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SOME PROPERTIES OF LAPLACIAN EIGENVALUES FOR GENERALIZED STAR GRAPHS

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Abstract. In this paper, we discuss all the Laplacian eigenvalues for generalized star graphs. When it is not possible to find the exact eigenvalues, we have given the upper and lower bounds. Moreover, we compare these bounds with the existing bounds in the literature [8, 10].

1. INTRODUCTION

Suppose $K_{1,n-1} \subseteq S_n \subseteq K_n$, where S_n is a graph of order n obtained by adding some edges (if exists) to $K_{1,n-1}$ (star graph of order n) or deleting some edges (if exists) to K_n (complete graph of order n). In other words S_n is a graph such that the highest degree is $n-1$. Let $S^j_{n_j} = (V_j, E_j),\ j = 1, 2, \ldots, k$ be k such graphs with $n_1 \geq n_2 \geq \ldots \geq n_k \geq 2$, where $V_j = \{v_{j1}, v_{j2}, \ldots, v_{jn_j}\}$. Let $\lambda^j_i,\ i = 1, 2, \ldots, n_j$ be a non-increasing sequence of eigenvalues of $L(S^j_{n_j}),\ j = 1, 2, \ldots, k$. Also let v_{1n_j} be the central vertex (degree of that vertex is $n_j - 1$) of the graph $S^j_{n_j},\ j = 1, 2, \ldots, k$.

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Let $A(G)$ be the adjacency matrix of a graph G of order n and $D(G)$ be its diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin's theorem, it follows that its eigenvalues are non-negative real numbers. Moreover since its rows sum to 0, 0 is the smallest eigenvalue of $L(G)$.

A pendant of G is a vertex of degree 1. A pendant neighbor (abbreviated as "neighbor") of G is a vertex adjacent to a pendant. Denote by $q(G)$ the number of neighbors. If I is some interval of the real line, write $m_G(I)$ for the number of eigenvalues of $L(G)$, multiplicity included, that belong to I. In the degenerate case, denote by $m_G(\lambda)$ the multiplicity of λ as an eigenvalue of $L(G)$. It is proved in [1] that $m_G[0, n] = n$, i.e., $n \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0$. The multiplicity of 0 as a Laplacian eigenvalue of G equals to the number of components of G, and the multiplicity of n equals to one less than the number of components of the complement of G. If $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ is an eigenvector corresponding to the eigenvalue λ of $L(G)$ then

$$(d_i - \lambda)x_i = \sum_j \{x_j : \ v_i v_j \in E\}, \ i = 1, 2, \ldots, n. \tag{1.1}$$

For the Laplacian eigenvalues of simple graphs, it has been established that there are a lot of bounds on the Laplacian eigenvalues of a graph (see, for example, [3, 4, 5, 6, 7, 12] and the references therein). Grone et al. [8] and Merris [10] studied the bounds of $m_G(I)$ for some certain I's, especially for $I = (2, n]$. Ming et al. [11] gave a lower bound for $m_G(2, n]$ in terms of the matching number of G when G has no perfect matchings.

The rest of the paper is structured as follows. In Section 2, we discuss some useful lemmas and results which will be used in Section 3 when we prove our main result in this paper, bounds on the Laplacian eigenvalues for generalized star graphs.

2. LEMMAS AND RESULTS

Let G be a graph and let $G^{'} = G + e$ be a graph obtained from G by inserting a new edge e into G. The following Lemmas are noted here from [2, 5, 8, 9, 10] to make this paper self-content.

Lemma 2.1 [2] *The Laplacian eigenvalues of G and $G^{'} = G + e$ interlace, that is,*

$$\lambda_1(G^{'}) \geq \lambda_1(G) \geq \lambda_2(G^{'}) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G^{'}) = \lambda_n(G) = 0.$$

Lemma 2.2 [8, 10] *Let G be a connected graph satisfying $2q(G) < n$. Then*

$$\text{(i) } m_G[0,1) \geq q(G), \ \text{(ii) } m_G(2,n] \geq q(G).$$

Lemma 2.3 [9] *If X_i is a Laplacian eigenvector corresponding to the eigenvalue λ_i of the graph G, then X_i is also a Laplacian eigenvector corresponding to the eigenvalue $n - \lambda_i$ of the graph G^c.*

Lemma 2.4 *Let $\lambda_i, \ i = 1, 2, \ldots, n$ be eigenvalues of $L(S_n)$. Then there exist $n-2$ eigenvectors of the eigenvalues $\lambda_i, \ i = 2, 3, \ldots, n-1$ such that the eigencomponent corresponding to the central vertex is 0.*

Proof. Let n be an eigenvalue of multiplicity k (≥ 1) of $L(S_n)$. Then we can easily construct $k-1$ linearly independent eigenvectors of the eigenvalue n such that the eigencomponent corresponding to the central vertex is zero.

Let v_1 be the central vertex of the graph S_n and v_1^c be the corresponding vertex of the complement graph S_n^c. Therefore vertex v_1^c is the isolated vertex in the complement graph S_n^c.

