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ON THE EXPONENTIAL STABILITY OF A CERTAIN LURIE SYSTEM

Olufemi Adeyinka Adesina

Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria.

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Abstract. We provide necessary and sufficient conditions that ensure the existence of a bounded solution, which is globally exponentially stable, periodic or almost periodic, for a special Lurie system with a single differentiable nonlinear term.

1. INTRODUCTION

Consider a special Lurie direct control system:

$$\begin{cases} \frac{dx}{dt} = Ax + bf(\sigma) \\ \sigma = c^T x \end{cases}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $c, b \in \mathbb{R}^n$, $f(\sigma) \in F$ with

$$F = \{f : f(0) = 0, 0 \leq \frac{f(\sigma)}{\sigma} \leq k < +\infty \text{ for } \sigma \neq 0\}. \tag{1.2}$$

It is well known that if there exists a real number $q \geq 0$, such that

$$Re\{(1 + i\omega q)W(i\omega)\} \geq 0 \text{ for all } \omega \geq 0, \tag{1.3}$$

where $W(i\omega) = -c^T(i\omega I - A)^{-1}b$, then the zero solution of the system (1.1) is absolutely stable. This is the Popov's criterion [20], obtained by using the frequency domain technique.

Impressive results have been obtained on the stability of control systems using frequency domain ideas over the years. Outstanding examples of such work can be found in the articles of Kalman [16], Popov [20] and Yacubovich [23-25], arising in their quests to solve Lurie's problems [19] in automatic controls. There exist generally two different methods in the generating of stability results in frequency domain form. One is the application of Yacubovich-Kalman lemma about solvability of matrix inequality, which uses the tools of Lyapunov functions of the type "quadratic form plus the integral of the nonlinear term". The other is the constructing of Popov functionals in the L^2-space of two functions depending on the solutions of the systems in order to get their upper estimates. However, it was Yacubovich [25] who first realized that the results in [16], [20] and [23] can be used to study forced oscillations of nonlinear control systems. Eversince [25] appeared, there have been a lot of articles on generalizations, applications and extensions of its results (see e.g [1-7] and [9-14]). More expository results can be found in [8, 15, 17-18 and 21-22].

The stimulation of this note comes from the relatively recent paper [26] (and as contained in [18]) where the system (1.1) for the special case

$$A = \begin{pmatrix} -\lambda & 1 & 0 & \dots & 0 \\ 0 & -\lambda & 0 & \dots & 0 \\ 0 & 0 & -\lambda & 0\dots & 0 \\ \vdots & \vdots & 0 \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & -\lambda \end{pmatrix}, \qquad \lambda > 0, \tag{1.4}$$

was considerd and Popov's criterion used in obtaining

$$c^T b \leq 0, \quad c^T A^{-1} b \geq 0 \tag{1.5}$$

as the necessary and sufficient conditions for the absolute stability of the system. Our purpose is to consider in this paper a more general system:

$$\begin{cases} \frac{dx}{dt} = Ax - bf(\sigma) + P(t) \\ \sigma = c^T x \end{cases} \tag{1.6}$$

where A is of the form (1.4), $P(t)$ bounded, c, b and f are as given in (1.1) and (1.2).

We shall use the Yacubovich's approach to obtain necessary and sufficient conditions under which there exists a solution that is bounded, globally exponentially stable, periodic (or almost periodic) according as $P(t)$ is periodic (or almost periodic) for the system (1.6). The results obtained in this work generalized some of those contained in [26] where the Popov's criterion was used. Let us now state without proof, the generalized theorem of Yacubovich as given in [12]:

