x) - \sum_{|\alpha|\le m_i} \frac{1}{\alpha!} D^\alpha A_i(y)(x-y)^\alpha.$$

The multilinear singular integral operators with variable Calderón-Zygmund kernel are defined by

$$T^{A_1,\cdots,A_l}(f)(x) = p.v.\int_{R^n} \frac{\Omega(x,x-y)}{|x-y|^{n+m}} \prod_{i=1}^{l} R_{m_i+1}(A_i;x,y) f(y)dy$$

and $T_\star^{A_1,\cdots,A_l}(f)(x) = \sup_{\varepsilon>0} |T_\varepsilon^{A_1,\cdots,A_l}(f)(x)|$, *where*

$$T_\varepsilon^{A_1,\cdots,A_l}(f)(x) = \int_{|x-y|>\varepsilon} \frac{\Omega(x,x-y)}{|x-y|^{n+m}} \prod_{i=1}^{l} R_{m_i+1}(A_i;x,y) f(y)dy,$$

and $\Omega(x,y)/|y|^n$ *is the variable Calderón-Zygmund kernel. We also define*

$$T(f)(x) = p.v.\int_{R^n} \frac{\Omega(x,x-y)}{|x-y|^n} f(y)dy,$$

which is the singular integral operator with variable Calderón-Zygmund kernel [1].

Note that when $m = 0$, $T^{A_1,\cdots,A_l}$ is just the commutator of T and $A_1,\cdots,A_l$ [14]. While $m > 0$, it is non-trivial generalization of the commutator. It is well known that commutators and multilinears operator are of great interest in harmonic analysis and have been widely studied by many authors [8, 9, 12]. The purpose of this paper is to find the good λ estimate for the multilinear singular integral operators with variable Calderón-Zygmund kernel, and with this result to find (L^p, L^q)-boundedness for the multilinear operators.

Theorem 1. *Let* T *be the singular integral operator with variable Calderón-Zygmund kernel,* $0 < \beta < 1$ *and* $D^\alpha A_i \in \dot{\Lambda}_\beta$ *for all* α *where* $|\alpha| = m_i$ *and* $i = 1,\cdots,l$.

Suppose $1 < r < p < \infty$. *Then there exist* $\gamma_0 > 0$ *such that, for any* $0 < \gamma < \gamma_0$ *and* $\lambda > 0$

$$\left|\left\{x \in R^n : T_\star^{A_1,\cdots,A_l}(f)(x) > 3\lambda,\ \prod_{i=1}^{l}\left(\sum_{|\alpha_j|=m_i} ||D^{\alpha_i}A_i||_{Lip_\beta}\right) M_{l\beta,p}(f)(x) \le \gamma\lambda\right\}\right|$$
$$\le \quad C\gamma^r|\{x \in R^n : T_\star^{A_1,\cdots,A_l}(f)(x) > \lambda\}|.$$

Theorem 2. *Let* T *be the singular integral operators with variable Calderón-Zygmund kernel,* $0 < \beta < 1$ *and* $D^\alpha A_i \in \dot{\Lambda}_\beta$ *for all* α *where* $|\alpha| = m_i$ *and* $i = 1, \cdots, l$. *Then* $T^{A_1,\cdots,A_l}$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$ *for* $1 < p < n/l\beta$ *and* $1/p - 1/q = l\beta/n$.

3. PROOFS OF THEOREMS

To prove the theorem, we need the following lemmas.

Lemma 1. [14] *Let* $0 < \beta < 1$ *and* $1 \le p \le \infty$, *then*

$$||b||_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q|dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|}\int_Q |b(x) - b_Q|^p dx\right)^{1/p}.$$

Lemma 2. [7] *Let* A *be a function on* R^n *and* $D^\alpha A \in L^q(R^n)$ *for all* α *where* $|\alpha| = m$ *and* $q > n$. *Then*

$$|R_m(A; x, y)| \le C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|Q(x,y)|}\int_{Q(x,y)} |D^\alpha A(z)|^q dz\right)^{1/q},$$

where $Q(x, y)$ *is the cube centered at* x, *with side length equal to* $5\sqrt{n}|x - y|$.

