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ON LOCALLY BOUNDED SPACES AND THEIR PRODUCTS*

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Abstract. In this paper we present a new characterization of locally bounded topological vector spaces, which generalize earlier characterizations of Aoki [1] and Rolewicz [13]. Further we shall prove that Topological vector space is Φ-paranormable (class introduced by S. Kasahara in 1973) if and only if it is a product of locally bounded spaces.

1. INTRODUCTION

Locally bounded spaces are very important in the theory of topological vector spaces. For example all normed linear spaces are locally bounded. Many characterizations of this class of metrizable spaces exist as metric linear space is locally bounded if and only if it is p-normable, or metric linear space is locally bounded if and only

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if it is quasi-normable. In this paper we present a new characterization of locally bounded topological vector spaces, which generalize earlier characterizations of Aoki [1] and Rolewicz [13].

Products of locally bounded spaces are also an important class in functional analysis, because they contain classes of locally convex spaces.

Many approaches to the theory of uniform spaces have been made by Tychonov, Kurepa, Weil, Efremovich, and many others. One of them was introduced by Antonovski, Boltjanski and Sarymsakov in 1960. They considered uniform spaces as "metric spaces" in which the distance between points belongs to a topological semifield. S. Kasahara [8], [9] considered topological vector spaces over topological semifield, and introduced the notion of Φ-paranormed spaces. Kasahara's approach was applied in papers of O. Hadžić [4], [5], Lj. Gajić [3], O. Hadžić, Lj. Gajić [6] which contained some results in the fixed point theory and related topics of nonlinear analysis. We shall prove that Topological vector space is Φ-paranormable if and only if it is the product of locally bounded spaces.

2. PRELIMINARIES

Let X be a metric linear space. Then there exists (see Rolewicz [14]) a metric d on X which is equivalent with the original metric on X such that function $|.\| : X \to [0, +\infty)$ defined by $|x\| = d(x, 0)$ has the following properties:

1) $|x\| = 0$ if and only if $x = 0$;
2) $|x\| = |-x\|$;
3) $|x + y\| \leq |x\| + |y\|$;
4) $0 < \alpha < \beta$ implies $|\alpha x\| < |\beta x\|$.

The mapping $|.\|$ is said to be a *$F-$ norm* or *paranorm.* If there exists a number p, $0 < p \leq 1$, such that $|tx\| = |t|^p|x\|$ for any scalar t and $x \in X$ it is said that $|.\|$ is a *p-norm* and X is a *p-normed* space.

Let X be a topological linear space. X is the quasi-normed space if there exists (see Köte [11]) a continuous function $\|.| : X \to [0, +\infty)$ such that:

1) $\|x| \geq 0$

2) $\|x| = 0$ if and only if $x = 0$;

3) $\|tx| = |t|\|x|$;

4) there exists $k \geq 0$ such that $\|x + y| \leq k(\|x| + \|y|)$.

Then mapping $\|.|$ is said to be quasi norm.

Let X be a Hausdorff topological vector space. A set $A \subseteq X$ is *bounded* if for each neighborhood of zero U there is a scalar α such that $A \subseteq \alpha U$. The space X is *locally bounded* if it contains a bounded neighborhood of zero. Each locally bounded space is metrizable (see [11], [14] or [15]).

Proposition 01. *Let X be a Hausdorff topological vector space. Then the following statements are equivalent:*

1) X is the locally bounded space;

2) X is a p-normed space for some p-norm $|.\|$;

3) X is a quasi normed space for some quasi norm $\|.|$.

The characterization 1) $\Leftrightarrow$ 2) was obtained by Aoki [1] and Rolewicz [13] and 1) $\Leftrightarrow$ 3) was obtained by Bourgin [2] and Hyers [7] (see also [11], [14] or [15]).

In the proof of our result we need the following well known statement on continuous solutions of the multiplicative form of Cauchy's functional equation.

Proposition 02. *Let $\varphi : [0, \infty) \to [0, \infty)$ be a function such that*

$$\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$$

for each $\alpha, \beta \in [0, \infty)$. Then $\phi(\alpha) = 0$ for each $\alpha \geq 0$ or there exists real number p such that $\phi(\alpha) = \alpha^p$ for each $\alpha > 0$.

Proof of this proposition can be found in [12].