Theorem O. *Consider the system*

$$X' = AX - B\varphi(\sigma) + P(t), \quad \sigma = C^T X \tag{1.7}$$

where A is an $n \times n$ real matrix, B and C are $n \times m$ real matrices with C^T as the transpose of C, $\varphi(\sigma) = col\varphi_j(\sigma_j), (j = 1, 2, ..., m)$ and $P(t)$ is an n-vector.
Suppose that in the system (1.7), the following assumptions are true:
(i) A is a stable matrix,
(ii) $P(t)$ is bounded for all t in $I\!R$,
(iii) for some constants $\hat{\mu_j} \geq 0, (j = 1, 2, ..., m)$

$$0 \leq \frac{\varphi_j(\sigma_j) - \varphi_j(\hat{\sigma_j})}{\sigma_j - \hat{\sigma_j}} \leq \hat{\mu}_j, (\sigma_j \neq \hat{\sigma}_j), \tag{1.4}$$

(iv) there exists a diagonal matrix $D > 0$, such that the frequency-domain inequality

$$\pi(\omega) = MD + ReDG(i\omega) > 0 \tag{1.5}$$

holds for all ω in $I\!R$, where $G(i\omega) = C^T(i\omega I - A)^{-1}B$ is the transfer function and $M = diag(\frac{1}{\hat{\mu}_j})$, $(j = 1, 2, ..., m)$. Then, the system (1.7) has a bounded solution which is:
(α) globally exponentially stable,
(β) periodic (or almost periodic) if $P(t)$ is periodic (or almost periodic).

2. MAIN RESULTS

Theorem 2.1. *Consider the system (1.6) where A is of the form (1.4). Let $P(t)$ be bounded, $c, b \in I\!R^n$, f continuous and $f(0) = 0$. Suppose there exists a nonnegative* *constant μ such that for all $z, \hat{z} \in I\!R$,*

$$0 \le \frac{f(z) - f(\hat{z})}{z - \hat{z}} \le \mu, \quad (z \ne \hat{z})$$

and

$$\mu > \frac{c_1 b_2 \lambda^2}{(c^T b)^2 \lambda^2 + (c_1 b_2)^2 - 2 c_1 b_2 c^T b \lambda}$$

with $c^T b < 0$. *Then there exists a bounded solution which satisfies properties* (α) *and* (β) *of Theorem O.*

Theorem 2.2. *If there exists a real similarity transformation which transforms the matrix* $\tilde{A}$ *of the system*

$$\begin{cases} \frac{dy}{dt} = \tilde{A}x - \tilde{b} f(\tilde{\sigma}) + \tilde{P}(t) \\ \sigma = \tilde{c}^T x \end{cases}, \tag{2.1}$$

where $\tilde{A} \in I\!R^{n\times n}$, $\tilde{b}, \tilde{c} \in I\!R^n$ *and* $f \in F$ *into the form presented in the Theorem 2.1, then the frequency domain inequalities for both systems (1.6) and (2.1) are equivalent and consequently, the system (2.1) has a bounded solution which satisfies properties* (α) *and* (β) *of Theorem O.*

3. PROOF OF THEOREM 2.1

From the Theorem O, $(i\omega I - A)^{-1}$ for the system (1.6) becomes

$$(i\omega I - A)^{-1} = \begin{pmatrix} \frac{1}{i\omega+\lambda} & \frac{1}{(i\omega+\lambda)^2} & 0 & \dots & 0 \\ 0 & \frac{1}{i\omega+\lambda} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{i\omega+\lambda} & 0\dots & 0 \\ \vdots & \vdots & 0 \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \frac{1}{i\omega+\lambda} \end{pmatrix} \tag{3.1}$$

and the transfer function $G(i\omega)$ is given by

$$\begin{aligned} G(i\omega) &= c^T (i\omega I - A)^{-1} b \\ &= (c_1, c_2, \dots, c_n)(i\omega I - A)^{-1}(b_1, b_2, \dots, b_n)^T \\ &= \textstyle\sum_{j=1}^n \frac{b_j c_j}{i\omega+\lambda} + \frac{c_1 b_2}{(i\omega+\lambda)^2} \\ &= \frac{c^T b(\lambda - i\omega)}{\lambda^2+\omega^2} + \frac{c_1 b_2 (\lambda^2-\omega^2) - 2i\omega c_1 b_2 \lambda}{(\lambda^2-\omega^2)^2} + 4\lambda^2\omega^2. \end{aligned} \tag{3.2}$$