Lemma 3. [2] *Let* $0 \le \delta < n$, $1 \le r < p < n/\delta$ *and* $1/q = 1/p - \delta/n$, *then*

$$||M_{\delta,r}(f)||_{L^q} \le C||f||_{L^p}.$$

Proof of Theorem 1. Without loss of generality, we may assume $l = 2$. By the Whitney decomposition, $\{x \in R^n : T_\star^{A_1,A_2}(f)(x) > \lambda\}$ may be written as a union of cubes $\{Q_j\}$ with mutually disjoint interiors and with distance from each to $R^n \setminus \bigcup_j Q_j$ comparable to the diameter of Q_j. It is enough to find a good λ estimate for each Q_j. There exists a constant $C = C(n)$ such that for each j, the cube $\tilde{Q}_j$ intersects $R^n \setminus \bigcup_j Q_j$, where $\tilde{Q}_j$ denotes the cube with the same center as Q_j and with the diam $\tilde{Q}_j = C$ diam Q_j. For each j, there exists a point $x_0 = x_0(j) \in \tilde{Q}_j$ such that

$$T_\star^{A_1,A_2}(f)(x_0) \le \lambda.$$

Now, we fix the cube Q_j. Without loss of generality, we may assume there exists a point $z = z(j)$ and

$$\prod_{i=1}^{2}\left(\sum_{|\alpha|=m_i} ||D^\alpha A_i||_{\dot{\Lambda}_\beta}\right) M_{2\beta,p}(f)(z) \le \gamma\lambda.$$

holds.

Let $\overline{Q}_j = \tilde{\tilde{Q}}_j$ and $f = f_1 + f_2$ for $f_1 = f\chi_{\overline{Q}_j}$ and $f_2 = f\chi_{R^n\setminus\overline{Q}_j}$. We turn to the estimates on f_1 and f_2.

The estimates on f_1. Choose $\varphi \in C^\infty$ such that $\varphi(x) \equiv 1$ for $x \in \overline{Q}_j$, $\varphi(x) \equiv 0$ for $x \notin \overline{\overline{Q}}_j$, $|\varphi(x)| \le 1$ for all x, and $|\varphi(x)| \le C(\mathrm{diam}\overline{\overline{Q}}_j)^{-|\alpha|}$ for any multiindex α where $|\alpha| \le m$. Let

$$A_1^\varphi(y) = R_{m_1}\left(A_1(\cdot) - \sum_{|\alpha|=m_1}\frac{1}{\alpha!}(D^\alpha A_1)_{Q_j}(\cdot)^\alpha; y, z\right)\cdot\varphi(y)$$

and

$$A_2^\varphi(y) = R_{m_2}\left(A_2(\cdot) - \sum_{|\alpha|=m_2}\frac{1}{\alpha!}(D^\alpha A_2)_{Q_j}(\cdot)^\alpha; y, z\right)\cdot\varphi(y).$$

Then, for $x \in Q_j$

$$T_\star^{A_1,A_2}(f_1)(x) = T_\star^{A_1^\varphi,A_2^\varphi}(f_1)(x).$$

It is obtaind [3, 7]

$$||D^\alpha A_1^\varphi||_{L^q} \le C\sum_{|\alpha|=m_1} ||D^\alpha A_1||_{\dot{\Lambda}_\beta}|\overline{\overline{Q}}_j|^{\beta/n+1/q} \text{ for } |\alpha| = m_1$$

and

$$||D^\alpha A_2^\varphi||_{L^q} \le C \sum_{|\alpha|=m_2} ||D^\alpha A_2||_{\dot\Lambda_\beta} |\overline{\overline{Q}}_j|^{\beta/n+1/q} \text{ for } |\alpha| = m_2.$$

