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SEQUENCES RELATED TO THE SUM OF DIVISORS[1]

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Abstract. In this paper are given some formulas for $\sigma(n)$, the sum of divisors of positive integer n. According to these formulas $\sigma(n)$ can be evaluated via some sequences without knowledge of prime factorization of n.

1. INTRODUCTION

Let n be a positive integer. $\sigma(n)$ denotes the sum of all positive divisors of n, and can be given symbolically by

$$\sigma(n) = \sum_{d|n} \sigma(d).$$

If we know the prime factorization of n, i.e. if

$$n = \prod_{j=1}^{k} p_j^{\alpha_j}, \quad p_j \in P, \alpha_j \in N, \tag{1}$$

then $\sigma(n)$ can be evaluated by the formula:

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$$\sigma(n)=\prod_{j=1}^{k}\frac{p_j^{\alpha_j+1}-1}{p_j-1}. \tag{2}$$

If we extend formula (1) to all $\alpha_j=0$, $\sigma(1)=1$ can be found from (2).

For $\sigma(n)$ we can give the formulas:

$$\sigma(n)=\sum_{k=1}^{n}\sum_{j=0}^{k-1}e^{\frac{2\pi nji}{k}}, \tag{3}$$

$$\sigma(n)=\frac{1}{n!}\sum_{k=1}^{n}k\left(\frac{x^k}{1-x^k}\right)_{x=0}^{(n)}, \tag{4}$$

$$\sigma(n)=n+\sum_{1\leqslant q\leqslant\frac{n}{2}}\left(\left[\frac{n}{q}\right]-\left[\frac{n-1}{q}\right]\right)q \quad (n\geqslant 2), \tag{5}$$

$$\sigma(n)=2n-1-S_n+S_{n-1} \quad (n\geqslant 3), \tag{6}$$

where S_k is the sum of the residues obtained by dividing k by each integer less than k.

Formula (5) was obtained by G. L. Dirichlet [3], while formula (6) was stated by E. Cesaro [6] and proved by E. Catalan [7]. From formula (6) can be derived the formula

$$\sigma(n)=\begin{cases} 2n-1-T_n+T_{n-1}-\left[\frac{n}{2}\right], & n\geqslant 3, \quad n\text{-odd}\\ 2n-1-T_n+T_{n-1}, & n\geqslant 4, \quad n\text{-even}\end{cases} \tag{7}$$

where T_k is the sum of the residues obtained by dividing k by each integer $\leqslant\left[\frac{k}{2}\right]$.

There are some formulas with recursive relations for $\sigma(n)$. Detailed information about them can be found in L. E. Dickson's book [8], chapter X.

First of them is Euler's [1] formula:

$$\sigma(n)+\sum_{j\geqslant 1}(-1)^j\left\{\sigma\left(n-\frac{3j^2-j}{2}\right)+\sigma\left(n-\frac{3j^2+j}{2}\right)\right\}=$$
$$=\begin{cases}(-1)^{k+1}n, & n=\frac{3k^2\pm k}{2},\\ 0, & \text{otherwise.}\end{cases} \tag{8}$$

The second is the formula:

$$\sum_{j\geqslant 0}(-1)^j(2j+1)\sigma\left(n-\frac{j(j+1)}{2}\right)\begin{cases}(-1)^{k+1}\frac{k(k+1)(2k+1)}{6}, & n=\frac{k(k+1)}{2},\\ 0, & \text{otherwise.}\end{cases} \tag{9}$$

Formula (9) was obtained by C. Sardi [5].

The third is the formula:

$$\sum_{j\geqslant 0}[n-1-5j(j+1)]\sigma(n-j(j+1)) = \begin{cases} \frac{k(k+1)(2k+1)^2}{6}, & n=k(k+1), \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Formula (10) was obtained by V. Bouniakowsky [4].

2. STATEMENTS AND RESULTS

From the formulas which will be given in the second part of this paper, $\sigma(n)$ can be evaluated via some sequences, without knowing prime factors of n.

Theorem 1.

$$\begin{gathered} \sum_{1\leqslant j\leqslant\sqrt{n}} (-1)^{j+1} j^2 \Delta_{n-j^2} = \begin{cases} \sigma(n), & n\text{-}odd, \\ \sigma(n)-\sigma\left(\frac{n}{2}\right), & n\text{-}even, \end{cases} \\ \Delta_s = 2\sum_{1\leqslant t\leqslant\sqrt{s}} (-1)^{t+1}\Delta_{s-t^2}, \quad \Delta_0 = 1. \end{gathered} \tag{11}$$

