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ON THE B-SCROLLS IN THE 3-DIMENSIONAL LORENTZIAN SPACE $\mathbb{L}^3$

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Abstract. The purpose of this paper is introduce nondegenerate ruled surfaces in $\mathbb{L}^3$ which are said to be B-scrolls. We defined the central point, the curve of striction, pseudo-orthogonal trajectory in a B-scroll and obtained some theorems related to these structures in the 3-dimensional Lorentzian space $\mathbb{L}^3$. We gave also the distribution parameter of a B-scroll and some theorems in $\mathbb{L}^3$.

1. INTRODUCTION

L^3 is by definition the 3-dimensional vector space R^3 with the inner product of signature $(1,2)$ given by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

for any colomn vectors $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in R^3$. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis of L^3 given by

$$e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1).$$

A basis $F = \{X, Y, Z\}$ of L^3 is called a (proper) null frame if it satisfies the following conditions:

$$\langle X, X\rangle = \langle Y, Y\rangle = 0,\ \langle X, Y\rangle = -1,$$

$$Z = X\Lambda Y = \sum_{i=1}^{3} \varepsilon_i \det [X, Y, e_i]\, e_i,$$

where $\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$. Hence we obtain that

$$\langle X, Z\rangle = \langle Y, Z\rangle = 0,\ \langle Z, Z\rangle = 1.$$

A vector V in L^3 is said to be null if $\langle V, V\rangle = 0$, ([2,3]).

A surface in the 3-dimensional Lorentz-Minkowski space L^3 is called a timelike surface if the induced metric on the surface is a Lorentzian metric. A ruled surface is a surface swept out by a straight line Y moving along a curve α. The various positions of the generating line Y are called the rulings of the surface. Such a surface has a parametrization in ruled form as follows:

$$\varphi(t, v) = \alpha(t) + vY(t).$$

We call α to be the base curve and Y to be the director curve. Alternatively, we may visualize Y as a vector field on α. Frequently, it is necessary to restrict v to some interval, so the rulings may not be entire straight lines. If the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable surface. The remaining ruled surfaces are called skew surfaces. If there exists a common perpendicular to two preceding rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the curve of striction, [7].

2. B-SCROLLS IN L^3

Let $\alpha = \alpha(t)$ be a null curve in L^3, that is, a smooth curve whose tangent vectors $\alpha'(t)$, for every $t \in I$ are null. For a given smooth positive function $d = d(t)$ let us put

$X = X(t) = d^{-1}\alpha'$. Then X is a null vector field along α. Moreover, there exists a null vector field $Y = Y(t)$ along α satisfying $\langle X, Y\rangle = -1$. Here if we put $Z = X \wedge Y$, then we can obtain a (proper) null frame field $F = \{X, Y, Z\}$ along α. In this case the pair (α, F) is said to be a (proper) framed null curve.

Let α be a (proper) framed null curve and ∇ be Levi-Civita connection on L^3. Then a framed null curve α satisfies the following so called the Frenet equations:

$$\begin{aligned} \nabla_X X &= aX + bZ, \\ \nabla_X Y &= -aY + cZ, \\ \nabla_X Z &= cX + bY, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} a &= -\langle \nabla_X X, Y\rangle \\ b &= \langle \nabla_X X, Z\rangle \\ c &= \langle \nabla_X Y, Z\rangle \end{aligned} \tag{2.2}$$

are smooth functions ([4]).

A framed null curve (α, F) with $d = 1$ and $a = 0$ is called a Cartan framed null curve and the frame field $F = \{X, Y, Z\}$ is called a Cartan frame.Then the Frenet equations (2.1) can be written as follows:

$$\begin{aligned} \nabla_X X &= bZ, \\ \nabla_X Y &= cZ, \\ \nabla_X Z &= cX + bY, \end{aligned} \tag{2.3}$$

Let α be a null curve and $F = \{X, Y, Z\}$ be a Cartan frame along α. If the null vector Y moves along α, then the ruled surface is given by the parametrization $(I \times R, \varphi)$ where

$$\varphi : I \times \mathbb{R} \to \mathbb{L}^3$$

is given by

$$(t, v) \to \varphi(t, v) = \alpha(t) + vY(t), t \in I, v \in J$$

which can be obtained in the 3-dimensional Lorentz-Minkowski space L^3. Then the ruled surface is called a B-scroll and is denoted by M. It is easy to check that M is a timelike surface.

If we fix the parameter v, then the curve $\varphi_v : I \times \{v\} \to M$ sending (t, v) to $\alpha(t) + vY(t)$ can be obtained on M, the tangent vector field of which is given by

$$A = X + cvZ,$$

(cf. [1,5].