Let x_1^c be the eigencomponent of an eigenvector of λ $(\neq 0)$ of $L(S_n^c)$ corresponding to the vertex v_1^c. In $L(S_n^c)$, the eigencomponent of an eigenvector of non-zero eigenvalue λ corresponding to v_1^c is zero, as $\lambda x_1^c = 0$. Now, the number of non-zero eigenvalues of $L(S_n^c)$ are $n-k-1$. Using Lemma 2.3 we conclude that the eigencomponent corresponding to the central vertex of $n-k-1$ eigenvalues (these eigenvalues are strictly less than n) of $L(S_n)$ are zero.

Hence the Lemma. $\square$

Corollary 2.5 *Let S_{n_1} be a graph of order n_1 and H be a graph of order n. If any number of vertices of H is connected to the central vertex of S_{n_1}, then all the Laplacian eigenvalues of S_{n_1} are the Laplacian eigenvalues of the resulting graph except the largest Laplacian eigenvalue.*

Proof. This result follows from Lemma 2.4. $\square$

Lemma 2.6[**5**] *Let $G = (V, E)$ be a graph with vertex subset $V' = \{v_1, v_2, \ldots, v_k\}$ having the same set of neighbors $\{v_{k+1}, v_{k+2}, \ldots, v_s\}$, where $V = \{v_1, \ldots, v_k, \ldots, v_s, \ldots, v_n\}$. Then this graph G has at least $k-1$ equal eigenvalues and they are equal to the cardinality of the neighbor set. Also the corresponding $k-1$ eigenvectors are*

$$(\underbrace{1,-1}_{2},0,\ldots,0)^T,\ (\underbrace{1,0,-1}_{3},0,\ldots,0)^T\ ,\ldots,\ \text{and}\ (\underbrace{1,0,\ldots,-1}_{k},0,\ldots,0)^T.$$

Lemma 2.7 [**5**] *Let T be a tree. If λ_1 is the largest eigenvalue of $L(T)$, then*

$$\lambda_1 \geq \max\Big\{\frac{d_i+m_i+1+\sqrt{(d_i+m_i+1)^2-4(d_im_i+1)}}{2}\ :\ v_i \in V\Big\},$$

where d_i is the degree of the vertex v_i and m_i is the average of the degrees of the adjacent vertices of vertex v_i. Moreover, the equality holds if and only if T is a tree $T(d_i, d_j)$, where $T(d_i, d_j)$ is formed by joining the centres of d_i copies of K_{1,d_j-1} to a new vertex v_i, that is, $T(d_i, d_j) - v_i = d_iK_{1,d_j-1}$.

3. MAIN RESULTS

We denote a star graph of order n with $K_{1,n-1}$. Let $G(K_{1,n_1-1}, K_{1,n_2-1}, \ldots, K_{1,n_k-1})$ be a resultant graph such that the central vertices of k star graphs $K_{1,n_1-1},\ K_{1,n_2-1}, \ldots$

and K_{1,n_k-1} are completely connected (that means any two central vertices of k star graphs are adjacent). Let $G = (V, E)$, where $V = \{v_{11}, v_{12}, \ldots, v_{1n_1}; v_{21}, v_{22}, \ldots, v_{2n_2}; \ldots; v_{k1}, v_{k2}, \ldots, v_{kn_k}\}$.

Lemma 3.1 *Let $G(K_{1,n_1-1}, K_{1,n_2-1}, \ldots, K_{1,n_k-1})$ be a graph defined above. Then each eigenvalue of $L(G)$ is 1 of multiplicity $n_1 + n_2 + \ldots + n_k - 2k$ and the other eigenvalues satisfy the following system of equations:*

$$\left.\begin{aligned} \lambda x_{2i-1} &= (k + n_i - 2)x_{2i-1} - (n_i - 1)x_{2i} - \sum_{j=1}^{k}\{x_{2j-1} : \ j \neq i\}, \ i = 1, 2, \ldots, k; \\ \lambda x_{2i} &= x_{2i} - x_{2i-1}, \ i = 1, 2, \ldots, k. \end{aligned}\right\} (A)$$

Proof. By Lemma 2.6, 1 is an eigenvalue of multiplicity $n_1 + n_2 + \ldots + n_k - 2k$.

Let λ ($\neq 1$) be an eigenvalue of $L(G)$. Since $\lambda \neq 1$, all the eigencomponents corresponding to the pendant vertices, those are connected to the same vertex with an eigenvalue λ, are equal. So, we can assume that λ is an eigenvalue corresponding to an eigenvector $\mathbf{X} = (\underbrace{x_1, x_2, x_2, \ldots, x_2}_{n_1}; \underbrace{x_3, x_4, x_4, \ldots, x_4}_{n_2}; \ldots; \underbrace{x_{2k-1}, x_{2k}, x_{2k}, \ldots, x_{2k}}_{n_k})^T$ of $L(G)$.