By choosing $D = \tau$ and $M = \frac{1}{\mu}$, the frequency domain inequality (1.5) for the system (1.6) becomes

$$\pi(\omega) = \frac{\tau}{\mu} + \tau \left[\frac{c^T b\lambda}{\lambda^2 + \omega^2} + \frac{c_1 b_2(\lambda^2 - \omega^2)}{(\lambda^2 - \omega^2)^2 + 4\lambda^2\omega^2} \right] > 0.$$

Further simplifications of the right hand side of the above equation gives

$$\frac{\tau}{\mu} \frac{[\omega^6 + (3\lambda^2 + \mu c^T b\lambda - \mu c_1 b_2)\omega^4 + (3\lambda^4 + 2\mu c^T b\lambda^3)\omega^2 + (\lambda^6 + \mu c^T b\lambda^5 + \mu c_1 b_2 \lambda^4)]}{(\lambda^2 + \omega^2)[(\lambda^2 - \omega^2)^2 + 4\lambda^2\omega^2]} > 0. \tag{3.3}$$

For inequality 3.3 to be satisfied, we shall use the following well known result (see e.g [7])

Lemma 3.1. *Let*

$$Q(v) = v^3 + k_1 v^2 + k_2 v + k_3, \tag{3.4}$$

then $Q(v)$ attains its minimum value at $v = \bar{v}$ (say), where

$$\bar{v} = -\left(\frac{k_1}{3}\right)\left[2 - \left(\frac{3k_2}{2{k_1}^2}\right) + o\left((\frac{3k_2}{{k_1}^2})^2\right)\right], \tag{3.5}$$

provided k_1, k_2, k_3 are real constants with $k_1 < 0$ and $|3k_2| \leq {k_1}^2$. $Q(\bar{v})$ is positive if

$$-k_1 k_2 \left[\frac{2}{3} - \frac{1}{2}(\frac{3k_2}{{k_1}^2})\right] + \frac{4{k_1}^3}{27} + k_3 > 0. \tag{3.6}$$

Remark 3.2. Lemma 3.1 is a powerful result on third order polynomials with real constant coefficients. The proof can be obtained by locating the point at which equation (3.4) attains its minimum. See [7] for the proof.

We shall now employ Lemma 3.1 to show that the inequality (3.3) holds. Let $\omega^2 = v$ in the inequality (3.3), then it is equivalent to the third order polynomial

$$v^3 + (3\lambda^2 + \mu c^T b\lambda - \mu c_1 b_2)v^2 + (3\lambda^4 + 2\mu c^T b\lambda^3)v + (\lambda^6 + \mu c^T b\lambda^5 + \mu c_1 b_2\lambda^4) > 0. \tag{3.7}$$

Let

$$\begin{aligned} k_1 &= 3\lambda^2 + \mu c^T b\lambda - \mu c_1 b_2, \\ k_2 &= 3\lambda^4 + 2\mu c^T b\lambda^3, \\ k_3 &= \lambda^6 + \mu c^T b\lambda^5 + \mu c_1 b_2 \lambda^4 \end{aligned}$$

in the equation (3.4), then it follows from Lemma 3.1 that

$$3\lambda^2 + \mu c^T b\lambda - \mu c_1 b_2 < 0, \tag{3.8}$$

$$|3(3\lambda^4 + 2\mu c^T b\lambda^3)| < (3\lambda^2 + \mu c^T b\lambda - \mu c_1 b_2)^2 \tag{3.9}$$

and

$$\begin{aligned}&(\mu c_1 c_2 - \mu c^T b\lambda - 3\lambda^2)(2\mu c^T b\lambda^3 + 3\lambda^4)\left[\tfrac{2}{3} - \tfrac{1}{12}\tfrac{(6\mu c^T b\lambda^3 + 9\lambda^4)}{(\mu c^T b\lambda + 3\lambda^2 - \mu c_1 b_2)^2}\right]\\ &+\tfrac{4}{27}(\mu c^T b\lambda + 3\lambda^2 - \mu c_1 b_2)^3 + (\lambda^6 + \mu c^T b\lambda^5 + \mu c_1 b_2 \lambda^4) > 0.\end{aligned} \tag{3.10}$$