Theorem 2.1. Let M be a B-scroll. Then the tangent planes along a ruling of M coincide if and only if $c = 0$.

Proof. It is straightforward.

Then we have following:

Corollary 2.2. The B-scroll M is developable if and only if $c = 0$.

Lemma 2.3. For the B-scroll M,

$$c = -\det(X, Y, \nabla_X Y). \tag{2.4}$$

Proof. If we use equations (2.3), then the proof easily can be done. □

3. POSITION VECTOR OF A CENTRAL POINT AND PSEUDO-ORTHOGONAL TRAJECTORY FOR THE B-SCROLLS

If the distance between the central point and the base curve of a B-scroll which is a skew timelike surface, is $\overline{u}$, then the position vector $\overline{\alpha}(t)$ can be expressed by $\overline{\alpha}(t, \overline{u}) = \alpha(t) + \overline{u}Y(t)$, where $\alpha(t)$ is the position vector of the base curve and $Y(t)$ is the directed vector belonging to the ruling. The parameter $\overline{u}$ can be expressed in terms of position vector of the base curve and directed vector of the ruling. Given three preceding rulings of a B-scroll such that the first one is $Y(t)$, and the second one is $Y(t)+dY(t)$. Let P, P' and Q, Q' be the feet on the rulings of common perpendicular to two preceding rulings. The common perpendicular to $Y(t)$ and $Y(t)+dY(t)$ is $Y(t)\Lambda dY(t)$.

The vector $\overrightarrow{PQ}$ coincides with the vector $\overrightarrow{PP'}$ in the limiting position and $\overrightarrow{PQ}$ will be the tangent vector of the curve of striction. Thus, we have

$$< \nabla_X Y dt, \overrightarrow{PQ} >= 0.$$

In case limiting position,

$$\overrightarrow{PQ} = \frac{d\overline{\alpha}}{dt}.$$

Therefore, if we consider the equations (2.3), we get $\overline{u} = 0$. Thus, position vector of the striction curve is

$$\overline{\alpha}(t) = \alpha(t).$$

Hence we have following:

Theorem 3.1. Let M be a nondevelopable B-scroll in L^3. Then the base curve of a B-scroll is also a striction curve.

Corollary 3.2. Let M be a nondevelopable B-scroll. Then the curve of striction is a null curve.

Theorem 3.3. Let M be a nondevelopable B-scroll. Then $\varphi(t, v_0)$ on the ruling through the point $\alpha(t)$ is a central point if and only if $\nabla_X Y$ is a normal vector of the tangent plane at $\varphi(t, v_0)$.

Proof. Let M be a nondevelopable B-scroll and $\nabla_X Y$ be a normal of the tangent plane at $\varphi(t, v_0)$ on the ruling through $\alpha(t)$. The tangent vector field of the curve

$$\varphi_{v_0} : IX\{v_0\} \to M$$

is

$$A = X + cv_0 Z.$$

Thus

$$\langle \nabla_X Y, A \rangle = 0$$

Then we get

$$v_0 = 0,$$

this means that $\varphi(t, v_0)$ is a central point of M.

Conversely, let $\varphi(t, v_0)$ be a central point on the ruling through $\alpha(t)$. Then we obtain

$$\langle \nabla_X Y, A \rangle = 0.$$

Thus $\nabla_X Y$ is a normal vector of the tangent plane at $\varphi(t, v_0)$. $\square$

Definition 3.1. Let M be a B-scroll in L^3. If there exists a curve which makes constant angle with each one of the rulings, then this curve is called a pseudo-orthogonal trajectory of M.

Theorem 3.4. Let M be a B-scroll in L^3. Then there exists unique pseudo-orthogonal trajectory of M through each point of M.

Proof. Let $\varphi : IXJ \to L^3$ defined by

$$\varphi(t, v) = \alpha(t) + vY(t)$$

be a parametrization of M. A pseudo-orthogonal trajectory of M is given by $\beta : \widetilde{I} \to M$, where $\beta(t) = \alpha(t) + f(t)Y(t)$, $t \in \widetilde{I}$ and $\langle \beta', Y \rangle = const$. We may assume that $\widetilde{I} \subset I$.

