NONEXPANSIVE MAPPINGS AND CONVEX SEQUENCES

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Abstract. The paper studies the convergence of one convex sequence to a fixed point of nonexpansive mapping on $g$–orbitally complete normed spaces with $\lambda$–uniform convex sphere

1. INTRODUCTION, SOME NEW NOTIONS AND THE MAIN RESULT

Let $X$ be metric space. For mapping $f : E \to E$, $E \subset X$ we say that it is nonextensive if

$$d(f(x), f(y)) < d(x, y)$$

(1)

Nonextensive mappings have been widely studied in relation to existential fixed point in normed spaces, e.g. by Browder [1], Karlovitz [2], Söneberg [9], Kirk [3], Reinermann [8], Opial [7], etc. A considerable contribution to calculating a fixed point of nonextensive compact operator $f : E \to E$, where $E$ is a closed, limited and convex subset of normed space $X$, over the sequence

$$x_{n+1} = \frac{x_n + f(x_n)}{2},$$
has been given by Krasnoselskij [4].

This paper introduces notions like $g$–orbital completeness of space and spaces with $\lambda$–uniform convex sphere.

**Definition.** *Normed space $X$ is considered to be a space with $\lambda$–uniform convex sphere if for every $\varepsilon > 0$ there is $\delta > 0$ so that for all $x, y \in X$ and $\| x - y \| > \varepsilon$ it is true that

$$\| \lambda x + (1 - \lambda)y \| \leq (1 - \delta) \max\{\| x \|, \| y \|\}, \lambda \in (0, 1)$$

(2)***

Let there be a mapping $f : E \to E, E \subset X$ where $X$ is normed space. Let us define a function

$$g(x, f(x)) = (\lambda_1 + \lambda_2 + \ldots + \lambda_p)^{-1}(\lambda_1 x + \lambda_2 f(x) + \ldots + \lambda_p f^{p-1}(x))$$

$\lambda_i \in \mathbb{R}, \lambda_i > 0, i = 1, 2, \ldots, p.$

The set

$$Og(x, f) = \{g_0(x, f(x)), g_1(x, f(x)), g_2(x, f(x)), \ldots\}$$

where $g_0(x, f(x)) = x$ and $g_n(x, f(x)) = g(g_{n-1}(x, f(x)), f(g_{n-1}(x, f(x))))$, is called a sequence of convex orbits given by the function $g$. If each Cauchy’s sequence of $Og(x, f)$ converges to $X$ then space $X$ is $g$–orbitally complete.

Introduction of convex sequences for calculating fixed points of certain mapping is justified by the fact that it is not always possible to reach a fixed point [6].

The point $x$ of a convex set $E \subset X$, where $X$ is normed linear vector space, it is called an extremal point, if $x = \lambda x_1 + (1 - \lambda)x_2, \lambda \in (0, 1), x_1, x_2 \in E$, then follows $x_1 = x_2 = x$.

**Lemma.** *Let $f : E \to E$ be completely continual linear operator, where $E$ is a limited subset of normed space $X$, and $J$ is a set of solutions of equation $x = f(x)$ in $E$.***
Let there be
\[ R(J, \alpha) = \{\alpha \mid x \in E, d(x, J) \geq \alpha\}, \alpha > 0. \]

Then for every \( x \in R(J, \alpha) \) and every \( \alpha > 0 \) there is \( \varepsilon = \varepsilon(\alpha) > 0 \) so that
\[ \| f(x) - x \| > \varepsilon, \]
and the set of solutions \( J \) of equation \( x = f(x) \) is convex.

The proof of this lemma can be found in [5].

**Theorem.** Let there be a complete continual operator \( f = E \to E \) where \( E \) is a closed, limited and convex subset of normed space \( X \) with \( \lambda \)-uniformly convex sphere. If \( O \) is an extremal point of the set \( E \), and for \( p \geq 3 \), \( X \) is \( g \)-orbitally complete space, and for all \( x, y \in E \) operator \( f \) satisfies the condition
\[ \| f(x) - f(y) \| \leq \| x - y \|, \]
then the sequence
\[ x_n = \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \left( \sum_{i=1}^{p} \lambda_i f^{i-1}(x_{n-1}) \right), \]
for \( n \in N \), \( \lambda_i \in R \), \( \lambda_i \geq 0 \), \( i = 1, 2, \ldots, p \) and for arbitrary \( x_0 \in E \), converges at least to one solution of the equation \( x = f(x) \)

**Proof.** We introduce the following mark
\[ J = \{x \mid x \in E, x = f(x)\} \]

On the basis of nonexpansiveness of operator \( f \) and by definition of a sequence \( \{x_n\}_{n \in \mathbb{N}} \) we obtain that
\[ d(x_{n+1}, J) = \inf_{y \in J} \| x_{n+1} - y \| \leq \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \cdot \inf_{y \in J} \sum_{i=1}^{p} \lambda_i \| f^{i-1}(x_n) - f^{i-1}(y) \| \]
\[ \leq \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \cdot \inf_{y \in J} \sum_{i=1}^{p} \lambda_i \| x_n - y \| \]
\[ = \inf_{y \in J} \sum_{i=1}^{p} \lambda_i \| x_n - y \| \]
\[ = d(x_n, y) \]
Consequently, the sequence \( \{d(x_n, J)\}, n \in N \), nonincreasing.