We can deduce from the inequality (3.8) that $c^T b < 0$ and on further simplifications of the inequality (3.9), we have

$$\left[(c^T b\lambda)^2 - 2c_1 b_2 c^T b\lambda + (c_1 b_2)^2\right]\mu^2 - c_1 b_2 \lambda^2 \mu > 0, \tag{3.11}$$

from which we obtained $\mu > 0$ or

$$\mu > \frac{c_1 b_2 \lambda^2}{(c^T b\lambda)^2 - 2c_1 b_2 c^T b\lambda + (c_1 b_2)^2}. \tag{3.12}$$

Thus the solution of the system (1.6) has the qualitative properties (α) and (β) of the Theorem O.

4. PROOF OF THEOREM 2.2

Let $x = Zy$ be a nonsingular transformation with $Z \in \mathbb{R}^{n\times n}$. Then, the system (1.6) is transformed into

$$\frac{dy}{dt}Z = AZy - bf(\sigma) + P(t); \quad \sigma = c^T Zy, \tag{4.1}$$

which is equivalent to

$$\begin{aligned}\tfrac{dy}{dt} &= Z^{-1}AZy - Z^{-1}bf(c^T Zy) + Z^{-1}P(t)\\ &= \tilde{A}y - \tilde{b}f(\tilde{c}^T y) + \tilde{P}(t),\end{aligned} \tag{4.2}$$

where $\tilde{A} = Z^{-1}AZ$, $\tilde{b} = Z^{-1}b$, $\tilde{c} = Z^T c$, $\tilde{P}(t) = Z^{-1}P(t)$. We shall show that $c^T b$ and the inequality (3.11) are not changed by the similarity transformation.

Obviously, $\tilde{c}^T\tilde{b} = c^T ZZ^{-1}b = c^T b$. By applying the generalized theorem of Yacubovich on the system (4.2), we have

$$\tilde{\mu} > \frac{\tilde{c}_1\tilde{b}_2\lambda^2}{(\tilde{c}^T\tilde{b}\lambda)^2 - 2\tilde{c}_1\tilde{b}_2\tilde{c}^T\tilde{b}\lambda + (\tilde{c}_1\tilde{b}_2)^2} \tag{4.3}$$

as a condition to satisfy the frequency domain inequality (1.5) for the system (1.6). Inequality (4.3) is equivalent to

$$\begin{aligned} \tilde{\mu} &> \frac{Z^T c_1 Z^{-1} b_2 \lambda^2}{(c^T ZZ^{-1}b\lambda)^2 - 2Z^T c_1 Z^{-1} b_2 c^T ZZ^{-1} b\lambda + (Z^T c_1 Z^{-1} b_2)^2} \\ &= \frac{c_1 ZZ^{-1} b_2 \lambda^2}{(c^T b)^2\lambda^2 - 2{c_1}^T ZZ^{-1} b_2 c^T ZZ^{-1} b\lambda + ({c_1}^T ZZ^{-1} b_2)^2} \\ &= \frac{c_1 b_2 \lambda^2}{(c^T b)^2\lambda^2 - 2{c_1}^T b_2 c^T b\lambda + ({c_1}^T b_2)^2}. \end{aligned} \tag{4.4}$$

Hence the conclusion of the proof.

References

[1] O. A. Adesina, *Periodicity and stability results for solutions of certain fifth order non-linear differential equations*, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica, **40** (2001), 7–16.

[2] A. U. Afuwape, O. A. Adesina, *Conditions on the qualitative behaviour of solutions of a certain class of fifth order non-linear differential equations*, An. Stii ale Univ. 'Al.I.Cuza', Iasi. t. XLVI s.I a Mathematica, f.2 (2000), 277–294.

[3] A. U. Afuwape, *Uniform dissipative solutions for some fourth-order nonlinear differential equations* , Ann. Stii. Univ. Al. I. Cuza, Iasi. Mathematica, **4** (1991), 431–445.