Therefore the remaining eigenvalues satisfy the system of equations (A). □

Corollary 3.2 *Let λ be an eigenvalue with corresponding eigenvector $\mathbf{X} = (x_1, x_2, x_3, x_4, \ \ldots, x_{2k-1}, x_{2k})^T$ of the system of equations (A). Then all x_{2i-1}, $i = 1, 2, \ldots, k$ can not be zero.*

Proof. If possible, let all x_{2i-1}, $i = 1, 2, \ldots, k$ be zero. We have $\lambda \neq 1$, then we can easily get $x_{2i} = 0$, $i = 1, 2, \ldots, k$. Hence all x_{2i-1}, $i = 1, 2, \ldots, k$ are not zero. □

Corollary 3.3 *Let $n_1 = n_2 = \ldots = n_k = m$. Then the eigenvalues of the system of equations (A) are*

$$\lambda = \frac{k + m + \sqrt{(k + m)^2 - 4k}}{2} \qquad \textit{of multiplicity } k - 1,$$
$$\mu = \frac{k + m - \sqrt{(k + m)^2 - 4k}}{2} \qquad \textit{of multiplicity } k - 1,$$

and the remaining two eigenvalues are 0 and m.

Proof. In this case the system of equations are as follows:

$$\left.\begin{aligned} \lambda x_{2i-1} &= (k+m-2)x_{2i-1} - (m-1)x_{2i} - \sum_{j=1}^{k}\{x_{2j-1} :\ j \neq i\},\ i = 1, 2, \ldots, k; \\ \lambda x_{2i} &= x_{2i} - x_{2i-1},\ i = 1, 2, \ldots, k. \end{aligned}\right\} (B)$$

From the system of equations (B), we can easily get

$$\lambda = \frac{k+m+\sqrt{(k+m)^2-4k}}{2},$$

as an eigenvalue of multiplicity $k-1$ corresponding to linearly independent eigenvectors $\Big(\underbrace{1, \frac{1}{1-\lambda}}_{1}, \underbrace{-1, -\frac{1}{1-\lambda}}_{2}, 0, \ldots, 0\Big)^T, \Big(\underbrace{1, \frac{1}{1-\lambda}}_{1}, 0, 0, \underbrace{-1, -\frac{1}{1-\lambda}}_{3}, 0, \ldots, 0\Big)^T, \ldots,$ and $\Big(\underbrace{1, \frac{1}{1-\lambda}}_{1}, 0, \ldots, 0, \underbrace{-1, -\frac{1}{1-\lambda}}_{k}\Big)^T$ respectively.

Similarly,

$$\lambda = \frac{k+m-\sqrt{(k+m)^2-4k}}{2},$$

is an eigenvalue of multiplicity $k-1$ corresponding to linearly independent eigenvectors $\Big(\underbrace{1, \frac{1}{1-\lambda}}_{1}, \underbrace{-1, -\frac{1}{1-\lambda}}_{2}, 0, \ldots, 0\Big)^T$, $\Big(\underbrace{1, \frac{1}{1-\lambda}}_{1}, 0, 0, \underbrace{-1, -\frac{1}{1-\lambda}}_{3}, 0, \ldots, 0\Big)^T, \ldots,$ and $\Big(\underbrace{1, \frac{1}{1-\lambda}}_{1}, 0, \ldots, 0, \underbrace{-1, -\frac{1}{1-\lambda}}_{k}\Big)^T$ respectively.

Also m is an eigenvalue with eigenvector $(1, \frac{1}{1-m}, 1, \frac{1}{1-m}, \ldots, 1, \frac{1}{1-m})^T$ and 0 is an eigenvalue with eigenvector $(1, 1, \ldots, 1)^T$ satisfy (B). $\square$

Corollary 3.4 *Let $n_1 = n_2 = \ldots = n_r = m$, $r \leq k$. Then λ and μ are two eigenvalues of multiplicities at least $r-1$ and are given by*

$$\lambda = \frac{k+m+\sqrt{(k+m)^2-4k}}{2}, \quad \mu = \frac{k+m-\sqrt{(k+m)^2-4k}}{2}.$$

of the system of equations (A).

Theorem 3.5 *Let $G(K_{1,n_1-1}, K_{1,n_2-1}, \ldots, K_{1,n_k-1})$ be a graph. Then the eigenvalues of $L(G)$ are 1 of multiplicity $n_1 + n_2 + \ldots + n_k - 2k$, and the remaining $2k$*

eigenvalues are as follows:

(i) the one set of $k-1$ eigenvalues are bounded by

$$\frac{k+n_1-\sqrt{(k+n_1)^2-4k}}{2} \quad \textit{and} \quad \frac{k+n_k-\sqrt{(k+n_k)^2-4k}}{2},$$

(ii) the another set of $k-1$ eigenvalues are bounded by

$$\frac{k+n_k+\sqrt{(k+n_k)^2-4k}}{2} \quad \textit{and} \quad \frac{k+n_1+\sqrt{(k+n_1)^2-4k}}{2},$$

and (iii) the remaining two eigenvalues are 0 and μ, $n_k \le \mu \le n_1$.

Moreover, the lower and upper bounds for each eigenvalue are equal if and only if $n_1 = n_2 = \ldots = n_k$.