Now we want to get a curve which is pass the point $p_0 = \varphi(t_0, v_0)$. Thus we can write

$$p_0 = \alpha(t) + f(t)Y(t)$$

$$p_0 = \alpha(t_0) + v_0 Y(t_0)$$

Therefore we get

$$\alpha(t) = \alpha(t_0)$$

and

$$f(t) = v_0.$$

If we choose I such that it is one to one, then we have $t = t_0$. Therefore the pseudo-orthogonal trajectory of M through the point p_0 is unique. Since this pseudo-orthogonal trajectory of M makes constant angle with each one of the rulings of M, we have $\widetilde{I} = I$. Thus the proof is completed. □

Theorem 3.5. Let M be a B-scroll in L^3. The shortest distance between two rulings is measured only on the curve of striction which is one of the pseudo-orthogonal trajectories.

Proof. We consider two rulings which are pass the points $\alpha(t_1)$ and $\alpha(t_2)$, where $t_1, t_2 \in I$ and $t_1 < t_2$. We compute the length $\ell(v)$ of an pseudo-orthogonal trajectory between these two rulings given by

$$\ell(v) = \int_{t_1}^{t_2} \|A\| \, dt = \int_{t_1}^{t_2} (c^2 v^2)^{\frac{1}{2}} dt.$$

Let us find the value of t which minimizes $\ell(v)$ and we get

$$\frac{\partial \ell(v)}{\partial v} = 0.$$

Thus we have

$$v = 0.$$

This completes the proof. □

Definition 3.2. Let M be a nondevelopable B-scroll in L^3. we know that $\overline{u} = 0$. Thus, we have $\nabla_X Y$ and Z are linearly dependent, that is,

$$\lambda \nabla_X Y = X \Lambda Y.$$

Hence we get

$$\lambda = \frac{< X\Lambda Y, \nabla_X Y >}{\|\nabla_X Y\|^2} = -\frac{\det(X, Y, \nabla_X Y)}{\|\nabla_X Y\|^2} = \frac{1}{c^2}.$$

λ is called the distribution parameter of M and denoted by λ or P_Y, where $< \nabla_X Y, \nabla_X Y > \neq 0$.

4. THE GAUSS MAP AND THE SHAPE OPERATOR OF B-SCROLLS IN L_1^3

The Gauss map can be directly obtained from $\varphi_t \Lambda \varphi_v$ getting

$$G(t,v) = -cvY(t) + Z(t). \tag{4.1}$$

As for the shape operator S of M we have that

$$G_v = -c\varphi_v$$

$$G_t = - < \nabla_X X \Lambda Y, X > \varphi_v - c\varphi_t.$$

So we write down as

$$S = \begin{bmatrix} -c & 0 \\ - < \nabla_X X \Lambda Y, X > & -c \end{bmatrix} \tag{4.2}$$

Theorem 4.1. Let M be a B-scroll in L_1^3. The Gaussian curvature K of M is positive.

Proof. The Gaussian curvature of a surface in Lorentzian space is defined by

$$K = \varepsilon \det S,$$

where $\varepsilon = 1$ or $\varepsilon = -1$ according to the surface is timelike or spacelike, respectively. Then we have

$$K = c^2$$

and this completes the proof of the theorem. □

Theorem 4.2. Let M be a B-scroll in L^3. Each one of the ruling of M is an asymptotic line and a geodesic in M.

Proof. Each one of the rulings is geodesic in R_1^3. Since each one of the rulings is a straight line in L^3. Thus we have $\nabla_Y Y = 0$, that is, Y is a geodesic in M. Since

$$S(Y) = S(\varphi_v) = -cY,$$

we have $< S(Y), Y >= 0$ which means that Y is an asymptotic line.This completes the proof. □

Lemma 4.1. Let M be a B-scroll in L^3. $\{X, Y, Z\}$ be a Cartan frame along the base curve α of M. Then

$$X \Lambda Y = Z, \quad Y \Lambda Z = -Y, \quad X \Lambda Z = X$$

and

$$X \Lambda \xi = X = -cvZ, \quad Y \Lambda \xi = -Y, \quad Z \Lambda \xi = -cvY,$$

where ξ is the unit normal vector field of M defined by

$$\xi = -cvY + Z.$$

Proof. It can be proved easily.

Theorem 4.3. Let M be a B-scroll in L^3. Then M is developable if and only if the Gaussian curvature function of M is zero.

Proof. The proof can be done using Corollary 2.2 and Theorem 4.1.

Theorem 4.4. Let M be a nondevelopable B-scroll in L^3. Then

$$\lambda = \frac{1}{\sqrt{K}},$$

where λ and K are the distribution parameter and the Gaussian curvature function of M, respectively.

Proof. Considering the equation (3.1) and (4.3) we get

$$\lambda = \frac{1}{\sqrt{K}}$$

which completes the proof. □

Now we will give a theorem (without loose of generality) Chasles Theorem for the B-scroll in L^3.