For $1/r = 1/p + 2/q < 1$ and $\eta > 0$,

$$\begin{aligned} |\{x \in R^n \ &: \ T_\star^{A_1,A_2}(f_1)(x) > \eta\lambda\}| \\ &= |\{x \in R^n : T_\star^{A_1^\varphi,A_2^\varphi}(f_1)(x) > \eta\lambda\}| \\ &\le C(\eta\lambda)^{-r} \left\| T_\star^{A_1^\varphi,A_2^\varphi}(f_1) \right\|_{L^r}^r \\ &\le C(\eta\lambda)^{-r} \left[\prod_{i=1}^2 \left(\sum_{|\alpha|=m_i} ||D^\alpha A_i^\varphi||_{L^q} \right) ||f_1||_{L^p} \right]^r \\ &\le C(\eta\lambda)^{-r} \left[\prod_{i=1}^2 \left(\sum_{|\alpha|=m_i} ||D^\alpha A_i||_{\dot\Lambda_\beta} \right) M_{2\beta,p}(f)(z) \right]^r |\overline{\overline{Q}}_j|^{r(1/p+2/q)} \\ &\le C(\eta\lambda)^{-r}(\gamma\lambda)^r |\overline{\overline{Q}}_j| \le C(\gamma/\eta)^r |Q_j|. \end{aligned}$$

The estimates on f_2. Let $H = H(n)$ be a large positive integer depending only on n. We consider the following two cases:

Case 1. $\mathrm{diam}(\tilde{Q}_j) \le \varepsilon \le H\mathrm{diam}(\tilde{Q}_j)$. Let

$$A_1^j(x) = A_1(x) - \sum_{|\alpha|=m_1} \frac{1}{\alpha!} (D^\alpha A_1)_{Q_j} \cdot x^\alpha$$

and

$$A_2^j(x) = A_2(x) - \sum_{|\alpha|=m_2} \frac{1}{\alpha!} (D^\alpha A_2)_{Q_j} \cdot x^\alpha,$$

then $T_\varepsilon^{A_1,A_2}(f_2)(x) = T_\varepsilon^{A_1^j,A_2^j}(f_2)(x)$. Set

$$K(x,y) = \frac{\Omega(x, x-y)}{|x-y|^{n+m}} \prod_{i=1}^2 R_{m_i+1}(A_i^j; x, y).$$

Choose $x_0 \in \tilde{Q}_j$ such that $x_0 \in R^n \setminus \bigcup_j Q_j$. For $x \in Q_j$ [7], we have

$$\begin{aligned} |T_\varepsilon^{A_1^j,A_2^j}(f)(x)| &\le \left| \int_{|x-y|>\varepsilon} (K(x,y) - K(x_0,y)) f(y) dy \right| \\ &\quad + \int_{R(x)} |K(x_0,y) f(y)| dy + \int_{R(x_0)} |K(x_0,y) f(y)| dy + |T_\varepsilon^{A_1,A_2}(f)(x_0)| \\ &= I + II + III + IV, \end{aligned}$$

where $R(u)=\{y\in R^n:\mathrm{diam}(\tilde{Q}_j)<|u-y|\le h\mathrm{diam}(\tilde{Q}_j)\}$. It holds [10]

$$\begin{aligned}T_\varepsilon^{A_1,A_2}(f)(x) &= \sum_{k=1}^{\infty}\sum_{h=1}^{g_k}a_{hk}(x)\int_{|x-y|>\varepsilon}\frac{Y_{hk}(x-y)}{|x-y|^{n+m}}\prod_{i=1}^{2}R_{m_i+1}(A_i^j;x,y)f(y)dy\\ &= \sum_{k=1}^{\infty}\sum_{h=1}^{g_k}a_{hk}(x)S_{hk}^{A}(f)(x),\end{aligned}$$

where $g_k\le Ck^{n-2}$, $||a_{hk}||_{L^\infty}\le Ck^{-2n}$, $|Y_{hk}(x-y)|\le Ck^{n/2-1}$ and

$$\left|\frac{Y_{hk}(x-y)}{|x-y|^n}-\frac{Y_{hk}(x_0-y)}{|x_0-y|^n}\right|\le Ck^{n/2}|x-x_0|/|x-y|^{n+1}$$

for $|x_0-y|>2|x-x_0|>0$.