Proof. The eigenvalues of $L(H)$ are 1 of multiplicity $kn_k - 2k$, $\frac{k+n_k+\sqrt{(k+n_k)^2-4k}}{2}$ of multiplicity $k-1$, $\frac{k+n_k-\sqrt{(k+n_k)^2-4k}}{2}$ of multiplicity $k-1$ and n_k, 0, where $H = H(K_{1,n_k-1}, K_{1,n_k-1}, \ldots, K_{1,n_k-1})$.

Using above result and Lemma 2.1 we conclude that the $k-1$ eigenvalues of $L(G)$ lie between

$$\frac{k+n_k+\sqrt{(k+n_k)^2-4k}}{2} \quad \text{and} \quad \frac{k+n_1+\sqrt{(k+n_1)^2-4k}}{2}$$

and the other eigenvalue lies between n_k and n_1. By Lemma 2.6, 1 is an eigenvalue of multiplicity $n_1 + n_2 + \ldots + n_k - 2k$ of $L(G)$.

Using Lemma 2.2 (i) and Corollary 3.3, we get $m_G[0,1) \ge k$. So we conclude that $m_G[0,1) = k$ is the number of remaining eigenvalues. Therefore one eigenvalue of the resulting graph is zero and $m_G(0,1) = k-1$ as G is a connected graph.

We can assume that

$$\mathbf{X} = (\underbrace{x_1, x_2, x_2, \ldots, x_2}_{n_1}; \underbrace{x_3, x_4, x_4, \ldots, x_4}_{n_2}; \ldots; \underbrace{x_{2k-1}, x_{2k}, x_{2k}, \ldots, x_{2k}}_{n_k})^T$$

be an eigenvector corresponding to an eigenvalue λ ($\in (0,1)$) of $L(G)$. Therefore the system of equations are as follows:

$$\left.\begin{array}{l} \lambda x_{2i-1} = (k+n_i-2)x_{2i-1} - (n_i-1)x_{2i} - \sum_{j=1}^{k}\{x_{2j-1} :\ j \ne i\},\ i = 1,2,\ldots,k; \\ \lambda x_{2i} = x_{2i} - x_{2i-1},\ i = 1,2,\ldots,k. \end{array}\right\} (D)$$

Therefore

$$\lambda x_{2i-1} = (k+n_i-2)x_{2i-1} + (n_i-1)\frac{x_{2i-1}}{\lambda-1} - \sum_{j=1}^{k}\{x_{2j-1} : j \neq i\},\ i=1,2,\ldots,k. \quad (3.1)$$

Since the sum of the eigencomponents corresponding to the eigenvalue λ is zero, we have

$$\sum_{j=1}^{k} x_{2j-1} + \sum_{j=1}^{k}(n_j-1)x_{2j} = 0,$$

$$\text{i.e.,}\quad \sum_{j=1}^{k}\Big(1-\frac{n_j-1}{\lambda-1}\Big)x_{2j-1} = 0,$$

$$\text{i.e.,}\quad \sum_{j=1}^{k}\frac{\lambda-n_j}{\lambda-1}x_{2j-1} = 0. \quad (3.2)$$

Since $\lambda \in (0,1)$, we get at least two eigencomponents of x_{2i-1}'s are of different signs. We can assume that x_{2i-1} and x_{2j-1} are of different signs, where $n_i \geq n_j$. From (3.1), we get

$$\begin{aligned} \lambda(x_{2i-1}-x_{2j-1}) &= n_i x_{2i-1} - n_j x_{2j-1} + (k-2)(x_{2i-1}-x_{2j-1}) \\ &\quad + \frac{n_i x_{2i-1} - n_j x_{2j-1}}{\lambda-1} + \frac{\lambda-2}{\lambda-1}(x_{2i-1}-x_{2j-1}), \end{aligned}$$

$$\text{i.e.,}\quad \lambda^2 - k\lambda + k = \lambda\frac{n_i x_{2i-1} - n_j x_{2j-1}}{x_{2i-1}-x_{2j-1}},$$

$$\text{i.e.,}\quad \lambda = \frac{k+r\pm\sqrt{(k+r)^2-4k}}{2},\quad \text{where}\quad r = \frac{n_i x_{2i-1} - n_j x_{2j-1}}{x_{2i-1}-x_{2j-1}}. \quad (3.3)$$

Since x_{2i-1} and x_{2j-1} are of different signs, $n_j \leq r \leq n_i$. Therefore $n_k \leq n_j \leq r \leq n_i \leq n_1$.

Hence $k-1$ non-zero eigenvalues (those are less than 1) lie between

$$\frac{k+n_1-\sqrt{(k+n_1)^2-4k}}{2} \quad \text{and} \quad \frac{k+n_k-\sqrt{(k+n_k)^2-4k}}{2}.$$

We can easily show that the lower and upper bounds for each eigenvalue are equal if and only if $n_1 = n_2 = \ldots = n_k$. $\square$

Corollary 3.6 *Let K_{1,n_i-1}, $i=1,2,\ldots,k$ be k star graphs. Suppose two star graphs are connected then the central vertices of these two star graphs are connected.*