Theorem 4.5. For the B-scroll in L^3, the normal vector ξ at a point of a ruling and the normal vector N at the striction point of this ruling are parallel.

Proof. The unit normal vector to the B-scroll M at (t, v) is given by

$$\xi = -cvY + Z$$

and the normal along the striction curve on M is given by

$$N = \frac{\nabla_X Y}{\|\nabla_X Y\|}.$$

Since N is a unit spacelike vector and ξ is a unit spacelike vector, thus if θ is the angle of rotation from the normal N to the normal ξ we get

$$\sin\theta = \|N\Lambda\xi\| = \left\|(-cvY + Z)\Lambda\frac{\nabla_X Y}{\|\nabla_X Y\|}\right\|.$$

By an routine calculation, one can obtain $\theta = 0$. Thus the proof is completed. □

The results in the study are confirmed by the following examples.

Example 4.1. Consider a null curve α of L^3 given by

$$\begin{aligned} \alpha : R &\to L^3 \\ t &\to \alpha(t) = (sht, t, cht) \end{aligned}$$

Then we choose the Cartan frame $\{X, Y, Z\}$ as follows

$$X(t) = \alpha'(t) = (cht, 1, sht)$$

$$Y(t) = \frac{1}{2}(cht, -1, sht)$$

$$Z(t) = (sht, 0, cht).$$

Thus a B-scroll on the curve α is given by

$$\varphi(t, v) = (sht, t, cht) + v(\frac{1}{2}cht, -\frac{1}{2}, \frac{1}{2}sht).$$

Here since $\det(X, Y, \nabla_X Y) = -1 \neq 0$, the B-scroll is nondevelopable. The striction curve is $\overline{\alpha}(t) = (sht, t, cht)$. The distribution parameter is $\lambda = 4$.

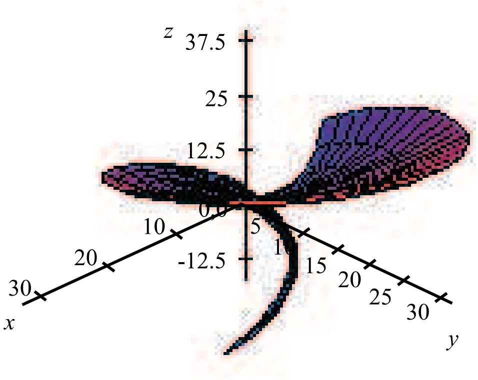

Fig. 1.

Example 4.2. Consider a null curve α of L^3 given by

$$\alpha : R \to L^3$$
$$t \to \alpha(t) = k(\tfrac{t^3}{3} + \tfrac{t}{4}, \tfrac{t^2}{2}, \tfrac{t^3}{3} - \tfrac{t}{4}),\ k \neq 0,$$

where k is the curvature of null curve α.Then we choose the Cartan frame $\{X, Y, Z\}$ as follows

$$X(t) = \alpha'(t) = k(t^2 + \frac{1}{4}, t, t^2 - \frac{1}{4})$$
$$Y(t) = \frac{2}{k}(1, 0, 1)$$
$$Z(t) = (2t, 1, 2t)$$

Thus a B-scroll on the curve α is parametrized by

$$\varphi(t, v) = (k(\frac{t^3}{3} + \frac{t}{4}) + \frac{2v}{k}, \frac{kt^2}{2}, k(\frac{t^3}{3} - \frac{t}{4}) + \frac{2v}{k})$$

Here since $\det(X, Y, \nabla_X Y) = 0$, the B-scroll is developable. The striction curve is $\overline{\alpha}(t) = k(\frac{t^3}{3} + \frac{t}{4}, \frac{t^2}{2}, \frac{t^3}{3} - \frac{t}{4}), k \neq 0.$

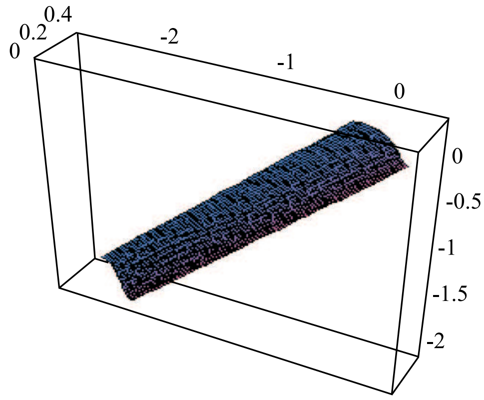

Fig. 2.

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