By $\mathbf{R}$ we shall denote the set of all real numbers. Further, let X be a Hausdorff topological vector space. Then there exists nonempty set Δ and family of metric linear spaces $\{X_i\}_{i\in\Delta}$ (see Klee [10]) such that

$$X = \prod_{i\in\Delta} X_i.$$

We shall denote by $\mathbf{R}_{\boldsymbol{\Delta}}$ the set of all mappings from Δ into $\mathbf{R}$ with Tychonoff product topology and operations $+$ and scalar multiplication as usual. If $f, g \in \mathbf{R}_{\boldsymbol{\Delta}}$ we shall say that $f \leq g$ if and only if $f(t) \leq g(t)$ for each $t \in \Delta$. By $\mathbf{P}_{\boldsymbol{\Delta}}$ we shall denote the cone of nonnegative elements in $\mathbf{R}_{\boldsymbol{\Delta}}$. For $q = (q_i)_{i \in \Delta} \in \mathbf{R}_{\boldsymbol{\Delta}}$ we define the $i-$projection as:

$$p_i(q) = q_i.$$

S. Kasahara [8], [9] introduced the following definition:

The triplet $(X, \|.], \Phi)$ is a Φ-paranormed space if and only if X is the Hausdorff topological vector space, $\|.] : X \to \mathbf{P}_{\boldsymbol{\Delta}}$, Φ is a continuous, linear, positive mapping from $\mathbf{R}_{\boldsymbol{\Delta}}$ into $\mathbf{R}_{\boldsymbol{\Delta}}$ such that the following conditions are satisfied:

1) $\|x] = 0$ if and only if $x = 0$;

2) $\|tx] = |t|\|x]$;

3) $\|x + y] \leq \Phi(\|x] + \|y])$.

Then mapping $\|.]$ is said to be Φ paranorm.

3. RESULTS

In this note we give the following characterization of locally bounded spaces, which generalize the results of Aoki [1] and Rolewicz [13].

Proposition 1. *Metric linear space X over field $\mathbf{F}$ ($\mathbf{F} \in \{\mathbf{R}, \mathbf{C}\}$) is locally bounded if and only if there exists paranorm (F-norm) $|.\|$ and function $\phi : [0, \infty) \to [0, \infty)$ such that*

$$|\lambda x\| = \varphi(\lambda)|x\|$$

for each $x \in X$ and $\lambda \in \mathbf{F}$.

Proof. If X is locally bounded then according to proposition 01. it follows that X is p-normed, so it satisfies the conditions of the statement.

Let X be a metrizable linear space. Let there exists $F-$norm $\|.\|$ and function φ which satisfies the conditions of the statement. If $\alpha, \beta \neq 0$ and $\|x\| = 1$ then

$$\|\alpha\beta x\| = \varphi(|\alpha||\beta|) = \varphi(|\alpha|)\|\beta x\| = \varphi(|\alpha|)\varphi(|\beta|).$$

Hence according to proposition 02. it follows that $\varphi(|\lambda|) = |\lambda|^p$ for a real number p. Condition 4) implies that φ is monotone increasing and thus $p > 0$. From triangle inequality it follows that $\varphi(2) = \|2x\| \leq 2\|x\| = 2$, because $\|x\| = 1$. Hence $\varphi(2) \leq 2$ and thus $p \leq 1$. Now we have that X is a p-normed space. According to Proposition 01. follows that X is locally bounded.

Proposition 2. *Topological vector space X is Φ-paranormable if and only if it is a product of locally bounded spaces.*

Proof. Let $X = \Pi_{i\in\Delta} X_i$ where X_i are locally bounded spaces. Then from Proposition 02. it follows that there exists family of quasi norms $\{\|.\|_i\}_{i\in\Delta}$ such that $(X_i, \|.\|_i$ is the quasi normed space, and a set of positive real numbers $(k_i)_{i\in\Delta}$ such that $\|x+y\|_i \leq k_i(\|x\|_i + \|y\|_i)$, for any $i \in \Delta$. For

$$x = (x_i)_{i\in\Delta}, \quad x_i \in X_i \quad \text{and} \quad p = (p_i)_{i\in\Delta} \in \mathbf{R}_{\mathbf{\Delta}}$$

we can define

$$\|x] = (\|x_i\|)_{i\in\Delta} \quad \text{and} \quad \Phi(p) = (k_i p_i)_{i\in\Delta}.$$

Now let $(X, \|.], \Phi)$ be a Φ-paranormed space. For any $i \in \Delta$ function $\Phi_i : \mathbf{R} \to \mathbf{R}$ is defined by

$$\Phi_i(t) = p_i(\Phi(t^*)),$$

where

$$t^* = (t_j)_{j\in\Delta} \quad \text{and} \quad t_j = \begin{cases} 0 & j \neq i \\ t & j = i \end{cases}$$

is a continuous, linear, positive mapping from $\mathbf{R}$ into $\mathbf{R}$ and so

$$\Phi_i(t) = k_i t$$

for some real number $k_i > 0$. We can define function $\|.\|_i : X \to [0, +\infty)$ by:

$$\|x\|_i = p_i(\|x])$$

and relation $\sim_i$ on $X \times X$ by: $x \sim_i y$ if and only if $\|x-y|_i = 0$. Now,

$$X = \prod_{i \in \Delta} X/\sim_i$$

and for each $i \in \Delta$ $(X/\sim_i, \|.|_i)$ is a quasi normed space and so it is locally bounded.

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