Then the Laplacian eigenvalues of the resulting connected graph G are 1 of multiplicity $n_1+n_2+\ldots+n_k-2k$*, and the remaining* $2k$ *eigenvalues are as follows:*
(i) upper bound of the one set of $k-1$ *non-zero eigenvalues is*

$$\frac{k+n_k-\sqrt{(k+n_k)^2-4k}}{2},$$

(ii) the another set of $k-1$ *eigenvalues lie between*

$$n_k \quad \textit{and} \quad \frac{k+n_1+\sqrt{(k+n_1)^2-4k}}{2},$$

and (iii) one eigenvalue lies in (n_k, n_1)*, and another is zero.*

Proof. By Lemma 2.6, 1 is an eigenvalue of multiplicity $n_1+n_2+\ldots+n_k-2k$. Using Theorem 3.5 and Lemma 2.1 we conclude that the remaining eigenvalues of the resulting graph are as follows:
(i) upper bound of the one set of $k-1$ non-zero eigenvalues are

$$\frac{k+n_k-\sqrt{(k+n_k)^2-4k}}{2},$$

(ii) the another set of $k-1$ eigenvalues are lie between

$$n_k \quad \text{and} \quad \frac{k+n_1+\sqrt{(k+n_1)^2-4k}}{2},$$

and (iii) one eigenvalue lies in (n_k, n_1), and another is zero. □

Corollary 3.7 *The eigenvalues of the system of equations (A) are as follows:*
(i) the one set of $k-1$ *eigenvalues are bounded by*

$$\frac{k+n_1-\sqrt{(k+n_1)^2-4k}}{2} \quad \textit{and} \quad \frac{k+n_k-\sqrt{(k+n_k)^2-4k}}{2},$$

(ii) the another set of $k-1$ *eigenvalues are bounded by*

$$\frac{k+n_k+\sqrt{(k+n_k)^2-4k}}{2} \quad \textit{and} \quad \frac{k+n_1+\sqrt{(k+n_1)^2-4k}}{2},$$

and (iii) the remaining two eigenvalues are 0 and μ*,* $n_k \le \mu \le n_1$*.*

Moreover, the lower and upper bounds for each eigenvalue are equal if and only if $n_1 = n_2 = \ldots = n_k$.

Proof. Using Lemma 3.1 and Theorem 3.5, the required result follows. □

Theorem 3.8 *Let* $S^j_{n_j} = (V_j, E_j)$ *be an* S_n *graph, where* $V_j = \{v_{j1}, v_{j2}, \ldots, v_{jn_j}\}$, $j = 1, 2, \ldots, k$. *Let* $\lambda^j_i,\ i = 1, 2, \ldots, n_j$ *be a non-increasing sequence of eigenvalues of* $L(S^j_{n_j}),\ j = 1, 2, \ldots, k$. *Also let* v_{1n_j} *be the central vertex of the graph* $S^j_{n_j},\ j = 1, 2, \ldots, k$. *If any two central vertices are adjacent then the Laplacian eigenvalues of the resulting graph* $(G = (V, E))$ *are*

$$\lambda^j_i,\ i = 2, \ldots, n_j - 1;\ j = 1, 2, \ldots, k;$$

and the remaining $2k$ *eigenvalues are as follows:*
(i) the one set of $k - 1$ *eigenvalues are bounded by*

$$\frac{n_1 + k - \sqrt{(n_1 + k)^2 - 4k}}{2} \quad \textit{and} \quad \frac{n_k + k - \sqrt{(n_k + k)^2 - 4k}}{2},$$

(ii) the another set of $k - 1$ *eigenvalues are bounded by*

$$\frac{n_k + k + \sqrt{(n_k + k)^2 - 4k}}{2} \quad \textit{and} \quad \frac{n_1 + k + \sqrt{(n_1 + k)^2 - 4k}}{2},$$

(iii) the remaining two eigenvalues are 0 and μ, $n_k \leq \mu \leq n_1$.

Moreover, the lower and upper bounds for each eigenvalue are equal if and only if $n_1 = n_2 = \ldots = n_k$.

Proof. Using Corollary 2.5 we conclude that the Laplacian eigenvalues of the resulting graph are

$$\lambda^j_i,\ i = 2, 3, \ldots, n_j - 1;\ j = 1, 2, \ldots, k.$$

Let λ be an eigenvalue corresponding to an eigenvector

$$\mathbf{X} = (x_1, x_2, x_3, x_4, \ldots, x_{2k-1}, x_{2k})^T$$

of the system of equations (A). Therefore

$$\lambda x_{2i-1} = (k + n_i - 2)x_{2i-1} - (n_i - 1)x_{2i} - \sum_{j=1}^{k}\{x_{2j-1} :\ j \neq i\},\ i = 1, 2, \ldots, k;$$

and $\lambda x_{2i} = x_{2i} - x_{2i-1},\ i = 1, 2, \ldots, k;$

i.e., $\lambda x_{2i-1} = (k + n_i - 2)x_{2i-1} - (n_i - 1)x_{2i} - \sum_{j=1}^{k}\{x_{2j-1} :\ j \neq i\},\ i = 1, 2, \ldots, k;$

and $\lambda x_{2i} = d_{ij}x_{2i} - (d_{ij} - 1)x_{2i} - x_{2i-1},\ j = 2, 3, \ldots, n_i;\ i = 1, 2, \ldots, k;$

where d_{ij} is the degree of the j-th vertex of i-th $S^i_{n_i}$ graph.