Let

$$K_{hk}(x,y)=\frac{Y_{hk}(x-y)}{|x-y|^{n+m}}\prod_{i=1}^{2}R_{m_i+1}(A_i^j;x,y).$$

For $x\in Q_j$ [7] we have

$$\begin{aligned}|S_{hk}^{A_1,A_2}(f_2)(x)| &= |S_{hk}^{A_1^j,A_2^j}(f_2)(x)|\\ &\le \left|\int_{|x-y|>\varepsilon}(K_{hk}(x,y)-K_{hk}(x_0,y))f(y)dy\right|+\int_{R(x)}|K_{hk}(x_0,y)f(y)|dy\\ &\quad+\int_{R(x_0)}|K_{hk}(x_0,y)f(y)|dy+|S_{hk}^{A_1,A_2}(f)(x_0)|\\ &= I_{hk}+II_{hk}+III_{hk}+IV_{hk}\end{aligned}$$

and

$$\begin{aligned}I &\le \sum_{k=1}^{\infty}\sum_{h=1}^{g_k}|a_{hk}(x)|I_{hk}\\ II &\le \sum_{k=1}^{\infty}\sum_{h=1}^{g_k}|a_{hk}(x)|II_{hk}\\ III &\le \sum_{k=1}^{\infty}\sum_{h=1}^{g_k}|a_{hk}(x)|III_{hk}.\end{aligned}$$

$$\begin{aligned}I_{hk} &= \int_{|x-y|>\varepsilon}\left(\frac{Y_{hk}(x-y)}{|x-y|^{n+m}}-\frac{Y_{hk}(x_0-y)}{|x_0-y|^{n+m}}\right)\prod_{i=1}^{2}R_{m_i}(A_i^j;x,y)f(y)dy\\ &\quad+\int_{|x-y|>\varepsilon}\Big(R_{m_1}(A_1^j;x,y)-R_{m_1}(A_1^j;x_0,y)\Big)\frac{Y_{hk}(x_0-y)}{|x_0-y|^{m+n}}R_{m_2}(A_2^j;x,y)f(y)dy\end{aligned}$$

$$
\begin{aligned}
&+\int_{|x-y|>\varepsilon}\left(R_{m_2}(A_2^j;x,y)-R_{m_2}(A_2^j;x_0,y)\right)\frac{Y_{hk}(x_0-y)}{|x_0-y|^{m+n}}R_{m_1}(A_1^j;x_0,y)f(y)dy\\
&-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{|x-y|>\varepsilon}\left[\frac{R_{m_2}(A_2^j;x,y)(x-y)^{\alpha_1}}{|x-y|^{m+n}}Y_{hk}(x-y)-\right.\\
&\qquad\left.-\frac{R_{m_2}(A_2^j;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^{m+n}}Y_{hk}(x_0-y)\right]\ D^{\alpha_1}A_1^j(y)f(y)dy\\
&-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{|x-y|>\varepsilon}\left[\frac{R_{m_1}(A_1^j;x,y)(x-y)^{\alpha_2}}{|x-y|^{m+n}}Y_{hk}(x-y)\right.\\
&\qquad\left.-\frac{R_{m_1}(A_1^j;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^{m+n}}Y_{hk}(x_0-y)\right]\ D^{\alpha_2}A_2^j(y)f(y)dy\\
&+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{|x-y|>\varepsilon}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^{m+n}}Y_{hk}(x-y)\right.\\
&\qquad\left.-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^{m+n}}Y_{hk}(x_0-y)\right]\ D^{\alpha_1}A_1^j(y)D^{\alpha_2}A_2^j(y)f(y)dy\\
=\ & I_{hk}^{(1)}+I_{hk}^{(2)}+I_{hk}^{(3)}+I_{hk}^{(4)}+I_{hk}^{(5)}+I_{hk}^{(6)}.
\end{aligned}
$$

For $b\in\dot{\Lambda}_\beta$ and the cube $Q=Q(x_0,d)$, by using Lemma2 and the following inequality

$$|b(x)-b_Q|\le\frac{1}{|Q|}\int_Q||b||_{\dot{\Lambda}_\beta}|x-y|^\beta dy\le||b||_{\dot{\Lambda}_\beta}(|x-x_0|+d)^\beta,$$

we get

$$|R_{m_i}(A_i^j;x,y)|\le\sum_{|\alpha|=m_i}||D^\alpha A_i||_{\dot{\Lambda}_\beta}(|x-y|+d)^{m_i+\beta};$$