Proof. Here, we can use Jacobi's [2] identity:

$$\prod_{n=1}^{\infty}\left(1-x^{2n}\right)\left(1+x^{2n-1}\right)^2 = 1+2\sum_{n=1}^{\infty}x^{n^2} \equiv \Psi(x) = \sum_{j=0}^{\infty}a_j x^j \tag{12}$$

with

$$a_0 = 1, \quad a_j = \begin{cases} 2, & j=r^2 \quad (r\geqslant 1), \\ 0, & j\neq r^2 \quad (r\geqslant 1). \end{cases} \tag{13}$$

If we take the logarithmic derivative of (12), and multiply the resulting identity by x, we obtain:

$$x\frac{\Psi'(x)}{\Psi(x)} = \sum_{n=1}^{\infty}\left(-2n\frac{x^{2n}}{1-x^{2n}}\right) + 2\sum_{n=1}^{\infty}(2n-1)\frac{x^{2n-1}}{1+x^{2n-1}}, \tag{14}$$

$$\sum_{n=1}^{\infty}(-2n)\frac{x^{2n}}{1-x^{2n}} = -2\sum_{n=1}^{\infty}\sigma(n)x^{2n}, \tag{15}$$

$$
\begin{aligned}
2\sum_{n=1}^{\infty}(2n-1)\frac{x^{2n-1}}{1+x^{2n-1}} &= 2\sum_{n=1}^{\infty}(2n-1)x^{2n-1}\sum_{m=0}^{\infty}(-1)^m x^{(2n-1)m} \\
= \quad -2\sum_{n=1}^{\infty}(2n-1)\sum_{m=1}^{\infty}(-1)^m x^{(2n-1)m} &= -2\sum_{k=1}^{\infty}x^k\sum_{2n-1|k}(-1)^{\frac{k}{2n-1}}(2n-1) \\
= \quad -2\sum_{n=1}^{\infty}d_n x^n, \qquad d_n &= \sum_{2k-1|n}(-1)^{\frac{n}{2k-1}}(2k-1).
\end{aligned}
$$

For $n = 2^a q$, where q is odd, it is

$$d_n = (-1)^n \sigma(q). \tag{16}$$

For even n it is

$$
\begin{aligned}
\sigma(n) \quad &= \quad \left(2^{a+1}-1\right)\sigma(q), \quad \sigma\left(\frac{n}{2}\right) = \left(2^a-1\right)\sigma(q) \quad \Rightarrow \\
\Rightarrow \quad & \quad \sigma(q) = \begin{cases} \sigma(n), & n\text{-odd}, \\ \sigma(n) - 2\sigma\left(\frac{n}{2}\right), & n\text{-even}. \end{cases}
\end{aligned} \tag{17}
$$

From (14), (15), (16), (17) we have

$$x\frac{\Psi'(x)}{\Psi(x)} = -2\sum_{n=1}^{\infty}\sigma(n)x^{2n} - 2\sum_{n=1}^{\infty}(-1)^n\sigma(q)x^n = \sum_{j=1}^{\infty}e_j x^j, \tag{18}$$

where

$$e_j = \begin{cases} 2\sigma(j), & j\text{-odd}, \\ -2\left(\sigma(j) - \sigma\left(\frac{j}{2}\right)\right), & j\text{-even}. \end{cases} \tag{19}$$

From (12) and (18) it follows

$$
\begin{aligned}
\sum_{n=1}^{\infty} n a_n x^n = \sum_{j=0}^{\infty} a_j x^j \sum_{k=1}^{\infty} e_k x^k \quad \Rightarrow \\
\Rightarrow \quad n a_n = \sum_{k=0}^{n-1} a_k e_{n-k}.
\end{aligned} \tag{20}
$$

If in (20) we take $n \to n, n-1, \ldots, 1$, we can consider it as the system with unknown $e_1, e_2, \ldots, e_n$ with a_j given from (13). The determinant of that system is 1. Solving the system with respect to e_n by Kramer's rule, we get the desired formula (11).

Thus, we have proved theorem 1. $\square$

Theorem 2.

$$\sum_{j\geqslant 1}(-1)^{1+\frac{j(j+1)}{2}}\frac{j(j+1)}{2}\omega_{n-\frac{j(j+1)}{2}}=\left\{\begin{array}{rl}\sigma(n), & n\text{-}odd,\\ 4\sigma\left(\frac{n}{2}\right)-\sigma(n), & n\text{-}even.\end{array}\right. \tag{21}$$

$$\omega_s=\sum_{r\geqslant 1}(-1)^{1+\frac{r(r+1)}{2}}\omega_{s-\frac{r(r+1)}{2}},\quad \omega_0=1. \tag{21'}$$

Proof. Now, we use the following Jacobi's [2] identity:

$$\prod_{n=1}^{\infty}\left(1-x^{2n}\right)\left(1+x^{2n}\right)^2=1+\sum_{n=1}^{\infty}x^{n(n+1)}\equiv\Phi(x)=\sum_{j=0}^{\infty}b_jx^j, \tag{22}$$

with

$$b_0=1,\quad b_j=\left\{\begin{array}{lll}1, & j=r(r+1) & (r\geqslant 1),\\ 0, & j\neq r(r+1) & (r\geqslant 1).\end{array}\right. \tag{23}$$