From these system of equations we can conclude that λ is an eigenvalue corresponding to an eigenvector $\mathbf{X} = (x_1, \underbrace{x_2, \ldots, x_2}_{n_1-1}; x_3, \underbrace{x_4, \ldots, x_4}_{n_2-1}; x_{2k-1}, \underbrace{x_{2k}, \ldots, x_{2k}}_{n_k-1})^T$ of the resulting graph. We have all the eigencomponents of λ^j_i corresponding to the central vertices are zero, where $i = 2, 3, \ldots, n_j - 1;\ j = 1, 2, \ldots, k$. Using Corollary 3.2 we can show that λ is different from the above eigenvalues. Using Corollary 3.7 we get the required result. $\square$

Corollary 3.9 *Let $S^j_{n_j}$ be an S_n graph, $j = 1, 2, \ldots, k$. Suppose two $S^j_{n_j}$ graphs are connected then the central vertices of these two graphs are adjacent. Then the eigenvalues of the resulting connected graph G_1 are*

$$\lambda^j_i,\ i = 2, \ldots, n_j - 1;\ j = 1, 2, \ldots, k;$$

and the remaining $2k$ eigenvalues are as follows:
(i) upper bound of the one set of $k - 1$ non-zero eigenvalues are

$$\frac{k + n_k - \sqrt{(k + n_k)^2 - 4k}}{2},$$

(ii) the another set of $k - 1$ eigenvalues lie between

$$n_k \quad \text{and} \quad \frac{k + n_1 + \sqrt{(k + n_1)^2 - 4k}}{2},$$

(iii) one eigenvalue lies in (n_k, n_1), and another is zero.

Proof. Using Corollary 2.5 we conclude that the Laplacian eigenvalues of the resulting graph are

$$\lambda^j_i,\ i = 2, 3, \ldots, n_j - 1;\ j = 1, 2, \ldots, k.$$

Using Theorem 3.8 and Lemma 2.1 we conclude that the remaining eigenvalues of the resulting graph are as follows:
(i) upper bound of the one set of $k-1$ non-zero eigenvalues is

$$\frac{k+n_k-\sqrt{(k+n_k)^2-4k}}{2},$$

(ii) the another set of $k-1$ eigenvalues lie between

$$n_k \quad \text{and} \quad \frac{k+n_1+\sqrt{(k+n_1)^2-4k}}{2},$$

and (iii) one eigenvalue lies in (n_k, n_1), and another is zero. $\square$

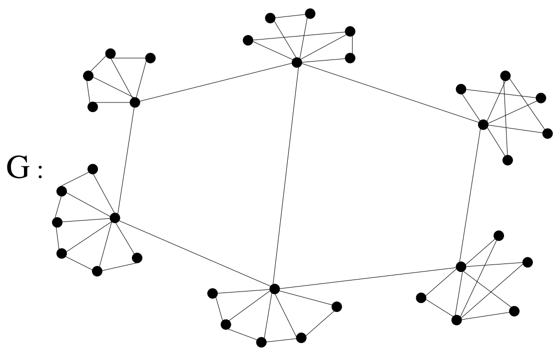

Fig. 1.

Remark 3.10 Let G be a graph as shown in Fig. 1. For this graph G both the lower bounds of $m_G[0,1)$ and $m_G(2,n]$ are 0 using Lemma 2.2, but using Corollary 3.9 we get both lower bounds are 6.

Lemma 3.11 *Let K_{1,n_1-1} $(n_1>1)$ and K_{1,n_2-1} $(n_2>1)$ be two star graphs of order n_1 and n_2 respectively, and $n_1 \geq n_2$. If one isolated vertex is connected to both the central vertices of K_{1,n_1-1} and K_{1,n_2-1}, then the Laplacian eigenvalues of the resulting graph are 1 of multiplicity n_1+n_2-4 and $0, \mu_1, \mu_2, \mu_3, \mu_4$, where $\mu_1, \mu_2, \mu_3, \mu_4$ are bounded as given below:*

$$\frac{n_2+3+\sqrt{(n_2-1)^2+4}}{2} \;\leq\; \mu_1 \leq \frac{n_1+3+\sqrt{(n_1-1)^2+4}}{2},$$

$$\begin{aligned}
\frac{n_2+1+\sqrt{(n_2+1)^2-4}}{2} &\leq \mu_2 \leq \frac{n_1+1+\sqrt{(n_1+1)^2-4}}{2},\\
\frac{n_1+3-\sqrt{(n_1-1)^2+4}}{2} &\leq \mu_3 \leq \frac{n_2+3-\sqrt{(n_2-1)^2+4}}{2},\\
\textit{and } \frac{n_1+1-\sqrt{(n_1+1)^2-4}}{2} &\leq \mu_4 \leq \frac{n_2+1-\sqrt{(n_2+1)^2-4}}{2}.
\end{aligned}$$

Moreover, the lower and upper bounds for each eigenvalue are equal if and only if $n_1 = n_2$.