Suppose now that for certain \( \alpha > 0, \ x_1, x_2, \ldots, x_k \in R(J, \alpha) \). On the basis of the previous lemma it follows that for this \( \alpha \) there is \( \varepsilon(\alpha) \) so that

\[
\|f(x_i) - x_i\| > \varepsilon(\alpha), \ i = 1, 2, \ldots, k.
\]

Since space \( X \) with \( \lambda \)-uniformly convex sphere, for each \( y \in J \) we obtain:

\[
\|x_2 - y\| = \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \left( \sum_{i=1}^{p} \lambda_i f^{i-1}(x_1) - \sum_{i=1}^{p} \lambda_i y \right)
\]

\[
\leq \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} (\lambda_1 + \lambda_2) \left\| \frac{\lambda_1}{\lambda_1 + \lambda_2} (x_1 - y) + \frac{\lambda_2}{\lambda_1 + \lambda_2} (f x_1 - f y) \right\| +
\]

\[
+ \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \sum_{i=3}^{p} \lambda_i \left\| f^{i-1}(x_1) - f^{i-1}(y) \right\|
\]

\[
\leq \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} (\lambda_1 + \lambda_2) (1 - \delta) \max \{\|x_1 - y\|, \|f x_1 - f y\|\} +
\]

\[
+ \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \sum_{i=3}^{p} \lambda_i \|x_1 - y\|
\]

\[
\leq \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \left( (\lambda_1 + \lambda_2) (1 - \delta) + \sum_{i=1}^{p} \lambda_i \right) \cdot 2M,
\]

where

\[
M = \sup_{t \in E} \|t\|
\]

Similarly, we prove that

\[
\|x_k - y\| \leq 2M \cdot \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \left( (\lambda_1 + \lambda_2) (1 - \delta) + \sum_{i=3}^{p} \lambda_i \right)^{k-1}
\]

then it follows that

\[
d(x_k, J) \leq 2M \cdot \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \left( (\lambda_1 + \lambda_2) (1 - \delta) + \sum_{i=3}^{p} \lambda_i \right)^{k-1}
\]

Since \( x_i \in R(J, \alpha) \), then also \( \|f(x_i) - x_i\| \geq \varepsilon \) for \( i = 1, 2, \ldots, k \), and consequently

\[
\varepsilon \leq \|f(x_i) - x_i\| \leq \|f(x_i) - f(y)\| + \|y - x_i\| \leq 2 \|x_i - y\|
\]
that is \( \| x_i - y \| \geq \frac{\varepsilon}{2} \) for \( i = 1, 2, \ldots, k \), \( y \in E \).

Now, we have

\[
2M \cdot \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \cdot \left( \left( \lambda_1 + \lambda_2 \right) \left( 1 - \delta \right) + \sum_{i=3}^{p} \lambda_i \right)^{k-1} \geq \frac{\varepsilon}{2}
\]

This inequality is valid if

\[
k < 1 + \left( \ln 4M - \ln \varepsilon \right) \cdot \left( -\ln \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \cdot \left( \left( \lambda_1 + \lambda_2 \right) \left( 1 - \delta \right) + \sum_{i=3}^{p} \lambda_i \right) \right)^{-1}
\]

Since the sequence \( \{d(x_n, J)\} \), \( n \in N \) is nonincreasing, for

\[
n > 1 + \left( \ln 4M - \ln \varepsilon \right) \cdot \left( -\ln \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \cdot \left( \left( \lambda_1 + \lambda_2 \right) \left( 1 - \delta \right) + \sum_{i=3}^{p} \lambda_i \right) \right)^{-1}
\]

it is valid that

\[d(x_n, J) < \alpha,\]

then it follows

\[\lim_{n \to \infty} d(x_n, J) = 0.\]  (3)

Based on the relation (3) it follows that for every \( \beta > 0 \) there is \( n_0 \) so that

\[d(x_{n_0}, J) < \frac{\beta}{2},\]

which implies that for certain \( y_0 \in J \) it is true \( d(x_{n_0}, y) < \frac{\beta}{2} \).

For \( m_1, m_2 > n_0 \) there are inequalities

\[
\| x_{m_1} - x_{m_2} \| \leq \| x_{m_1} - y_0 \| + \| y_0 - x_{m_2} \| \leq \frac{\beta}{2} + \frac{\beta}{2} = \beta
\]

Therefore, the sequence \( \{x_n\}_{n \in N} \) is Cauchy’s and since space \( X \) is \( g \)-orbitally complete, then it is convergent in \( E \). Let there be \( \lim_{n \to \infty} x_n = \xi \).

From the relation

\[
\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \left( \sum_{i=1}^{p} \lambda_i \right)^{-1} \sum_{i=1}^{p} \lambda_i f^i(x_n) = \xi
\]

we get that

\[
\sum_{i=1}^{p} \lambda_i \left( \xi - f^i(\xi) \right) = 0
\]

Since \( 0 \) is extremal point it is valid that \( \xi = f(\xi) \) so sequence \( \{x_n\}_{n \in N} \) converges to at least one solution of equation \( x = f(x) \).

This completes our proof.
References