Proceeding as in the proof of theorem 1, we get

$$x\frac{\Phi'(x)}{\Phi(x)}=-2\sum_{n=1}^{\infty}\sigma(n)x^{2n}-4\sum_{n=1}^{\infty}x^{2n}\sum_{k|n}(-1)^{\frac{n}{k}}k. \tag{24}$$

$$l_n=\sum_{k|n}(-1)^{\frac{n}{k}}k.$$

For $n=2^aq$, where q is odd, it is

$$l_n=-\sigma(q)=2\sigma\left(\frac{n}{2}\right)-\sigma(n). \tag{25}$$

Thus, we obtain

$$2nb_{2n}=\sum_{k=0}^{n-1}b_{2k}f_{2(n-k)}, \tag{26}$$

where is

$$f_{2j}=4\sigma(q)-2\sigma(j)=\left\{\begin{array}{rl}2\sigma(j)-8\sigma\left(\frac{j}{2}\right), & j\text{-even},\\ 2\sigma(j), & j\text{-odd}\end{array}\right. \tag{27}$$

In the same manner as in theorem 1, we obtain formula (21), thus proving the theorem 2. □

Theorem 3.

$$(-1)^n\sigma(n)=\sum_{j\geqslant 1}(-1)^{\frac{(j+1)(j+2)}{2}}\frac{j(j+1)(2j+1)}{6}\Omega_{n-\frac{j(j+1)}{2}}, \tag{28}$$

$$\Omega_s=\sum_{r\geqslant 1}(-1)^{\frac{(r+1)(r+2)}{2}}(2r+1)\Omega_{s-\frac{r(r+1)}{2}},\quad \Omega_0=1. \tag{28'}$$

Proof. We use another one Jacobi's [2] identity

$$\prod_{n=1}^{\infty}(1-x^n)^3 = 1+\sum_{n=1}^{\infty}(-1)^n(2n+1)x^{\frac{n(n+1)}{2}} \equiv \sum_{j=0}^{\infty} c_j x^j, \tag{29}$$

with

$$c_0 = 1, \quad c_j = \begin{cases} (-1)^k(2k+1), & j = \frac{k(k+1)}{2} \quad (k \geqslant 1), \\ 0, & j \neq \frac{k(k+1)}{2} \quad (k \geqslant 1). \end{cases} \tag{30}$$

From this identity we get the equality (9), which, as in the proof of theorem 1, yields to formula (28). $\square$

Further, we give a scheme based on the result of theorem 1.

On the scheme, in the first diagonal is

$$\lambda_1(n) = \begin{cases} \sigma(n), & n\text{-odd}, \\ \sigma(n) - \sigma\left(\frac{n}{2}\right), & n\text{-even}, \end{cases}$$

with

$$\lambda_1(2^a q) = 2^a \sigma(q), \quad q\text{-odd},$$

$$\begin{aligned}
\lambda_1(0) &= \frac{1}{2}, \\
\lambda_1(n) &= 2\lambda_1(n-1) - \lambda_2(n), \\
\lambda_2(n) &= 2\lambda_1\left(n-2^2\right) - \lambda_3(n), \\
\lambda_3(n) &= 2\lambda_1(n-3^2) - \lambda_4(n), \\
&\vdots \\
\lambda_k(n) &= 2\lambda_1(n-k^2) - \lambda_{k+1}(n),
\end{aligned}$$

for $n > k^2$ and

$$\begin{aligned}
\lambda_k(n) &= 0, \ n \leq k^2 - 1, \\
\lambda_k(k^2) &= k^2.
\end{aligned}$$

$\lambda_k(n)$ is number in the n-th column and in the k-th diagonal. Since $\lambda_k(k^2) = k^2 \quad (k \in N)$, initial values in diagonals are $1, 4, 9, 16, 25, ...$

According to the scheme 1, the following recurrence relation straightforwardly follows:

$$\lambda_1\left(m^2+k\right) = (-1)^{m+1} m^2 \delta_{k,0} + 2\sum_{j=1}^{m} (-1)^{j+1} \lambda_1\left(m^2+k-j^2\right), \quad (0 \leqslant k \leqslant 2m)$$

where $j_{\max} = m-1$ for $k=0$, and

$$\delta_{k,0} = \begin{cases} 1, & k=0 \\ 0, & k \neq 0. \end{cases}$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	1			4					9							16		
		2			2					2							2	
			4			4					4							4
				4			8					8						
					6			8					8					
						8			3					12				
							8			14					16			
								8			12					0		
									13			8					14	
										12			18					22
											12			12				
												16			8			
													14			32		
														16			14	
															24			10
																16		
																	18	
																		26

Scheme 1.

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