[4] A. U. Afuwape, *Uniform dissipativity of some fourth-order nonlinear differential equations* , Ann. Stii. Univ. Al. I. Cuza, Iasi. Mathematica, **35** (1989), 9–16.

[5] A. U. Afuwape, *On the applications of the frequency-domain method for uniformly dissipative nonlinear systems* , Math. Nachr., **130** (1987), 217–224.

[6] A. U. Afuwape, *An application of the frequency-domain criteria for dissipativity of a certain third-order nonlinear equation* , Analysis 1, (1981), 211–216.

[7] A. U. Afuwape, *Solutions of some third-order nonlinear differential equations: Frequency domain method* , Ann. Stii. Univ. Al. I. Cuza, Iasi. Mathematica, (1979), 297–302.

[8] M. A. Aizerman, F. R. Gantmacher, *Absolute stability of regulator systems*, Holden-day Inc. San Francisco, London, Amsterdam (1964).

[9] I. Barbalat, *Conditions pour un bon comportement de certaines equations differentielles du troisieme et du quatrieme ordre* , In equations differentielles et fonctionnelles nonlineares, (Eds., P. Janssens, J. Mawhin and N. Rouche), Hermann, Paris, (1973), 80–91.

[10] I. Barbalat, A. Halanay, *Conditions de comportment "presque lineaire" dans la theorie des oscillations* , Rev. Roum. Sci. Techn, **29** (1974) 321–341.

[11] I. Barbalat, A. Halanay, *Nouvelles applications de la methode frequentielle dans la theorie des oscillations* , Rev. Roum. Sci. Techn, **16** (1971) 689–702.

[12] I. Barbalat, A. Halanay, *Applications of the frequency- domain method to forced non-linear oscillations* , Math. Nachr., **44** (1970), 165–179.

[13] A. Halanay, *Frequency domain criteria for dissipativity*, In Ordinary Differential Equations, (Ed., L. Weiss), Academic press, (1972), 413–416.

[14] A. Halanay, *For and against the Liapunov function*, Symp. Math. VI (1971), 167–175.

[15] A. Halanay, *Differential Equations: stability, oscillations, time lags* , Academic Press, New York, (1966).

[16] R. E. Kalman, *Lyapunov functions for the problem of Lurie in automatic control*, Proc. Natn. Acad. Sci. U.S.A., **49** (2) (1963), 201–205.

[17] G. A. Leonov, D. V. Ponomarenko, V. B. Smirnova, *Frequency-domain methods for nonlinear analysis, theory and applications*, World Scientific, Singapore, New Jersey, London, Hongkong, (1996).

[18] X. Liao, *Absolute stability of nonlinear control systems*, Kluwer Academic Publishers, Dordrecht, Boston, London (1993).

[19] A. I. Lurie, *Nonlinear problems of automatic control theory*, Gostekhizdat, Moscow-Leningrad (1951).

[20] V. M. Popov, *Absolute stability of non-linear control systems* , Aut. Rem. Control 22, (1962), 867–875.

[21] R. Reissig, G. Sansone, R. Conti, *Non-linear differential equations of higher order* , Noordhoff, Groninger, (1974).

[22] N. Rouche, P. Habets, M. Laloy, *Stability theory by Liapunov's direct method* , Springer Verlag, New York, Heidelbreg, Berlin, (1977).

[23] V. A. Yacubovich, *The matrix-inequality method in the theory of the stability of non-linear control systems* , Aut. Rem. Control 25, (1964), 905–916.

[24] V. A. Yacubovich, *Frequency-domain conditions for absolute stability and dissipativity of control systems with one differentiable non-linearity* , Soviet Math. Dokl., **6** (1965), 81–101.

[25] V. A. Yacubovich, *Frequency domain criteria for oscillation in non-linear systems with one stationary component* , Sib. Mat. Journal, 14, no 5, (1966), 768–788.

[26] W. Zhang, *The criterion for absolute stability of direct control systems of Lurie type* , Appl. Math and Mech., **10** (1989), 955–967.