Proof. Let λ be an eigenvalue of the resulting graph distinct from 1 and the corresponding eigenvector $\mathbf{X}$. By Lemma 2.6, we can say that 1 is an eigenvalue of multiplicity n_1+n_2-4 of the resulting graph. Since $\lambda \neq 1$, therefore we can assume that $\mathbf{X} = (\underbrace{x_1,\ldots,x_1}_{(n_1-1)}, x_2, x_3, x_4, \underbrace{x_5,\ldots,x_5}_{n_2-1})^T$ and all the eigenvalues of the resulting graph distinct from 1 are obtained from the following matrix:

$$W = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -(n_1-1) & n_1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & n_2 & -(n_2-1) \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Then we get the set of equations:

$$\begin{aligned}
\lambda x_1 &= x_1 - x_2\\
\lambda x_2 &= n_1 x_2 - (n_1-1)x_1 - x_3\\
\lambda x_3 &= 2x_3 - x_2 - x_4\\
\lambda x_4 &= n_2 x_4 - (n_2-1)x_5 - x_3\\
\lambda x_5 &= x_5 - x_4,
\end{aligned}$$

that is,

$$\Big(\lambda - n_1 - \frac{n_1-1}{\lambda-1}\Big)x_2 = -x_3 \tag{3.4}$$

$$\lambda x_3 = 2x_3 - x_2 - x_4 \tag{3.5}$$

$$\Big(\lambda - n_2 - \frac{n_2-1}{\lambda-1}\Big)x_4 = -x_3. \tag{3.6}$$

From (3.4) and (3.6), we get

$$\lambda(\lambda-1)+1=\lambda\frac{n_1x_2-n_2x_4}{x_2-x_4}. \tag{3.7}$$

From (3.4), (3.5) and (3.6), we get

$$\lambda\Big(\lambda^2-3\lambda+1-(\lambda-2)\frac{n_1x_2+n_2x_4}{x_2+x_4}\Big)=0. \tag{3.8}$$

Eliminating x_2 and x_4 from these equations, we get one eigenvalue as 0 and the other eigenvalues satisfy the equation $f(\lambda)=0$, where $f(\lambda)=\lambda^4-(n_1+n_2+4)\lambda^3+(n_1n_2+3n_1+3n_2+5)\lambda^2-(2n_1n_2+2n_1+2n_2+4)\lambda+(n_1+n_2+1)$.

Let $\mu_1\geq\mu_2\geq\mu_3\geq\mu_4$ be the roots of the equation $f(\lambda)=0$.

Now, $f(0)=(n_1+n_2+1)>0$, $f(1)=-(n_1-1)(n_2-1)<0$, $f(2)=(n_1+n_2-3)>0$,

$f(n_2)=(n_2-1)^2(n_1-n_2+1)>0$ and $f(n_1+1)=-(n_1-n_2+1)<0$.

Using Lemma 2.7, we get

$$\mu_1>n_1+1.$$

Therefore $n_1+1>\mu_2>n_2$, $2>\mu_3>1$, and $1>\mu_4>0$.

Case (i) $\lambda=\mu_1$ and $\mu_1>n_1+1$.

Since $\mu_1>n_1+1$,

$$\Big(\mu_1-n_1-\frac{n_1-1}{\mu_1-1}\Big)>0, \text{ and } \Big(\mu_1-n_2-\frac{n_2-1}{\mu_1-1}\Big)>0.$$

Using these two results and from (3.4) and (3.6), we conclude that x_2 and x_4 are of the same sign.

Case (ii) $\lambda=\mu_2$ and $n_1+1>\mu_2>n_2$.

Two subcases are (a) $n_1\neq n_2$, (b) $n_1=n_2$.

Subcase (a) $n_1\neq n_2$.

We have $f(n_2+2)=-(n_1-n_2-1)[(n_2+1)^2-2]-2<0$,

and $f(n_2)=(n_2-1)^2(n_1-n_2+1)>0$.

Since $\mu_1>n_1+1\geq n_2+2$, using the above result we get $n_2+2>\mu_2>n_2$.

Using $n_1\geq n_2+1$, and $2<\mu_2<n_2+2$, we have that

$$\Big(\mu_2-n_1-\frac{n_1}{\mu_2-1}-\frac{1}{(\mu_2-1)(\mu_2-2)}\Big)<0. \tag{3.9}$$

From (3.4) and (3.5), we get

$$\Big(\mu_2 - n_1 - \frac{n_1}{\mu_2 - 1} - \frac{1}{(\mu_2-1)(\mu_2-2)}\Big)x_2 = \frac{x_4}{(\mu_2-2)}. \tag{3.10}$$

From equation (3.10) and using the fact (3.9), we conclude that x_2 and x_4 are of different signs.

Subcase (b) $n_1 = n_2$.

In this case $f(n_1+1) = -1 < 0$, and $f(n_1) = (n_1-1)^2 > 0$.

Since $\mu_1 > (n_1+1)$ and using the above result, we get $n_1 < \mu_2 < n_1+1$. In particular, $\mu_2 > 2$, and therefore (3.9) is also satisfied. From (3.10) and using the fact (3.9), we get x_2 and x_4 are of different signs.