By using Lemma 1 and the following formula [3, 7]

$$R_{m_i}(A_i^j;x,y)-R_{m_i}(A_i^j;x_0,y)=\sum_{|\theta|<m_i}\frac{1}{\theta!}R_{m_i-|\theta|}(D^\theta A_i^j;x,x_0)(x-y)^\theta$$

we have

$$
\begin{aligned}
I_{hk}^{(1)}\ &\le\ Ck^{n/2}\prod_{i=1}^2\left(\sum_{|\alpha|=m_i}||D^\alpha A_i||_{\dot{\Lambda}_\beta}\right)|\tilde{Q}_j|^{2\beta/n}\sum_{\nu=0}^\infty\nu^2\int_{2^\nu\varepsilon<|x-y|\le2^{\nu+1}\varepsilon}\frac{|x-x_0|}{|x-y|^{n+1}}|f(y)|dy\\
&\le\ Ck^{n/2}\prod_{i=1}^2\left(\sum_{|\alpha|=m_i}||D^\alpha A_i||_{\dot{\Lambda}_\beta}\right)\sum_{\nu=0}^\infty\nu^2 2^{-(1+2\beta)\nu}\left(\frac{1}{(2^{\nu+1}\varepsilon)^{n-2\beta}}\int_{|x-y|\le2^{\nu+1}\varepsilon}|f(y)|dy\right)\\
&\le\ Ck^{n/2}\prod_{i=1}^2\left(\sum_{|\alpha|=m_i}||D^\alpha A_i||_{\dot{\Lambda}_\beta}\right)M_{2\beta,p}(f)(z)\le Ck^{n/2}\gamma\lambda,
\end{aligned}
$$

$$
\begin{aligned}
I_{hk}^{(2)} &\le Ck^{n/2}\prod_{i=1}^{2}\left(\sum_{|\alpha|=m_i}||D^\alpha A_i||_{\dot\wedge_\beta}\right)|\tilde Q_j|^{2\beta/n}\sum_{\nu=0}^{\infty}\nu^2\int_{2^\nu\varepsilon<|x-y|\le 2^{\nu+1}\varepsilon}\frac{|x-x_0|}{|x-y|^{n+1}}|f(y)|dy\\
&\le Ck^{n/2}\prod_{i=1}^{2}\left(\sum_{|\alpha|=m_i}||D^\alpha A_i||_{\dot\wedge_\beta}\right)M_{2\beta,p}(f)(z)\le Ck^{n/2}\gamma\lambda,\\
I_{hk}^{(3)} &\le Ck^{n/2}\gamma\lambda,\\
I_{hk}^{(4)} &\le C\sum_{|\alpha_1|=m_1}\int_{|x-y|>\varepsilon}\left|\frac{(x-y)^{\alpha_1}Y_{hk}(x-y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}Y_{hk}(x_0-y)}{|x_0-y|^m}\right||R_{m_2}(A_2^j;x,y)|\\
&\qquad\times|D^{\alpha_1}A_1^j(y)||f(y)|dy\\
&\quad+C\sum_{|\alpha_1|=m_1}\int_{|x-y|>\varepsilon}|R_{m_2}(A_2^j;x,y)-R_{m_2}(A_2^j;x_0,y)|\frac{|(x_0-y)^{\alpha_1}Y_{hk}(x_0-y)|}{|x_0-y|^m}\\
&\qquad\times|D^{\alpha_1}A_1^j(y)||f(y)|dy\\
&\le Ck^{n/2}\prod_{i=1}^{2}\left(\sum_{|\alpha|=m_i}||D^\alpha A_i||_{\dot\wedge_\beta}\right)M_{2\beta,p}(f)(z)\le Ck^{n/2}\gamma\lambda,\\
I_{hk}^{(5)} &\le Ck^{n/2}\gamma\lambda,\\
I_{hk}^{(6)} &\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{|x-y|>\varepsilon}\left|\frac{(x-y)^{\alpha_1+\alpha_2}Y_{hk}(x-y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}Y_{hk}(x_0-y)}{|x_0-y|^m}\right|\\
&\qquad\times|D^{\alpha_1}A_1^j(y)||D^{\alpha_2}A_2^j(y)||f(y)|dy\\
&\le Ck^{n/2}\gamma\lambda.
\end{aligned}
$$

Therefore $I_{hk}\le Ck^{n/2}\gamma\lambda$.

