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ANOTHER CLASS OF EQUIENERGETIC GRAPHS

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Abstract. The energy of a graph is the sum of the absolute values of its eigenvalues. Let $\overline{G}$ and $L^2(G)$ denote the complement and the second iterated line graph, respectively, of the graph G. If G_1 and G_2 are two regular graphs, both on n vertices, both of degree $r \geq 3$, then $\overline{L^2(G_1)}$ and $\overline{L^2(G_2)}$ have equal energies, equal to $(nr-4)(2r-3)-2$.

INTRODUCTION

Using the notation and terminology of the preceding paper [1], we denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of a graph G and by n its order. The energy of the graph G is then defined as $E(G) = |\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|$. Two graphs G_1 and G_2 are said to be equienergetic if $E(G_1) = E(G_2)$.

Let $L^2(G)$ denote the second iterated line graph of the graph G.

In the paper [1] we proved the following:

Theorem 1. *Let G_1 and G_2 be two regular graphs, both on n vertices, both of degree $r \geq 3$. Then $L^2(G_1)$ and $L^2(G_2)$ are equienergetic, and $E\left(L^2(G_1)\right) = E\left(L^2(G_2)\right) = 2nr(r-2)$.*

Here we demonstrate the validity of a similar result. Let $\overline{G}$ denote the complement of the graph G.

Theorem 2. *Let G_1 and G_2 be two regular graphs, both on n vertices, both of degree $r \geq 3$. Then $\overline{L^2(G_1)}$ and $\overline{L^2(G_2)}$ are equienergetic, and $E\left(\overline{L^2(G_1)}\right) = E\left(\overline{L^2(G_2)}\right) = (nr-4)(2r-3)-2$.*

PROOF OF THEOREM 2

From [1] we know that if G is a regular graph of order n and degree r, then $L^2(G)$ is a regular graph of order $nr(r-1)/2$ and degree $4r-6$. If $\lambda_1 = r, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of G, then the spectrum of $L^2(G)$ consists of the numbers

$$\left.\begin{array}{rl} \lambda_i + 3r - 6 & i = 1, 2, \ldots, n \\ 2r - 6 & n(r-2)/2 \text{ times} \\ -2 & nr(r-2)/2 \text{ times} \end{array}\right\} \tag{1}$$

If G is regular of order n and degree r, then its complement $\overline{G}$ is a regular graph of order n and of degree $n-r-1$. The spectrum of $\overline{G}$ consists of the numbers (see [2] or Theorem 2.6 in [3]):

$$\left.\begin{array}{rl} n - r - 1 & \\ -\lambda_i - 1 & i = 2, 3, \ldots, n \end{array}\right\} \tag{2}$$

Combining (1) and (2), we obtain that the eigenvalues of the complement of $L^2(G)$ are:

$$\left.\begin{array}{rl} nr(r-1)/2 - 4r + 5 & \\ -\lambda_i - 3r + 5 & i = 2, 3, \ldots, n \\ -2r + 5 & n(r-2)/2 \text{ times} \\ 1 & nr(r-2)/2 \text{ times} \end{array}\right\} \tag{3}$$

The quantity $nr(r-1)/2-4r+5$ is necessarily positive–valued, because it is equal to the degree of $\overline{L^2(G)}$. Evidently, $-2r+5$ is negative–valued for $r \geq 3$. In order to determine the sign of $-\lambda_i - 3r + 5$, recall that all eigenvalues of a regular graph of degree r lie in the interval $[-r, +r]$. Therefore, $-\lambda_i \leq r$, i. e.,

$$-\lambda_i - r \leq 0 \ . \tag{4}$$

Because $r \geq 3$, we have

$$6 - 2r \leq 0 \ . \tag{5}$$

Summing (4) and (5) we obtain $-\lambda_i - 3r + 6 \leq 0$, from which it follows that the eigenvalues $-\lambda_i - 3r + 5$ are negative–valued for all $i = 2, 3, \ldots, n$.

Knowing the signs of all eigenvalues of $\overline{L^2(G)}$, from (3) we can express its energy:

$$\begin{aligned} E\left(\overline{L^2(G)}\right) &= [nr(r-1)/2 - 4r + 5] + \sum_{i=2}^{n} [-(-\lambda_i - 3r + 5)] \\ &+ [-(-2r+5)] \times \frac{1}{2}\, n(r-2) + [1] \times \frac{1}{2}\, nr(r-2) \end{aligned}$$

which, bearing in mind

$$\sum_{i=1}^{n} \lambda_i = 0 \quad \text{i. e.} \quad \sum_{i=2}^{n} \lambda_i = -\lambda_1 = -r$$

yields the formula

$$E\left(\overline{L^2(G)}\right) = (nr-4)(2r-3) - 2 \ . \tag{6}$$

From Eq. (6) we see that the energy of the complement of the second iterated line graph of a regular graph of order n and degree $r \geq 3$ depends only on the parameters n and r.

Theorem 2 follows.

DISCUSSION

In full analogy with the corollaries of Theorem 1 (stated in [1]), we now have:

Corollary 2.1. *Let G_1 and G_2 be two regular graphs, both on n vertices, both of degree $r \geq 3$. Then for any $k \geq 2$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are equienergetic.*

Corollary 2.2. *Let G_1 and G_2 be two connected and non-cospectral regular graphs, both on n vertices, both of degree $r \geq 3$. Then for any $k \geq 2$, both $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are regular, connected, non-cospectral and equienergetic. Furthermore, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ possess the same number of vertices, and the same number of edges.*

Within Theorem 2 we obtained the expression (in terms of n and r) for the energy of the complement of the second iterated line graph of a regular graph. Analogous (yet much less simple) expressions could be calculated also for $E\left(\overline{L^k(G)}\right)$, $k \geq 3$, i. e., the energy of the complement of the k-th iterated line graph, $k \geq 2$, of a regular graph on n vertices and of degree $r \geq 3$ is also fully determined by the parameters n and r .

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