Case (iii) $\lambda = \mu_3$ and $2 > \mu_3 > 1$.

Since $\Big(\mu_3 - n_1 - \frac{n_1-1}{\mu_3-1}\Big) < 0$ and $\Big(\mu_3 - n_2 - \frac{n_2-1}{\mu_3-1}\Big) < 0$, from (3.4) and (3.6) we conclude that x_2, x_3 and x_4 are of the same sign.

Case (iv) $\lambda = \mu_4$ and $1 > \mu_4 > 0$.

Without loss of generality, let x_1 and x_5 be positive. From $x_2 = (1-\mu_4)x_1$ and $x_4 = (1-\mu_4)x_5$, we get x_2 and x_4 are positive. From $x_2+x_4 = (2-\mu_4)x_3$, we get x_3 is positive. But it is not possible, because it is well known that all the eigencomponents are not of the same sign corresponding to a non-zero eigenvalue. Therefore x_1 and x_5, and hence x_2 and x_4 are of different signs.

From (3.7), we get

$$\lambda^2 - (s+1)\lambda + 1 = 0, \quad \text{where } s = \frac{n_1x_2 - n_2x_4}{x_2 - x_4}.$$

For $\lambda = \mu_2,\ \mu_4$; x_2 and x_4 are of different signs. Therefore $n_2 \leq s \leq n_1$. From above equation, we get

$$\begin{aligned} \frac{n_2+1+\sqrt{(n_2+1)^2-4}}{2} &\leq \ \mu_2 \leq \frac{n_1+1+\sqrt{(n_1+1)^2-4}}{2} \\ \text{and } \frac{n_1+1-\sqrt{(n_1+1)^2-4}}{2} &\leq \ \mu_4 \leq \frac{n_2+1-\sqrt{(n_2+1)^2-4}}{2}. \end{aligned}$$

From (3.8), we get

$$\lambda^2 - (t+3)\lambda + 2t + 1 = 0, \text{ where } t = \frac{n_1x_2 + n_2x_4}{x_2 + x_4}.$$

For $\lambda = \mu_1,\ \mu_3$; x_2 and x_4 are of the same sign. Therefore $n_2 \leq t \leq n_1$. From the above equation, we get

$$\frac{n_2+3+\sqrt{(n_2-1)^2+4}}{2} \leq \mu_1 \leq \frac{n_1+3+\sqrt{(n_1-1)^2+4}}{2},$$

and
$$\frac{n_1+3-\sqrt{(n_1-1)^2+4}}{2} \leq \mu_3 \leq \frac{n_2+3-\sqrt{(n_2-1)^2+4}}{2}.$$

We can easily show that the lower and upper bounds for each eigenvalue are equal if and only if $n_1 = n_2$. □

Theorem 3.12 *Let $S^1_{n_1}$ $(n_1 > 1)$ and $S^2_{n_2}$ $(n_2 > 1)$ be the two S_n graphs of order n_1 and n_2 respectively, with $n_1 \geq n_2$. Also let $\lambda_i,\ i = 1, 2, \ldots, n_1$ and $\lambda'_i,\ i = 1, 2, \ldots, n_2$ be the eigenvalues of $L(S^1_{n_1})$ and $L(S^2_{n_2})$ respectively. If one isolated vertex is connected to both the central vertices of $S^1_{n_1}$ and $S^2_{n_2}$, then the Laplacian eigenvalues of the resulting graph are $\lambda_i,\ i = 2, 3, \ldots, n_1-1$; $\lambda'_i,\ i = 2, 3, \ldots, n_2-1$ and $0, \mu_1, \mu_2, \mu_3, \mu_4$, where $\mu_1, \mu_2, \mu_3, \mu_4$ are bounded as given below:*

$$\frac{n_2+3+\sqrt{(n_2-1)^2+4}}{2} \leq \mu_1 \leq \frac{n_1+3+\sqrt{(n_1-1)^2+4}}{2},$$

$$\frac{n_2+1+\sqrt{(n_2+1)^2-4}}{2} \leq \mu_2 \leq \frac{n_1+1+\sqrt{(n_1+1)^2-4}}{2},$$

$$\frac{n_1+3-\sqrt{(n_1-1)^2+4}}{2} \leq \mu_3 \leq \frac{n_2+3-\sqrt{(n_2-1)^2+4}}{2},$$

and
$$\frac{n_1+1-\sqrt{(n_1+1)^2-4}}{2} \leq \mu_4 \leq \frac{n_2+1-\sqrt{(n_2+1)^2-4}}{2}.$$

Moreover, the lower and upper bounds for each eigenvalue are equal if and only if $n_1 = n_2$.

Proof. Using Corollary 2.5 we conclude that the Laplacian eigenvalues of the resulting graph are $\lambda_i,\ i = 2, 3, \ldots, n_1-1$ and $\lambda'_i,\ i = 2, 3, \ldots, n_2-1$. Similarly using Theorem 3.8, the remaining eigenvalues can be obtained from the following matrix:

$$W = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -(n_1-1) & n_1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & n_2 & -(n_2-1) \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Using above Lemma 3.11 we get the required result. □

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