When $y\in R(x)$ note that for II_{hk} and III_{hk}

$$|x-y|\le H\mathrm{diam}(\tilde Q_j).$$

We get

$$
\begin{aligned}
II_{hk} &\le \int_{R(x)}\frac{|Y_{hk}(x_0-y)|}{|x_0-y|^{n+m}}\prod_{i=1}^{2}|R_{m_i}(A_i^j;x,y)||f(y)|dy\\
&\quad+C\sum_{|\alpha_1|=m_1}\int_{R(x)}\frac{|R_{m_2}(A_2^j;x_0,y)|}{|x_0-y|^{m_2+n}}|Y_{hk}(x_0-y)||D^{\alpha_1}A_1^j(y)||f(y)|dy\\
&\quad+C\sum_{|\alpha_2|=m_2}\int_{R(x)}\frac{|R_{m_1}(A_1^j;x_0,y)|}{|x_0-y|^{m_1+n}}|Y_{hk}(x_0-y)||D^{\alpha_2}A_2^j(y)||f(y)|dy\\
&\quad+C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\int_{|x-y|>\varepsilon}\frac{|Y_{hk}(x_0-y)|}{|x_0-y|^n}|D^{\alpha_1}A_1^j(y)||D^{\alpha_2}A_2^j(y)||f(y)|dy
\end{aligned}
$$

$$\leq \; Ck^{n/2}\gamma\lambda.$$

Similar $III_{hk} \leq Ck^{n/2}\gamma\lambda$.

Thus

$$\begin{aligned} I + II + III &\leq C\sum_{k=1}^{\infty}\sum_{h=1}^{g_k} |a_{hk}(x)|k^{n/2}\gamma\lambda \\ &\leq C\sum_{k=1}^{\infty} k^{-2n+n/2+n-2}\gamma\lambda \leq C\gamma\lambda. \end{aligned}$$

For IV, since $x \notin \bigcup_j Q_j$ then $|T_\varepsilon^{A_1,A_2}(f)(x_0)| \leq \lambda$. For $x \in Q_j$,

$$\sup_{\varepsilon \approx \mathrm{diam}(\tilde{Q}_j)} |T_\varepsilon^{A_1,A_2}(f_2)(x)| \leq C\gamma\lambda + \lambda.$$

Case 2. $\varepsilon > H\mathrm{diam}(\tilde{Q}_j)$. Let Q_j^ε is the cube with the same center as Q_j, diam $Q_j^\varepsilon = \varepsilon$, and

$$A_1^\varepsilon(x) = A_1(x) - \sum_{|\alpha|=m_1} \frac{1}{\alpha!}(D^\alpha A_1)_{Q_j^\varepsilon} \cdot x^\alpha$$

and

$$A_2^\varepsilon(x) = A_2(x) - \sum_{|\alpha|=m_2} \frac{1}{\alpha!}(D^\alpha A_2)_{Q_j^\varepsilon} \cdot x^\alpha.$$

Similar to **Case 1**,

$$\sup_{\varepsilon > H\mathrm{diam}(\tilde{Q}_j)} |T_\varepsilon^{A_1,A_2}(f_2)(x)| \leq C\gamma\lambda + \lambda.$$

Thus, we have shown that for $x \in Q_j$,

$$T_\star^{A_1,A_2}(f_2)(x) \leq C\gamma\lambda + \lambda.$$

Now, choose γ_0 such that $C\gamma_0 < 1$ and $\eta = 1$. Combining the estimates on f_1 and f_2, we get

$$\begin{aligned} &\left|\left\{x \in Q_j : T_\star^{A_1,A_2}(f)(x) > 3\lambda, \; \prod_{i=1}^{2}\left(\sum_{|\alpha_i|=m_i} ||D^{\alpha_i}A_i||_{\dot{\Lambda}_\beta}\right) M_{2\beta,p}(f)(x) \leq \gamma\lambda\right\}\right| \\ &\leq |\{x \in Q_j : T_\star^{A_1,A_2}(f_1)(x) > 2\lambda - C\gamma\lambda\}| + |\{x \in Q_j : T_\star^{A_1,A_2}(f_2)(x) > \lambda + C\gamma\lambda\}| \\ &\leq C\gamma^r|\{x \in Q_j : T_\star^{A_1,A_2}(f_1)(x) > \lambda\}| \\ &\leq C\gamma^r|Q_j|. \end{aligned}$$

This completes the proof of Theorem 1.

Theorem 2 follows from Theorem 1 and Lemma 3.

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